15th Triennial World Congress, Barcelona, Spain

ABSOLUTE STABILITY ANALYSIS IN THE PRESENCE OF DISC UNCERTAINTIES

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Abstract: The famous Popov criterion is used for absolute stability analysis of uncertain nonlinear systems. Uncertainty is assumed to exist in the linear subsystem in terms of coefficient perturbations in complex plane discs. Existing results, which are based on strict positive realness (SPR) conditions, are generalized so as to cover a wider spectrum of systems. All the results are then restated using circular arithmetic which is proposed as an alternative to SPR conditions. The use of circular arithmetic leads to non-conservative results at the expense of more computational effort. This is in contrast with the conservative results yielded by the computationally more efficient approach based on SPR conditions.

Keywords: Absolute stability, Popov's criterion, strict positive realness, circular arithmetic, disc uncertainties.

1. INTRODUCTION

Control systems consisting of a known, linear, time-invariant element in the forward path and a sector-bounded nonlinearity in the feedback path have been long under consideration. The importance of this configuration stems from its very nature; it represents a family of systems arising from the uncertainty in the nonlinear part. The famous absolute stability (AS) theory, originally formulated and developed by Lur'e, stands as a major robustness result for systems of this sort. The term AS refers to the stability of a continuum of systems instead of just a specific one. The main idea of this theory is to deduce the stability of an entire family of systems by only studying some of its members (Narendra and Taylor, 1973).

Following the work of Lur'e, numerous solutions have been obtained to different versions of the AS problem. Among these, the most celebrated one is, probably, the Popov criterion (PC) (Popov, 1961) which provides sufficient-only conditions for global asymptotic stability under the assumption that the linear part of the system has a fixed structure. The main reasons underlying this fact are its close resemblance to the classical Nyquist stability condition and its generality in the sense that its special cases correspond to several other solutions to the AS problem (Hsu and Meyer, 1968).

Extensions of the PC have been given by several researchers for cases with one/two of the following three uncertainty types in the linear subsystem: parametric, disc or norm-bounded. Mori *et al.* (1994) and Dahleh *et al.* (1993) have recast it in the presence of interval parametric uncertainty. Soh and Foo (1992) and Tan and Atherton (1999) have considered it under the influence of affine linear parametric uncertainty. Cases involving additive or multiplicative norm-bounded perturbations have been dealt with by Tsypkin and Polyak (1992). Kamal and Dahleh (1993) and Impram and Munro (2001) have provided generalizations for systems with both parametric and norm-bounded perturbations by exploiting some fundamental results from the theory of robust control. With regard to disc uncertainties, on the other hand, the first results were obtained by Polyak and Tsypkin (1991) who derived a robust (but, in general, conservative) version of the PC by utilizing strict positive realness (SPR) conditions.

Disc uncertainty is an alternative to real parametric uncertainty, and it gives rise to transfer functions with complex coefficients. Given that in daily life there is no such thing as a complex number, the obvious question to ask at this point is whether or not such transfer functions make sense and, if they do, how they might arise. It is perhaps easier to give answers to the second question and use these to answer the first one. To this end, the simplest example is probably a system model which involves the describing function of a multi-valued nonlinear element such as a relay with hysteresis (Gelb and Vander Velde, 1968). Bose and Shi (1987) have given more sophisticated examples such as models of whirling shafts, vibrational systems, electrical networks and filters. Filters are also mentioned in the papers by Barros and Lind (1986) and Crystal and Ehrman (1968), where it is claimed that complex coefficient filters are computationally much more efficient when it comes to simulating band-pass processes. Therefore, transfer functions with complex coefficients are not merely theoretical objects, and the motivation for considering disc uncertainty is the same as the one for considering real parametric uncertainty; it is a device for taking into account variation of parameters in prescribed ranges.

The aims of this paper are to (i) extend the results of Polyak and Tsypkin (1991) to cases where the nonlinearity lies in a sector with non-zero lower bound, and (ii) develop a robust and non-conservative version of the PC by making use of circular arithmetic. A numerical example is given to illustrate the use of the mathematical results and to compare the approach based on SPR conditions to that based on circular arithmetic.

