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ROBUST OUTPUT REGULATION OF SINGULAR NONLINEAR SYSTEMS [1]

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Abstract: Singular systems are dynamical systems subject to algebraic constraints, and arise in many engineering disciplines. The output regulation problem for singular nonlinear systems has been studied recently for the ideal case where the mathematical model is exactly known. This paper will further consider the robust output regulation problem for a class of singular nonlinear systems which contain uncertain parameters. It will establish the conditions for the solvability of the problem, thus extending the existing results from the normal nonlinear systems to the singular nonlinear systems. *Copyright @ 2002 IFAC*

Keywords: Output Regulation, Singular Systems, Nonlinear Systems, Robust Control

1. INTRODUCTION

Singular systems arise in many engineering areas such as electrical networks, power system, aerospace engineering and chemical processing. Since the late 1970s singular systems have attracted attentions from many researchers. Several books and survey papers dealing with these systems have appeared addressing the issues of solvability, controllability and observability, pole assignment, the elimination of impulsive behavior, and so on (Cobb, 1984; Lewis, 1985; Dai, 1989; Campbell and Griepentrog, 1995). This paper will consider the robust output regulation problem for a class of singular nonlinear systems to be described in Section 2. Briefly, the output regulation problem aims to design control laws for a plant so that the output of the closed-loop system is able to asymptotically track a class of reference inputs and reject a class of disturbances. Both the disturbance and reference are generated by an autonomous differential equation called exosystem. When the controller is also required to tolerate certain plant uncertainty, the problem is called robust output regulation problem. For the class of linear systems, this problem was thoroughly studied for the normal systems in the 1970s in (Francis, 1977; Francis and Wonham, 1976) among others. A salient outcome of these research activities is the internal model principle which is the extension of the well known PID control. The problem was also investigated for linear singular systems in 1980s (Dai, 1989). Recently, a more clear-cut solution of this problem for linear singular systems was obtained in (Lin and Dai, 1996). For the class of nonlinear systems, the same problem was first treated for the normal systems. The special case in which the exogenous signals are constant were studied in (Francis and Wonham, 1976; Huang and Rugh, 1990). The general case with time varying exogenous signals was studied in (Isidori and Byrnes, 1990) without considering the parameter uncertainty. Subsequently, the robust version of the same problem was pursued in (Huang and Lin, 1991,1993; Huang, 1996; Byrnes, *et al.*, 1997). More recently, the output regulation problem for singular nonlinear systems has been formulated and solved in (Huang and Zhang, 1998). The objective of this paper is to further pursue the research initiated in (Huang and Zhang, 1998) by considering the plant uncertainty.

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2. PROBLEM DESCRIPTION AND STANDARD ASSUMPTIONS

Consider the plant described by

$$\begin{aligned} &\mathbf{E}\dot{x}(t) = f(x(t),u(t),v(t),w), x(0) = x_0 \\ &y(t) = h(x(t),v(t),w), t \geq 0 \end{aligned} \quad (2.1)$$

and an exosystem described by

$$\dot{v}(t) = a(v(t)), v(0) = v_0 \quad (2.2)$$

where $x(t) \in \Re^n$ is the plant state, $u(t) \in \Re^m$ the plant input, $y(t) \in \Re^p$ the plant output representing the tracking error, $v(t) \in \Re^q$ the exogenous signal representing the disturbance and/or the reference input, $w \in \Re^N$ the plant unknown parameters, and $\mathbf{E} \in \Re^{n\times n}$ a singular constant matrix, and $rank(\mathbf{E}) = n_E < n$. It is noted that if $n_E = n$, the plant is called *normal*. Also it is assumed that 0 is the nominal value of the uncertain parameters w.

This paper will focus on the dynamic output feedback controller as follows:

$$\begin{aligned} &u(t) = k(z(t),y(t)) \\ &\dot{z}(t) = g(z(t),y(t)) \end{aligned} \quad (2.3)$$

where $z(t)$ is the compensator state vector of dimension n_c. When the state of the system is available, the state feedback control law can be considered accordingly.

