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PARTIAL SYNCHRONIZATION THROUGH PERMUTATION SYMMETRY

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Abstract: In this paper we consider the problem of the existence and stability of invariant manifolds in a network of diffusively coupled identical systems. It is shown that the existence of a symmetry in the network implies the existence of linear invariant manifolds. This correspond to so called *partial synchronization*, or *clusterization*, a phenomenon occurring when some subsystems from the network operate in a synchronous manner. Conditions guaranteeing global asymptotic stability of the partial synchronization manifolds are presented. *Copyright © IFAC 2002*

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1. INTRODUCTION

The last decade has witnessed a growing attention of the scientific community on the study of synchronized behaviour of coupled dynamical systems (see Special Issue, 1997, and references therein), whether the dynamics is continuous, discrete, or spatiotemporal. The discovery that synchronization can be found (or even be induced) among coupled chaotic oscillators provided additional momentum for the developments of synchronization techniques, to possibly exploit properties of chaotic dynamics for application, for example, as analog carrier for encoding or masking messages in a secure manner (Cuomo *et al.*, 1993). For two identical systems suitably coupled, synchronous motion is most often understood as the equality of corresponding variables of the two systems. In other words, the trajectories of two (or more) identical systems will follow, after some transient, the same path in time.

The equality of corresponding state variables is not, of course, the only commonly understood situation of synchronization. It is not at all uncommon, for example, to find anti-phase synchronous behaviour in coupled oscillators. This feature is present in, for example, two or more parametrically excited pendula linearly coupled through their pivots. If $(\theta_1(t), \omega_1(t))$ and $(\theta_2(t), \omega_2(t))$ indicate angular position and velocity of the two oscillators, different initializations of the two pendula may result in in-phase synchronization $(\theta_1(t), \omega_1(t)) = (\theta_2(t), \omega_2(t))$, or anti-phase synchronization $(\theta_1(t), \omega_1(t)) = (-\theta_2(t), -\omega_2(t))$, that we both regard as synchronous behaviour. To further generalize the description of what synchronous motion can be, it has been further studied the case in which the relationship between different sets of variables of two or more coupled oscillators is not a linear function, or maybe not even a smooth function (Stark, 1997; Kocarev and Parlitz, 1999).

The scope of this paper is to analyze the relationship that occurs between a symmetry of a network composed by coupled identical oscillators, and the different linear invariant subsets that the system may possess, investigating their existence and stability. Important previous results on this topic have been obtained by (Dionne *et al.*, 1996), who investigated the symmetries of the network combined with the symmetries of the vector field, and by (Belykh *et al.*, 2000), who investigated the existence and stability of invariant manifolds in linear arrays of coupled systems, with identical coupling constants. In this paper, instead, we allow for coupling constants to be different, but symmetrically arranged. The paper is organized as follows: after we list some preliminary notions in Section 2, the problem statement is expressed in Section 3. The symmetries of a given network is analyzed in Section 4, and the stability of invariant manifolds is studied in Section 5. Some conclusions close the paper.

2. PRELIMINARIES

The Euclidean norm in $\mathbb{R}^n$ is denoted simply as $|\cdot|$, $|x|^2 = x^\top x$, where $^\top$ defines transposition. The notation $\mathrm{col}(x_1, x_2, \ldots, x_n)$ stands for the column vector composed of the elements $x_1, \ldots, x_n$. This notation will also be used in case where the components x_i are vectors again.

A function $V : X \to \mathbb{R}_+$ defined on a subset X of $\mathbb{R}^n$, $0 \in X$ is *positive definite* if $V(x) > 0$ for all $x \in X \setminus \{0\}$ and $V(0) = 0$. It is *radially unbounded* (if $X = \mathbb{R}^n$) or *proper* if $V(x) \to \infty$ as $|x| \to \infty$. For two square matrices A and B $[A, B]$ stands for the commutator $[A, B] = AB - BA$, that is $[A, B] = 0$ if A and B commute. We also denote with I_k the $k \times k$ identity matrix.

If all solutions of the dynamics $\dot{x} = f(x)$ eventually end up within a bounded domain which can be chosen independently of the initial conditions then the system is referred to as *ultimately bounded*.

For matrices A and B the notation $A \otimes B$ (the Kronecker product) stands for the matrix composed of submatrices $A_{ij}B$, i.e.

$$A \otimes B = \begin{pmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1}B & A_{n2}B & \cdots & A_{nn}B \end{pmatrix} \tag{1}$$

where A_{ij}, $i, j = 1 \ldots n$, stands for the ij-th entry of the $n \times n$ matrix A.