2. ROBUST POPOV CRITERION BASED ON SPR CONDITIONS

The system configuration which is considered throughout this paper is depicted in Figure 1, where $C(s) = N_C(s)/D_C(s)$ is a fixed controller with real/complex coefficients and $G(s,\mathbf{q})$ is a proper plant with numerator $N_G(s,\mathbf{a}) = a_m s^m + a_{m-1}s^{m-1} + \cdots + a_1 s + a_0$, denominator $D_G(s,\mathbf{b}) = b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0$, and vector of uncertain parameters $\mathbf{q} = [\mathbf{a},\mathbf{b}] = [a_0,\ldots,a_m,b_0,\ldots,b_n]$. The domain in which $\mathbf{q}$ varies is defined as $\mathcal{Q} = \mathcal{A}_0 \times \cdots \times \mathcal{A}_m \times \mathcal{B}_0 \times \cdots \times \mathcal{B}_n$ where each $\mathcal{A}_i$ ($\mathcal{B}_i$) is a complex plane disc with center a_i^o (b_i^o) and radius $\gamma\alpha_i$ ($\gamma\beta_i$). The scalar γ sets the level of disc uncertainty around the nominal coefficients a_i^o and b_i^o. The input/output (I/O) characteristics of the nonlinearity φ are defined in the sequel.

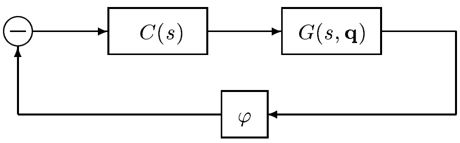

Fig. 1. Uncertain nonlinear system.

An important condition which arises in the solution of the AS problem is the property of SPR of a transfer function. This property is closely related to the passivity and robustness of the system, and is defined as below:

Definition 2.1: A proper transfer function is said to be strictly positive real if

(i) it has no poles in the closed right half plane,
(ii) its real part is positive for all $\omega \in \mathbb{R}$.

When there is disc uncertainty in the transfer function, a family of systems arises and, thus, the tests involved in the above definition need to be carried out infinitely many times. The theorem below is a considerable simplification of this task. Before stating it, however, let $L(s,\mathbf{q}) = C(s)G(s,\mathbf{q})$ and define the following frequency functions:

$$R_N(\omega) = \alpha_m|\omega|^m + \cdots + \alpha_0 \quad (1a)$$
$$R_D(\omega) = \beta_n|\omega|^n + \cdots + \beta_0 \quad (1b)$$
$$R_\alpha(\omega) = |N_C(j\omega)|\, R_N(\omega) \quad (1c)$$
$$R_\beta(\omega) = |D_C(j\omega)|\, R_D(\omega) \quad (1d)$$
$$Z_N(\omega) = R_\alpha(\omega)/\,|N_C(j\omega)N_G(j\omega,\mathbf{a}^o)| \quad (1e)$$
$$Z_D(\omega) = R_\beta(\omega)/\,|D_C(j\omega)D_G(j\omega,\mathbf{b}^o)| \quad (1f)$$
$$Y_N(j\omega) = \sqrt{1-(\gamma Z_N(\omega))^2} + j\gamma Z_N(\omega) \quad (1g)$$
$$Y_D(j\omega) = \sqrt{1-(\gamma Z_D(\omega))^2} + j\gamma Z_D(\omega) \quad (1h)$$
$$Z_{max}(j\omega) = L(j\omega,\mathbf{q}^o)Y_N(j\omega)Y_D(j\omega) \quad (1i)$$
$$Z_{min}(j\omega) = L(j\omega,\mathbf{q}^o)Y_N^*(j\omega)Y_D^*(j\omega) \quad (1j)$$

where $\mathbf{q}^o = [\mathbf{a}^o,\mathbf{b}^o]$ and $*$ denotes complex conjugation.

Theorem 2.1: $L(s,\mathbf{q})$ is strictly positive real for all $\mathbf{q} \in \mathcal{Q}$, if and only if $L(s,\mathbf{q}^o)$ is strictly positive real and the following conditions are satisfied for all $\omega \in \mathbb{R}$:

(i) $Z_D(\omega) < \gamma^{-1}$,
(ii) Re $\{Z_{max}(j\omega)\} > 0$,
(iii) Re $\{Z_{min}(j\omega)\} > 0$.