The closed-loop system composed of plant (2.1), (2.2) and control law (2.3) can be put into the following form:

$$\begin{aligned} &\mathbf{E}_c\dot{x}_c(t) = f_c(x_c(t),v(t),w), x_c(0) = x_{c0} \\ &y(t) = h_c(x_c(t),v(t),w) \end{aligned} \quad (2.4)$$

where

$$\begin{aligned} &x_c = \begin{bmatrix} x \\ z \end{bmatrix}, \mathbf{E}_c = \begin{bmatrix} \mathbf{E} & 0 \\ 0 & I_{n_c} \end{bmatrix} \\ &f_c(x_c,v,w) = \begin{bmatrix} f(x,k(z,h(x,v,w)),v,w) \\ g(z,h(x,v,w)) \end{bmatrix} \\ &h_c(x_c,v,w) = h(x,v,w) \end{aligned} \quad (2.5)$$

Throughout this paper, it is assumed that all the functions involved in this setup are sufficiently smooth and defined globally on the appropriate Euclidean spaces, and $a(0) = 0$, $f(0,0,0,w) = 0$, and $h(0,0,0,w) = 0$ for any $w \in W$ with W an open neighborhood of the origin of $\Re^N$. Our results will be stated locally in terms of V and W with V an open neighborhood of the origin in $\Re^q$. In the sequel, V and W are implicitly permitted to be made smaller to accommodate subsequent local arguments.

The linearization of the system (2.1) and (2.2) at $(x,u,v) = (0,0,0)$ will be frequently used, therefore, the following notation is given,

$$A(w) = \frac{\partial f}{\partial x}|_{x=0,u=0,v=0}, B(w) = \frac{\partial f}{\partial u}|_{x=0,u=0,v=0},$$

$$E(w) = \frac{\partial f}{\partial v}|_{x=0,u=0,v=0}, C(w) = \frac{\partial h}{\partial x}|_{x=0,u=0,v=0},$$

$$F(w) = \frac{\partial h}{\partial v}|_{x=0,u=0,v=0}, A_1 = \frac{\partial a(v)}{\partial v}|_{v=0},$$

$$A_c(w) = \frac{\partial f_c}{\partial x_c}|_{x_c0,v=0}$$

As a result, the system (2.1) and (2.2) can also be written as

$$\begin{aligned} &\mathbf{E}\dot{x} = A(w)x + B(w)u + E(w)v + o(x,u,v,w) \\ &y = C(w)x + F(w)v + o(x,u,v,w) \\ &\dot{v} = A_1 v + o(v) \end{aligned}$$

where $o(x,u,v,w)$ (or $o(v)$) is a sufficiently smooth function vanishing at $(x,u,v) = (0,0,0)$ (or $v = 0$) together with its first order derivative for any $w \in W$. For convenience, let $A, B, \cdots$, denote $A(0), B(0), \cdots$, respectively.

The robust output regulation problem: Find a control law such that the closed-loop (2.4) has the two properties:

(P1) the linearization at $x_c = 0$ of $\mathbf{E}_c\dot{x}_c(t) = f_c(x_c(t),0,0)$ is strongly stable, that is, $deg(det(\lambda\mathbf{E}_c - A_c) = rank(\mathbf{E}_c)$, and $\sigma(\mathbf{E}_c, A_c) \in c^-$, where $\sigma(\mathbf{E}_c,A_c) \triangleq \{\lambda | \det(\lambda\mathbf{E}_c - A_c) = 0\}$.

(P2) the trajectories starting from all sufficiently small initial state (x_{c0}, v_0) satisfy

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} h_c(x_c(t),v(t),w) = 0 \quad (2.6)$$

Remark 2.1: The above problem is clearly an extension of the problem studied in (Huang and Zhang, 1998) by taking into account the uncertainty. Viewing w as generated by an exosystem of the form $\dot{w} = 0$, a solvability condition can be obtained, by slightly modifying Lemma 4.1 of (Huang and Zhang, 1998), as follows:

Theorem 2.2: Assume the following,
A1: The equilibrium of exosystem (2.2) is stable and all the eigenvalues of $(\partial a/\partial v)(0)$ have zero real parts.

Then the controller (2.3) solves the robust output regulation problem of singular uncertain system (2.1) and (2.2) if it is such that the closed-loop system satisfies

(i) (P1) holds.

(ii) there exists a sufficiently smooth function $\mathbf{x}_c(v,w)$ locally defined in $V \times W$ satisfying $\mathbf{x}_c(0,0) = 0$ and

$$\begin{aligned} &\mathbf{E}_c\frac{\partial \mathbf{x}_c(v,w)}{\partial v}a(v) = f_c(\mathbf{x}_c(v,w),v,w) \\ &h_c(\mathbf{x}_c(v,w),v,w) = 0 \end{aligned} \quad (2.7)$$

To close this section, some standard assumptions will be listed.

A2: $(\mathbf{E},A,B)$ is strongly stabilizable, *i.e.*, there exists a matrix $K \in \Re^{m\times n}$ such that $(\mathbf{E},A+BK)$ is strongly stable.

A3: $(\mathbf{E},A,C)$ is strongly detectable, *i.e.*, there exists a matrix $K \in \Re^{n\times p}$ such that $(\mathbf{E},A+KC)$ is strongly stable.