Consider the nonlinear time-invariant affine system:

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases} \tag{2}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input which is assumed to be any continuous and bounded function of time: $u(\cdot) \in \mathcal{C}^0 \cap \mathcal{L}_\infty$, $y(t) \in \mathbb{R}^m$ is the output; $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(0) = 0$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are smooth enough to ensure existence of solutions at least on a finite time interval $0 < t < T_{x_0,u}$; $h : \mathbb{R}^n \to \mathbb{R}^m$ is the output mapping.

Suppose there exist a nonnegative differentiable *storage function* $V : \mathbb{R}^n \to \mathbb{R}_+$, $V(0) = 0$ such that for all admissible inputs u and initial conditions $x(0) = x_0$ and for all time instants $0 \le t < T_{x_0,u}$ the following dissipation inequality is valid:

$$\dot{V}(x, u) \le y^\top u - H(x) \tag{3}$$

where the function $H : \mathbb{R}^n \to \mathbb{R}$ is nonnegative outside some ball:

$$\exists \rho > 0,\ \forall |x| \ge \rho \Longrightarrow H(x) \ge \varrho(|x|) \tag{4}$$

for some continuous nonnegative function ϱ defined for $|x| \ge \rho$. Then the system (2) is called a *semipassive system*. This notion was introduced in (Pogromsky, 1998). Roughly speaking a semipassive system behaves like a passive system (Byrnes *et al.*, 1991) for sufficiently large $|x|$. If the function H is positive outside some ball, i.e. (4) holds for some continuous positive function ϱ, then the system (2) is said to be *strictly semipassive*.

The concept of semipassivity allows one to find simple conditions which ensure boundedness of the solutions of interconnected systems. Consider k (possibly different) systems of the form (2):

$$\begin{cases} \dot{x}_j = f_j(x_j) + g_j(x_j)u_j \\ y_j = h_j(x_j) \end{cases} \tag{5}$$

where $j = 1, \dots, k$.

Define the symmetric $k \times k$ matrix Γ as

$$\Gamma = \begin{pmatrix} \sum_{i=2}^{k} \gamma_{1i} & -\gamma_{12} & \cdots & -\gamma_{1k} \\ -\gamma_{21} & \sum_{i=1, i\neq 2}^{k} \gamma_{2i} & \cdots & -\gamma_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ -\gamma_{k1} & -\gamma_{k2} & \cdots & \sum_{i=1}^{k-1} \gamma_{ki} \end{pmatrix} \tag{6}$$

where $\gamma_{ij} = \gamma_{ji} \geq 0$ and all row sums are zero. The matrix Γ is symmetric and therefore all its eigenvalues are real. Moreover applying Gerschgorin's theorem about localization of eigenvalues (see, e.g. (Stewart and Sun, 1990)) one can see that all eigenvalues of Γ are nonnegative, that is, the matrix Γ is positive semidefinite.

There is an important advantage in using semipassive systems. It has been proven in (Pogromsky *et al.*, 1999) that semipassive systems (5), interconnected through dissipative coupling, have bounded solutions.

3. PROBLEM STATEMENT

We consider k identical systems of the form

$$\begin{cases} \dot{x}_j = f(x_j) + Bu_j \\ y_j = Cx_j \end{cases} \tag{7}$$

where $j = 1, \dots, k$, $x_j(t) \in \mathbb{R}^n$ is the state of the j-th system, $u_j(t) \in \mathbb{R}^m$ is the input, $y_j(t) \in \mathbb{R}^m$ is the output of the j-th system, $f(0) = 0$, and B, C are constant matrices of appropriate dimension. The connections between the given k systems are expressed in their input terms as

$$u_j = -\gamma_{j1}(y_j - y_1) - \gamma_{j2}(y_j - y_2) - \dots - \gamma_{jk}(y_j - y_k) \tag{8}$$

where $\gamma_{ij} = \gamma_{ji} \geq 0$ are constants such that $\sum_{j\neq i}^{k} \gamma_{ji} > 0$ for all $i = 1, \dots, k$.