Proof: Necessity of nominal SPR is obvious since $L(s,\mathbf{q}^o) \in L(s,\mathcal{Q})$. Similarly, condition (i) is nothing but a stability test for $L(s,\mathbf{q})$ (Impram, 2000). Therefore, assume that these two requirements are met. Now, an alternative way of expressing condition (ii) in Definition 2.1 is

$$|\angle L(j\omega,\mathbf{q})| < \pi/2 \ , \ \ \mathbf{q} \in \mathcal{Q} \quad (2)$$

where

$$\begin{aligned}\angle L(j\omega,\mathbf{q}) &= \angle C(j\omega) + \angle G(j\omega,\mathbf{q})\\ &= \angle C(j\omega) + \angle N_G(j\omega,\mathbf{a}) - \angle D_G(j\omega,\mathbf{b})\\ &= \angle C(j\omega) + \angle N_G(j\omega,\mathbf{a}^o) \pm \angle Y_N(j\omega)\\ &\quad - \angle D_G(j\omega,\mathbf{b}^o) \pm \angle Y_D(j\omega)\\ &= \angle L(j\omega,\mathbf{q}^o) \pm \angle Y_N(j\omega) \pm \angle Y_D(j\omega)\end{aligned}$$

Since $\angle$ is essentially the inverse tangent function, which has a monotonically increasing nature, it is straightforward to see that the extreme values of $\angle L(j\omega, \mathbf{q})$ are given by

$$\max \angle L(j\omega,\mathbf{q}) = \angle L(j\omega,\mathbf{q}^o) + \angle Y_N(j\omega) + \angle Y_D(j\omega)$$
$$\min \angle L(j\omega,\mathbf{q}) = \angle L(j\omega,\mathbf{q}^o) - \angle Y_N(j\omega) - \angle Y_D(j\omega)$$

which can be thought of as the phases of two different complex-valued frequency functions. Indeed, comparing these with

$$\begin{aligned}\angle Z_{max}(j\omega) &= \angle L(j\omega,\mathbf{q}^o) + \angle Y_N(j\omega) + \angle Y_D(j\omega)\\ \angle Z_{min}(j\omega) &= \angle L(j\omega,\mathbf{q}^o) + \angle Y_N^*(j\omega) + \angle Y_D^*(j\omega)\\ &= \angle L(j\omega,\mathbf{q}^o) - \angle Y_N(j\omega) - \angle Y_D(j\omega)\end{aligned}$$

yields $\max \angle L(j\omega,\mathbf{q}) = \angle Z_{max}(j\omega)$ and $\min \angle L(j\omega,\mathbf{q}) = \angle Z_{min}(j\omega)$. From (2), it follows that $\angle Z_{max}(j\omega) < \pi/2$ and $\angle Z_{min}(j\omega) > -\pi/2$ which are equivalent to conditions (ii) and (iii), respectively.

The sufficiency proof is easier to give. Whenever conditions (i)-(iii) are satisfied, SPR of $L(s,\mathbf{q})$ is implied. ♢

Assume now that the nonlinearity φ in Figure 1 has a memoryless and time-invariant I/O characteristic lying in the sector $[0, k_2]$, and that $\mathbf{q} = \mathbf{q}^o$ with $\mathbf{q}^o$ being real. Assume also that $L(s,\mathbf{q}^o)$ is strictly proper and strictly stable. The resulting system is then absolutely stable in the sense of the PC, if there exists an $\eta \in \mathbb{R}$ such that

$$\text{Re}\,\{(1+j\omega\eta)L(j\omega,\mathbf{q}^o)\} + k_2^{-1} > 0 \qquad (3)$$

holds for all $\omega \in \mathbb{R}$. This inequality can be seen as the SPR condition for the transfer function

$$H(s,\mathbf{q}^o) = (1+\eta s)L(s,\mathbf{q}^o) + k_2^{-1} \qquad (4)$$

Defining the frequency functions

$$R_\eta(\omega) = \sqrt{1+(\eta\omega)^2}R_\alpha(\omega) + R_\beta(\omega)/k_2 \qquad (5a)$$
$$Z_N(\omega) = R_\eta(\omega)/\,|\text{num}\,[H(j\omega,\mathbf{q}^o)]| \qquad (5b)$$
$$Z_{max}(j\omega) = H(j\omega,\mathbf{q}^o)Y_N(j\omega)Y_D(j\omega) \qquad (5c)$$
$$Z_{min}(j\omega) = H(j\omega,\mathbf{q}^o)Y_N^*(j\omega)Y_D^*(j\omega) \qquad (5d)$$

where num$[\cdot]$ denotes the numerator of its argument, and assuming that $L(s,\mathbf{q})$ is strictly proper and strictly stable for all $\mathbf{q} \in \mathcal{Q}$, the PC can be stated for systems with disc uncertainties in the linear part as follows:

Theorem 2.2: Let $L(s,\mathbf{q}^o)$ satisfy the PC for an $\eta \in \mathbb{R}$. Then, $L(s,\mathbf{q})$ satisfies the PC for all $\mathbf{q} \in \mathcal{Q}$, if and only if the conditions of Theorem 2.1 hold with $Z_{max}(j\omega)$ and $Z_{min}(j\omega)$ being as in (5c) and (5d), respectively.