A4: There exist sufficiently smooth functions $\mathbf{x}(v,w)$ and $\mathbf{u}(v,w)$ such that $\mathbf{x}(0,0)=0$ and $\mathbf{u}(0,0)=0$ and satisfy, for $v \in V, w \in W$

$$\begin{aligned} &\mathbf{E}\frac{\partial \mathbf{x}(v,w)}{\partial v}a(v) = f(\mathbf{x}(v,w),\mathbf{u}(v,w),v,w) \\ &h(\mathbf{x}(v,w),v,w) = 0 \end{aligned} \quad (2.8)$$

Remark 2.3: When **E** is an identity matrix, Assumptions **A2** and **A3** reduce to exactly the same ones assumed by (Isidori and Byrnes, 1990) for the *normal* systems. Equation (2.8) becomes the so called regulator equations discovered by Isidori and Byrnes.

3. A PRELIMINARY RESULT

Recall that, in the normal case, the way to handle the robust output regulation problem is much more complicated than the way to handle the output regulation problem. This is because, when there is no uncertainty, the solution of equation (2.8) or its estimation can be used as a feedforward function to cancel the steady state error output. But this is impossible when the uncertain parameter w is present since the solution of the equation (2.8) also depends on w which cannot appear in the control law. As a result, the problem has to be approached with the employment of a nonlinear version of the internal model principle (Huang, 1996; Huang and Lin, 1991, 1993; *etc*). Here this technique will be further extended to the singular nonlinear systems. To begin with, some notations that have been used frequently in (Huang, 1996; Huang and Lin, 1991,1993) are first introduced. For any matrix M, $k=1,2,\cdots$ define

$$M^{(0)} = I, M^{(1)} = M, \cdots, M^{(k)} = \underbrace{M\otimes\cdots\otimes M}_{k\ factors}$$

where $\otimes$ denotes the Kronecker product. Also let $v^{[l]}$ denote the vector

$$v^{[l]} = [v_1^l, v_1^{l-1}v_2, \cdots, v_1^{l-1}v_q, v_1^{l-2}v_2^2, v_1^{l-2}v_2v_3, \cdots, v_1^{l-2}v_2v_q, \cdots, v_q^l]^T$$

It was shown in (Huang, 1996) that if v satisfies $\dot v = A_1 v$ for some square matrix A_1, then there exist square matrices A_l, $l=2,3,\cdots$, such that

$$\dot v^{[l]}(t) = A_l v^{[l]}(t),\ l=2,3,\cdots$$

In fact, A_l can be explicitly given as

$$A_l = M_l\left[\sum_{i=1}^{l} I_q^{(i-1)}\otimes A_1 \otimes I_q^{(l-i)}\right]N_l$$

where M_l and N_l are such that $v^{[l]} = M_l v^{(l)}, v^{(l)} = N_l v^{[l]}$ and I_q denote the q dimensional identity matrix. As in (Huang , 1996), the following autonomous system is called K-fold exosystem.

$$\begin{bmatrix} \dot v^{[1]} \\ \dot v^{[2]} \\ \vdots \\ \dot v^{[k]} \end{bmatrix} = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & A_k \end{bmatrix} \begin{bmatrix} v^{[1]} \\ v^{[2]} \\ \vdots \\ v^{[k]} \end{bmatrix} \quad (3.1)$$

A linear result which will play an important role in establishing the major result is first stated.

Lemma 3.1: Given any square matrix $\hat A_1$ such that all the eigenvalues of $\hat A_1$ are on the closed right complex plane, let $\beta_i \in \Re^{n_i\times n_i}, \sigma_i \in \Re^{1\times n_i}$, $i=1,\cdots,r$, for some positive integers $n_1,\cdots,n_r,r$, satisfy the following

(i) β_i and σ_i are controllable, and

(ii) the minimal polynomial of $\hat A_1$ divides the characteristic polynomial of β_i,

let

$$\begin{aligned} G_1 &= block\ diag\ [\beta_1,\cdots,\beta_r], \\ G_2 &= block\ diag\ [\sigma_1,\cdots,\sigma_r] \end{aligned}$$

and finally, let matrices g_1 and g_2 admit the following form

$$g_1 = T\begin{bmatrix} S_1 & S_2 \\ 0 & G_1 \end{bmatrix}T^{-1}, \quad g_2 = T\begin{bmatrix} S_3 \\ G_2 \end{bmatrix}$$

where S_1,S_2,S_2 are any matrices with proper dimensions, and T is a nonsingular matrix. Then, for any matrices $\hat A,\hat B,\hat C,\hat D$ with appropriate dimensions, if the matrix