The systems (7) coupled through (8) are referred to as *diffusively coupled* provided the product CB is similar to a diagonal matrix with *positive entries*. This definition was introduced in (Pogromsky *et al.*, 1999) and was inspired by the paper of Smale on interaction between two cells (Smale, 1976). We will understand a diffusive medium as a dynamical system consisting of a number of interconnected identical dynamical systems. Each separate system has inputs and outputs of the same dimension. The diffusive coupling is described by a static relation between inputs and outputs.

Note that if we define the coupling matrix Γ as in (6), the feedback (8) can be written in a matrix notation as

$$u = -(\Gamma \otimes I_m)y.$$

with $y = \mathrm{col}(y_1, y_2, \dots, y_m)$, $u = \mathrm{col}(u_1, u_2, \dots, u_m)$. It is easy to observe that the matrix Γ is singular (all row sums are zero). Additionally, if there exists only one eigenvector associated with the zero eigenvalue of Γ then the network of the diffusively coupled systems can not be divided into two or more disconnected networks (the dimension of $\ker\Gamma$ is the number of disconnected networks). Networks of this kind will be referred to as *diffusive cellular networks* (Pogromsky and Nijmeijer, 2001).

It has been proved in (Pogromsky and Nijmeijer, 2001) that under some additional assumptions if the lowest nonzero eigenvalue of Γ exceeds some threshold value the closed loop system (7,8) has globally asymptotically stable compact subset of the invariant set

$$\mathcal{A} = \{x_i \in \mathbb{R}^n \ : \ x_i = x_j, \ i, j = 1, \dots, k\}$$

Asymptotic stability of this set is usually referred to as full synchronization. However, as was shown in (Pogromsky and Nijmeijer, 2001) the occurrence of full synchronization in large networks is significantly limited. Indeed, if all diffusive coefficients

γ_{ij} are bounded and each cell is connected with no more than N other cells, then zero is an accumulation point in the spectrum of Γ when $k \to \infty$ (Pogromsky and Nijmeijer, 2001).

In this paper we are focusing on the existence and stability of linear invariant manifolds of diffusive cellular networks, a phenomenon usually called *partial synchronization*. Recall that for the dynamics

$$\dot{x} = F(x),$$

$x \in \mathbb{R}^n$, the linear manifold $\mathcal{A}_D = \{x \in \mathbb{R}^n \ : \ Dx = 0\}$ is invariant if $D\dot{x} = 0$ whenever $Dx = 0$. Particularly we will study the relation between the symmetries of Γ and conditions guaranteeing the existence and stability of linear invariant manifolds for the system (7,8).

4. SYMMETRIES AND INVARIANT MANIFOLDS

The matrix Γ defined in (6) describes the geometry of the extended system as a network of coupled identical oscillators (7,8). If the network possesses some symmetry, this information can be retrieved by the symmetries of Γ. Let $\Pi \in \mathbb{R}^{k\times k}$ be a permutation matrix. Their properties are already well known, so we briefly state that if $\varepsilon_1, ...\varepsilon_k$ denote the columns of I_k, a permutation matrix Π is a matrix obtained from I_k by permuting its columns, that is, the columns of Π are $\varepsilon_{\alpha(1)}, ...\varepsilon_{\alpha(k)}$, where α is a permutation of the set $\{1, 2, ..., k\}$. If S_k is the set of all permutations of $\{1, 2, ..., k\}$ it is possible to prove that the set of all permutation matrices form a group that is isomorphic to S_k (Rotman, 1994). Permutation matrices are orthogonal, i.e. $\Pi^\top \Pi = I_k$, and they form a group with respect to the multiplication, so for any two permutation matrices Π_i, Π_j, $\Pi_i \Pi_j$ is a permutation matrix too.

Theorem 1. Suppose there is a permutation Π commuting with Γ. Then the set $\ker(I_{kn} - \Pi \otimes I_n)$ is an invariant manifold for system (7,8).

Proof: First we rewrite the system equation (7,8) as

$$\dot{x} = F(x) + Gx \tag{9}$$

where we denoted $x = \mathrm{col}(x_1, \dots, x_k)$, $F(x) = \mathrm{col}(f(x_1), \dots, f(x_k))$ and $G = -(I_k \otimes B)(\Gamma \otimes I_m)(I_k \otimes C)$ that, after some algebra can be rewritten as $G = -\Gamma \otimes BC$. Define the matrix $\Sigma = \Pi \otimes I_n$. Since $[\Pi, \Gamma] = 0$ it follows that Σ and G commute as well. Therefore, using (9) we obtain

$$(I_{nk} - \Sigma)\dot{x} = (I_{nk} - \Sigma)F(x) + G(I_{nk} - \Sigma)x = 0. \tag{10}$$