Proof: The proof can be given by making use of the same arguments as in the proof of Theorem 2.1. ♢

Although applying this result to systems with a fixed level of disc uncertainty is quite straightforward, when γ_{max} is to be determined, the iterative procedure below is called for:

(1) Plot the Popov locus of $L(s,\mathbf{q}^o)$.
(2) Choose an η and draw a line through the points $(-k_2^{-1}, 0)$ and $\big(0, (\eta k_2)^{-1}\big)$.
(3) Assume a γ and observe the behaviour of $Z_D(\omega)$, Re $\{Z_{max}(j\omega)\}$ and Re $\{Z_{min}(j\omega)\}$.
(4) Repeat the above step for different γ values.

The main difficulty associated with this procedure is that γ_{max} may vary according to the choice of η. This was not pointed out by Polyak and Tsypkin (1991) who have stated the above results by remaining in the complex plane. They have set $C(s) = 1$, and with specific reference to Theorem 2.1, they have defined

$$Z_N(j\omega) = N_G(j\omega,\mathbf{a}^o)/R_N(\omega)$$
$$Z_D(j\omega) = D_G(j\omega,\mathbf{b}^o)/R_D(\omega)$$
$$Y_N(j\omega) = \sqrt{1-\big(\gamma Z_N^{-1}(j\omega)\big)^2} + j\gamma Z_N^{-1}(j\omega)$$
$$Y_D(j\omega) = \sqrt{1-\big(\gamma Z_D^{-1}(j\omega)\big)^2} + j\gamma Z_D^{-1}(j\omega)$$

instead of (1e)-(1h). These frequency functions, where $R_N(\omega)$ and $R_D(\omega)$ are respectively as in (1a) and (1b), lead to condition (i) of Theorem 2.1 requiring the non-intersection of $Z_D(j\omega)$ and the disc $|z| \le \gamma$. They have used these conditions also in their version of Theorem 2.2, which requires only $Z_N(j\omega)$ above to be modified.

The PC, in its original form, is not applicable to systems having a marginally stable or an unstable linear part and/or a nonlinearity bounded by a sector $[k_1, k_2]$. Of the proposed solutions to this problem, the most straightforward ones are based on simple transformations of the system elements. One such technique is the pole-shifting transformation which consists of changing $[k_1, k_2]$ into $[k_1 - k_{ps}, k_2 - k_{ps}]$, and $L(s,\mathbf{q}^o)$ into $L^{ps}(s,\mathbf{q}^o) = L(s,\mathbf{q}^o)/\big(1+k_{ps}L(s,\mathbf{q}^o)\big)$ which has the same zeros as $L(s,\mathbf{q}^o)$, but different poles. Since the PC requires the lower bound of the sector to be zero, it is common practice to set $k_{ps} = k_1$ provided that $L^{ps}(s,\mathbf{q}^o)$ is stable. Then, satisfaction of

$$\text{Re}\,\{(1+j\omega\eta)L^{ps}(j\omega,\mathbf{q}^o)\} + k^{-1} > 0 \qquad (7)$$

for an $\eta \in \mathbb{R}$ and for all $\omega \in \mathbb{R}$ implies AS of the original system. Here, $k = k_2 - k_1$. Note that since the linear subsystem is now given by $L^{ps}(s,\mathbf{q}^o)$, $H(s,\mathbf{q}^o)$ in (4) becomes

$$H(s,\mathbf{q}^o) = (1+\eta s)L^{ps}(s,\mathbf{q}^o) + k^{-1} \qquad (8)$$

from which

$$R_\eta(\omega) = \sqrt{(k_2/k)^2 + (\eta\omega)^2}R_\alpha(\omega) + k^{-1}R_\beta(\omega) \qquad (9a)$$
$$Z_N(\omega) = R_\eta(\omega)/\,|\text{num}\,[H(j\omega,\mathbf{q}^o)]| \qquad (9b)$$
$$Z_D(\omega) = \big(k_1R_\alpha(\omega) + R_\beta(\omega)\big)\,/\,|\text{den}\,[H(j\omega,\mathbf{q}^o)]| \qquad (9c)$$

are obtained. Theorem 2.2 can then be restated as follows:

Theorem 2.3: Let $L^{ps}(s,\mathbf{q}^o)$ satisfy the PC for an $\eta \in \mathbb{R}$. Then, $L^{ps}(s,\mathbf{q})$ satisfies the PC for all $\mathbf{q} \in \mathcal{Q}$, if and only if the conditions of Theorem 2.1 hold with $Z_D(\omega)$ being as in (9c), and $Z_{max}(j\omega)$ and $Z_{min}(j\omega)$ being as in (5c) and (5d), respectively.

Proof: Similar to that of Theorem 2.2. $\diamondsuit$

Remark: The numerator and denominator polynomials in (4) and (8) involve common uncertain parameters. Consequently, any AS result obtained via Theorems 2.2 and 2.3 is, in general, conservative.

3. CIRCULAR ARITHMETIC

Let $\mathcal{C}(c, r)$ be a circular region in the complex plane defined as $\mathcal{C}(c, r) = \{z \in \mathbb{C} : |z - c| \leq r\}$. Then, the simplest formulae for the addition of such regions, their multiplication by complex numbers and their inversion are as follows (Polyak *et al.*, 1994):

$$\sum_{k=1}^{n} d_k \mathcal{C}_k(c_k, r_k) = \mathcal{C}\left(\sum_{k=1}^{n} d_k c_k, \sum_{k=1}^{n} |d_k| r_k\right) \quad (10)$$

$$\mathcal{C}^{-1}(c, r) = \left(|c|^2 - r^2\right)^{-1} \mathcal{C}\left(|c|^2 c^{-1}, r\right) \quad (11)$$

In (10), the d_k are arbitrary complex numbers, and in (11), it is assumed that $|c| > r$. Unfortunately, the product of two circular regions does not describe a similar region. Assuming $c_1 \neq 0$ and $c_2 \neq 0$,

$$\mathcal{C}_1(c_1, r_1)\mathcal{C}_2(c_2, r_2) = c_1 c_2 \mathcal{O}(\rho_1, \rho_2) \quad (12)$$

where $\rho_i = r_i/|c_i|$ and $\mathcal{O}(\rho_1, \rho_2)$ is a so-called oval. The points $z = \mu e^{j\phi}$ making up $\mathcal{O}(\rho_1, \rho_2)$ are determined according to the rules below:

if $\max\{\rho_1, \rho_2\} > 1$, then $\mu \in [0, \mu^+]$, $\phi \in [0, 2\pi]$
if $\max\{\rho_1, \rho_2\} \leq 1$, then $\mu \in [\mu^-, \mu^+]$, $\phi \in [-\theta, \theta]$

The extreme points μ^-, μ^+ and θ are respectively given by

$$\mu^- = \rho_1\rho_2 + \cos\phi - \sqrt{\rho_{12} + 2\rho_1\rho_2\cos\phi - \sin^2\phi}$$

$$\mu^+ = \rho_1\rho_2 + \cos\phi + \sqrt{\rho_{12} + 2\rho_1\rho_2\cos\phi - \sin^2\phi}$$

$$\theta = \cos^{-1}\left(-\rho_1\rho_2 + \sqrt{(1 - \rho_1^2)(1 - \rho_2^2)}\right)$$

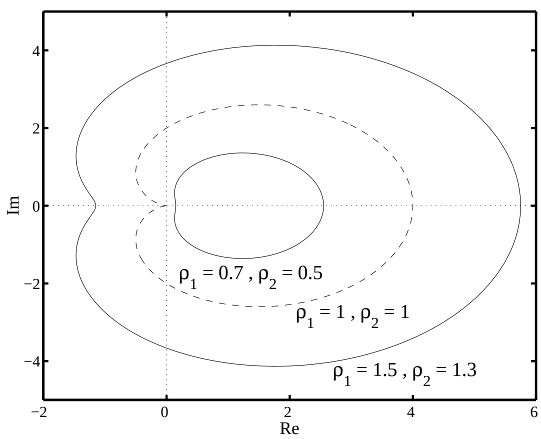

Fig. 2. $\mathcal{O}(\rho_1, \rho_2)$ for different values of ρ_1 and ρ_2.