Since Σ is a permutation matrix, $\Sigma F(x) = F(\Sigma x)$. Suppose $x \in \ker(I_{kn} - \Sigma)$, then $x = \Sigma x$ and therefore $F(x) = \Sigma F(x)$. Hence, $\ker(I_{kn} - \Sigma)$ is an invariant manifold. □

Of course, these are not the only possible linear invariant subsets that a network of coupled systems (7,8) may possess. Additional invariant subsets may arise when, for instance, possible symmetries of $F(x)$ in (9) (hence $f(x_j)$ in (7,8)) are taken into consideration. A popular case is represented by the anti-phase synchronization that can happen in systems whose vector fields are odd functions of the state vector. In this case we are able to formulate the following result.

Theorem 2. Suppose there is a permutation Π commuting with Γ and f displays odd symmetry ($i.e. f(x) = -f(-x)$). Then the set $\ker(I_{kn} + \Pi \otimes I_n)$ is an additional invariant manifold for system (7,8).

Proof: Using (9) we obtain

$$(I_{nk} + \Sigma)\dot{x} = (I_{nk} + \Sigma)F(x) + G(I_{nk} + \Sigma)x = 0.$$

Suppose $x \in \ker(I_{kn} + \Sigma)$, then $x = -\Sigma x$ and therefore

$$(I_{nk} + \Sigma)F(x) = \Sigma(F(-x) + F(x)) = 0$$

Consequently, $x \in \ker(I_{kn} + \Sigma)$ implies $(I_nk + \Sigma)\dot{x} = 0$. □

4.1 *Example*

Let us consider the example of four coupled systems (7,8) in a ring, as shown schematically in Figure 1. In this Figure we have imposed the following symmetry in coupling constants: $\gamma_{12} = \gamma_{34} = K_0$, and $\gamma_{14} = \gamma_{23} = K_1$. The particular geometry of

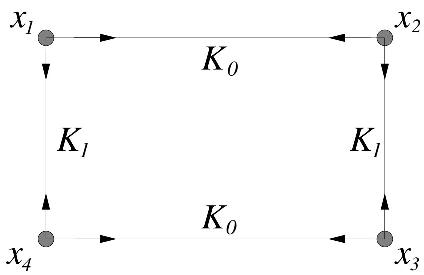

Fig. 1. A network of four coupled identical systems with symmetric coupling at the opposite sides.

the coupling defines the following coupling matrix:

$$\Gamma = \begin{pmatrix} K_0+K_1 & -K_0 & 0 & -K_1 \\ -K_0 & K_0+K_1 & -K_1 & 0 \\ 0 & -K_1 & K_0+K_1 & -K_0 \\ -K_1 & 0 & -K_0 & K_0+K_1 \end{pmatrix}.$$

The four permutation matrices for which $[\Pi, \Gamma] = 0$ are

$$\Pi_1 = I_4, \quad \Pi_2 = \begin{pmatrix} J & O \\ O & J \end{pmatrix},$$
$$\Pi_3 = \begin{pmatrix} O & I_2 \\ I_2 & O \end{pmatrix}, \quad \Pi_4 = \begin{pmatrix} O & J \\ J & O \end{pmatrix} \qquad (11)$$

where we denoted

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and O is the 2×2 zero matrix. Let us analyse what is the action of these matrices Π. Π_1 is the identity, therefore leaves everything unchanged. The action of Π_2 is to switch *simultaneously* x_1 with x_2 and x_3 with x_4. One can easily notice from Fig. 1 that this operation leaves the network unchanged, with respect to the coupling constants. The same result holds for Π_3 and Π_4. From Theorem 1 we derive that the invariant linear manifolds associated with Π_2, Π_3 and Π_4 are, respectively,

$$\begin{aligned} \mathcal{A}_2 &= \{x \in \mathbb{R}^{4n} \;:\; x_1 = x_2, x_3 = x_4\} \\ \mathcal{A}_3 &= \{x \in \mathbb{R}^{4n} \;:\; x_1 = x_3, x_2 = x_4\} \\ \mathcal{A}_4 &= \{x \in \mathbb{R}^{4n} \;:\; x_1 = x_4, x_2 = x_3\}. \end{aligned}$$

The intersection of any two of these linear manifolds gives the linear manifold describing full synchronization (*i.e.* $x_1 = x_2 = x_3 = x_4$).

5. ON STABILITY OF PARTIAL SYNCHRONIZATION MANIFOLDS

Consider the coupled system (7, 8). Suppose it forms a diffusive cellular network, that is, there is only one eigenvector of Γ which corresponds to zero eigenvalue. We denote eigenvalues of the coupling matrix Γ as γ_i, $i = 1, \dots, k$ ordered in increasing order:

$$0 = \gamma_1 < \gamma_2 \leq \dots \leq \gamma_k.$$

Suppose there is a nontrivial permutation Π, $\Pi \neq I_k$ commuting with Γ. We are going to investigate stability of the partial synchronization manifold $\ker(I_{kn} - \Pi \otimes I_n)$. Let γ' be a minimal eigenvalue of Γ under restriction that the eigenvectors of Γ are taken from the set range$(I_k - \Pi)$. It is easy to see that $\ker\Gamma \subset \ker(I_k - \Pi)$ and hence

$$0 < \gamma_2 \leq \gamma' \leq \gamma_k$$

Using nonsingularity of CB it is possible to find a coordinate transformation which brings each free system from (7) into the form

$$\begin{cases} \dot{z}_j = q(z_j, y_j) \\ \dot{y}_j = a(z_j, y_j) + CBu_j \end{cases}$$

where $z_j \in \mathbb{R}^{n-m}$, $y_j, u_j \in \mathbb{R}^m$, $j = 1, \dots, k$. We now formulate the following result:

Theorem 3. Consider the k smooth diffusively coupled systems (7, 8) with CB similar to a positive definite matrix. Assume that

A1. Each free system (7) is strictly semipassive with respect to the input u_j and output y_j with a radially unbounded storage function $V : \mathbb{R}^n \to \mathbb{R}_+$.

A2. There exist a $\mathcal{C}^2$-smooth positive definite function $V_0 : \mathbb{R}^{n-m} \to \mathbb{R}_+$ and a positive number α such that the following inequality is satisfied

$$(\nabla V_0(z_1 - z_2))^\top (q(z_1, y_1) - q(z_2, y_1)) \leq -\alpha |z_1 - z_2|^2.$$

for all $z_1, z_2 \in \mathbb{R}^{n-m}, y_1 \in \mathbb{R}^m$.

Then there exists a positive $\bar{\gamma}$ such that if $\gamma' > \bar{\gamma}$ the set $\ker(I_{kn} - \Pi \otimes I_n)$ contains a globally asymptotically stable compact subset.

We can now apply this theorem to the example we previously introduced. The eigenvalues

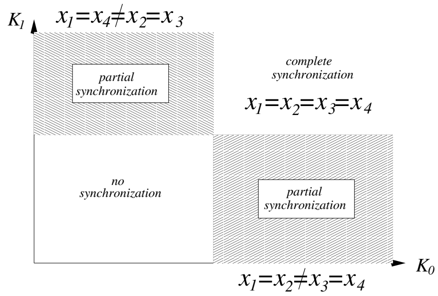

Fig. 2. Stability of different invariant manifolds.

of the matrix Γ are given by $\gamma_1 = 0, \gamma_2 = \min\{2K_0, 2K_1\}, \gamma_3 = \max\{2K_0, 2K_1\}, \gamma_4 = 2(K_0 + K_1)$. It is not difficult to compute the value γ' for different permutations: $\gamma' = 2K_0$ for the permutation Π_2, $\gamma' = \min\{2K_0, 2K_1\}$ for the permutation Π_3 and $\gamma' = 2K_1$ for the permutation Π_4. Hence, according to Theorem 3 for large K_0 and small K_1 one should expect asymptotic stability of the set $\mathcal{A}_2$, for small K_0 and large K_1 one should expect asymptotic stability of the set $\mathcal{A}_4$, while the full synchronization occurs for large both K_0 and K_1 (see Figure 2). The set $\mathcal{A}_3$ is stable only as a stable intersection of $\mathcal{A}_2$ and $\mathcal{A}_4$, which is the set $\mathcal{A}$. This theoretical observation is in agreement with the results of computer simulation which will be reported elsewhere.

6. CONCLUSION

In this paper we have studied how to derive some linear invariant subsets of a network of coupled identical oscillators from the symmetry under permutation of the elements that form the network. We have presented conditions that ensure stability of partial synchronization manifolds.

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