where $\rho_{12} = \rho_1^2 + \rho_2^2$. Although the set $\mathcal{O}(\rho_1, \rho_2)$ is always closed and symmetric with respect to the real axis, depending on ρ_1 and ρ_2 it may or may not have a smooth and convex boundary (see Figure 2). Note that, when $\max\{\rho_1, \rho_2\} > 1$, the boundary of $\mathcal{O}(\rho_1, \rho_2)$ is obtained by fixing μ at μ^+, whereas if $\max\{\rho_1, \rho_2\} \leq 1$, it is generated by setting $\mu = \mu^-$ and $\mu = \mu^+$.

If either of the circles, say $\mathcal{C}_1$, has its center at the origin, then (12) becomes

$$\mathcal{C}_1(0, r_1)\mathcal{C}_2(c_2, r_2) = \mathcal{C}\left(0, r_1\left(|c_2| + r_2\right)\right)$$

If $\mathcal{C}_2$ is also located at the origin, then the result is obvious from the above. The division of $\mathcal{C}_1$ and $\mathcal{C}_2$, on the other hand, is essentially the multiplication of $\mathcal{C}_1$ by the inverse of $\mathcal{C}_2$.

4. ROBUST POPOV CRITERION BASED ON CIRCULAR ARITHMETIC

From (10), it should be clear that, at a fixed $\omega \in \mathbb{R}$, as the parameter vector $\mathbf{q}$ varies in $\mathcal{Q}$, $N_G(j\omega, \mathbf{a})$ and $D_G(j\omega, \mathbf{b})$ respectively describe the circular regions

$$\mathcal{C}_N\left(N_G(j\omega, \mathbf{a}^o), \gamma R_N(\omega)\right) \quad (13a)$$

$$\mathcal{C}_D\left(D_G(j\omega, \mathbf{b}^o), \gamma R_D(\omega)\right) \quad (13b)$$

This suggests that $G(j\omega, \mathcal{Q})$; i.e. the complex plane image of $G(s, \mathcal{Q})$ at a fixed $\omega \in \mathbb{R}$; can be obtained by dividing two circles. The brute force solution to this problem is point-by-point division of the boundaries. A computationally more tractable approach, on the other hand, is to make use of the fundamentals of circular arithmetic.

Theorem 4.1: Let $G(s, \mathbf{q})$ be a transfer function as described in Section 2 such that $G(j\omega, \mathcal{Q})$ is bounded. Then, at each $\omega \in \mathbb{R}$,

$$\partial\left[G(j\omega, \mathcal{Q})\right] = G(j\omega, \mathbf{q}^o)H(\omega)\partial\left[\mathcal{O}\left(\rho_1(\omega), \rho_2(\omega)\right)\right] \quad (14)$$

where $\partial[\cdot]$ denotes the boundary of its argument, $H(\omega) = \left(1 - \rho_2^2(\omega)\right)^{-1}$ and $\mathcal{O}\left(\rho_1(\omega), \rho_2(\omega)\right)$ is an oval as given in (12) with

$$\rho_1(\omega) = \gamma R_N(\omega) / \left|N_G(j\omega, \mathbf{a}^o)\right| \quad (15a)$$

$$\rho_2(\omega) = \gamma R_D(\omega) / \left|D_G(j\omega, \mathbf{b}^o)\right| \quad (15b)$$

where $R_N(\omega)$ and $R_D(\omega)$ are as defined in (1a) and (1b), respectively.

Proof: As mentioned before stating the theorem, for a fixed ω, $G(j\omega, \mathcal{Q})$ can be computed by multiplying (13a) by the inverse of (13b) which, from (11), is a disc with center $H(\omega)/D_G(j\omega, \mathbf{b}^o)$ and radius $\gamma R_D(\omega)/(|D_G(j\omega, \mathbf{b}^o)|^2 - (\gamma R_D(\omega))^2)$. According to (12), this operation is equivalent to

$$G(j\omega, \mathcal{Q}) = G(j\omega, \mathbf{q}^o)H(\omega)\mathcal{O}\left(\rho_1(\omega), \rho_2(\omega)\right) \quad (16)$$

The proof is completed by noting that

$$\begin{aligned}&\partial\left[G(j\omega, \mathbf{q}^o)H(\omega)\mathcal{O}\left(\rho_1(\omega), \rho_2(\omega)\right)\right] \\ &\quad = G(j\omega, \mathbf{q}^o)H(\omega)\partial\left[\mathcal{O}\left(\rho_1(\omega), \rho_2(\omega)\right)\right]\end{aligned} \quad (17)$$

since $G(j\omega, \mathbf{q}^o)$ and $H(\omega)$ are respectively complex and real numbers and, as such, preserve the boundary of $\mathcal{O}\left(\rho_1(\omega), \rho_2(\omega)\right)$. $\diamondsuit$

The above frequency domain result enables one to check the Popov inequality directly and, thus, leads to non-conservative assessment of AS. However, due to the set operations involved, the computation times are higher than those required by the approach based on SPR conditions. These points can be seen in the theorem below which is a restatement of Theorems 2.2 and 2.3.

Theorem 4.2: The inequalities in (3) and (7) are satisfied for all $G(s, \mathbf{q}) \in G(s, \mathcal{Q})$ if and only if they are satisfied for $G(s, \mathbf{q}) \in \partial[G(s, \mathcal{Q})]$.

Proof: The proof for (3), which can be written as

$$\text{Re}\left\{H(j\omega, \mathbf{q})\right\} > 0 \tag{18}$$

where $H(j\omega, \mathbf{q})$ is as given in (4), follows directly from the fact that, for fixed η and ω in $\mathbb{R}$,

$$\begin{aligned}\partial\left[H(j\omega, \mathcal{Q})\right] &= \partial\left[(1+j\omega\eta)L(j\omega, \mathcal{Q}) + k_2^{-1}\right] \\ &= (1+j\omega\eta)\partial\left[L(j\omega, \mathcal{Q})\right] + k_2^{-1}\end{aligned} \tag{19}$$

where $\partial\left[L(j\omega, \mathcal{Q})\right] = C(j\omega)\partial\left[G(j\omega, \mathcal{Q})\right]$. The theorem can be proven similarly for (7). $H(j\omega, \mathbf{q})$ in (18) is now given by (8) which can be written as

$$H(j\omega, \mathbf{q}) = (1 + j\omega\eta)\left(L^{-1}(j\omega, \mathbf{q}) + k_1\right)^{-1} + k^{-1}$$

It is then easy to see that

$$\begin{aligned}\partial\left[H(j\omega, \mathcal{Q})\right] &= \partial\Big[f(j\omega)\big(L^{-1}(j\omega, \mathcal{Q})+k_1\big)^{-1}+k^{-1}\Big] \\ &= f(j\omega)\partial\Big[\big(L^{-1}(j\omega, \mathcal{Q})+k_1\big)^{-1}\Big]+k^{-1} \\ &= f(j\omega)\Big(\left(\partial\left[L(j\omega, \mathcal{Q})\right]\right)^{-1}+k_1\Big)^{-1}+k^{-1}\end{aligned} \tag{20}$$

where $f(j\omega) = 1 + j\omega\eta$. $\diamondsuit$

In practice, (19) is used as follows:

(1) Choose an ω from $\mathbb{R}$.
(2) Plot the Popov template

$$\begin{aligned}\partial\left[L_P(j\omega, \mathcal{Q})\right] = \text{Re}\left\{\partial\left[L(j\omega, \mathcal{Q})\right]\right\} \\ + \omega\text{Im}\left\{\partial\left[L(j\omega, \mathcal{Q})\right]\right\}\end{aligned}$$

(3) Repeat these steps for different ω values.
(4) See if an η can be found such that (3) is satisfied.

The boundary result in (20), on the other hand, means that the Popov templates are given by

$$\begin{aligned}\partial\left[L_P^{ps}(j\omega, \mathcal{Q})\right] = \text{Re}\left\{\partial\left[L^{ps}(j\omega, \mathcal{Q})\right]\right\} \\ + \omega\text{Im}\left\{\partial\left[L^{ps}(j\omega, \mathcal{Q})\right]\right\}\end{aligned}$$

where

$$\partial\left[L^{ps}(j\omega, \mathcal{Q})\right] = \Big(\left(\partial\left[L(j\omega, \mathcal{Q})\right]\right)^{-1} + k_1\Big)^{-1}$$

Remark: When the coefficients of $C(s)$ and $G(s, \mathbf{q}^o)$ are real, the ω sweeps in Theorems 2.1-2.3 and 4.2 can be restricted to $\mathbb{R}^+$.

5. NUMERICAL EXAMPLE

To illustrate the salient features of the procedure given immediately after Theorem 2.2, and compare the results it yields to those obtained by making use of circular arithmetic, consider the system of Figure 1 where the nonlinearity φ belongs to the sector $[0.75, 1.5]$, and

$$C(s) = \frac{(3+j2)s + (1+j2)}{(3+j6)s^2 + (3+j4)s + (1+j)}$$

$$G(s, \mathbf{q}) = \frac{a_2 s^2 + a_1 s + a_0}{b_3 s^3 + b_2 s^2 + b_1 s + b_0}$$

The uncertain parameters are such that $\mathbf{a}^o = [4 - j4, 1-j, 5-j4]$, $\boldsymbol{\alpha} = \gamma[1,3,5]$, $\mathbf{b}^o = [21+j12, 53+j13, 49+j18, 11+j]$, and $\boldsymbol{\beta} = \gamma[4,6,2,3]$. Now, because the lower bound of the sector is not zero, first, a pole-shifting transformation with $k_{ps} = 0.75$ is applied to the system. Next, the η value for which the PC is just satisfied with $L^{ps}(s, \mathbf{q}^o)$ is determined. It should be obvious that for this particular η, $\gamma_{max} = 0$. Then, η is decreased and the steps of the relevant procedure are carried out in order to obtain the corresponding γ_{max}. For each η, a γ_{max} is also computed via Theorem 4.2. The results are shown in Figure 3, where the conservatism of the approach based on SPR conditions is evident. The maximum point of the solid curve is 0.9616, whereas that of the dotted one is 1.1639. Figure 4, on the other hand, illustrates the application of Theorem 2.3 with $\eta = 0$ and $\gamma_{max} = 0.9616$.

6. CONCLUSIONS

Nonlinear systems with disc uncertainties have been considered and their AS has been studied using the PC. Some results in this direction already exist in the literature. However, they all make use of SPR conditions and are, in general, conservative. The main contribution of this paper is the use of circular arithmetic instead of SPR conditions in order to draw non-conservative conclusions regarding AS. It must be said, however, that this new approach is computationally more demanding, even though this demand is well within the capabilities of modern computing facilities.

Acknowledgements: This work was done while S. T. Impram was at UMIST, and was supported by the UK EPSRC under Grant GR/L63303. S. T. Impram's stay at the Università di Firenze was supported by the Nonlinear Control Network under the EC-TMR program.

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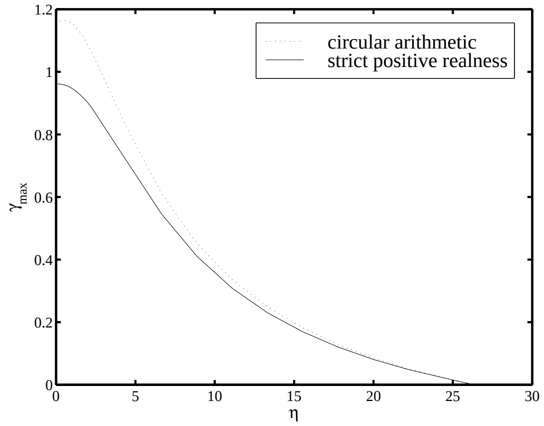

Fig. 3. Variation of γ_{max} with η for the transformed example system.

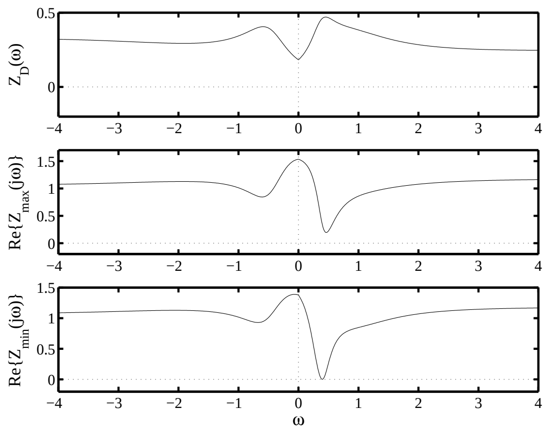

Fig. 4. The PC (based on SPR conditions) with $\eta = 0$ and $\gamma_{max} = 0.9616$ for the transformed example system .

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