$$\begin{bmatrix} \hat A & \hat B \\ g_2\hat C & g_1 + g_2\hat D \end{bmatrix} \quad (3.2)$$

is Hurwitz, then for any (U,V) with proper dimension, the linear matrix equation

$$\begin{aligned} \phi\hat A_1 &= \hat A\phi + \hat B\theta + U \\ \theta\hat A_1 &= g_1\theta + g_2(\hat C\phi + \hat D\theta + V) \end{aligned} \quad (3.3)$$

has a unique solution which satisfies

$$\hat C\phi + \hat D\theta + V = 0 \quad (3.4)$$

Proof: The proof is quite similar to Theorem 2.1 of (Huang, 1995). Since (3.3) is a Sylvester equation, it follows from the assumptions on matrices $\hat A_1$ and (3.2) that equation (3.3) has a unique solution. To show (3.4), let $\theta = [\hat\theta^T, \bar\theta^T]^T$, where $\bar\theta$ has the same dimension as that of G_1. Then the second equation of (3.3) implies

$$\bar\theta\hat A_1 - G_1\bar\theta = G_2 Y \quad (3.5)$$

where $Y = \hat C\phi + \hat D\theta + V$.

Due to the block diagonal structure of G_1, G_2, $r=1$ can be assumed without loss of generality. Therefore, we can write G_1, G_2 in the following form:

$$G_1 = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -\alpha_{n_k} & -\alpha_{n_k-1} & \cdots & -\alpha_2 & -\alpha_1 \end{bmatrix}, G_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Let $\theta_j, j = 1,\cdots,n_k$, denote the j^{th} row of $\bar{\theta}$. Then expanding (3.5) gives

$$Y = \theta_1(\hat{A}_1^{n_k} + \alpha_1\hat{A}_1^{n_k-1} + \cdots + \alpha_{n_k} I)$$

Noting that the characteristic polynomial of G_1 is divisible by the minimal polynomial of $\hat{A}_1$ gives $Y = 0$.

Lemma 3.2: Under assumption **A1**, for any positive integer k, let $\hat{A}_1 = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & A_k \end{bmatrix}$. Let a linear controller of the form

$$\begin{aligned} u &= K_1 z + K_2 y \\ \dot{z} &= g_1 z + g_2 y \end{aligned} \tag{3.6}$$

be given where g_1 and g_2 are given as in Lemma 3.1 with $r = p$. Then, if the controller (3.6) makes the pair

$$\left(\begin{bmatrix} \mathbf{E} & 0 \\ 0 & I_{n_c} \end{bmatrix}, \begin{bmatrix} A + BK_2C & BK_1 \\ g_2C & g_1 \end{bmatrix} \right) \tag{3.7}$$

strongly stable, then the closed-loop system composed of (2.1), (2.2) and controller (3.6) has the property that there exists a sufficiently smooth function $\mathbf{x}_c(v,w)$ locally defined in $V \times W$ satisfying $\mathbf{x}_c(0,0) = 0$, and

$$\begin{aligned} \mathbf{E}_c \frac{\partial \mathbf{x}_c(v,w)}{\partial v} a(v) &= f_c(\mathbf{x}_c(v,w),v,w) \\ h_c(\mathbf{x}_c(v,w),v,w) &= O\left(v^{k+1}\right) \end{aligned} \tag{3.8}$$

where $O\left(v^{k+1}\right)$ is such that $\lim_{v\to 0} \frac{\|O(v^{k+1})\|}{\|v\|^{k+1}}$ is a finite constant.

Proof: This result will be established by performing the standard coordinate transformation on the closed-loop system. To this end, first note that the closed-loop system composed of (2.1),(2.2) and (3.6) can be written as follows,

$$\begin{aligned} &\mathbf{E}\dot{x} = (A(w) + B(w)K_2C(w))x + B(w)K_1 z \\ &+(E(w) + B(w)K_2F(w))v + o(x,u,v,w) \\ &\dot{z} = g_2C(w)x + g_1 z + g_2F(w)v \\ &+o(x,u,v,w) \\ &y = C(w)x + F(w)v + o(x,u,v,w) \\ &\dot{v} = A_1 v + o(v) \end{aligned} \tag{3.9}$$

Let T_1 and T_2 be two nonsingular matrices such that $T_1\mathbf{E}T_2 = \begin{bmatrix} I_{n_E} & 0 \\ 0 & 0 \end{bmatrix}$. Let

$$T_1A(w)T_2 = \begin{bmatrix} A_{11}(w) & A_{12}(w) \\ A_{21}(w) & A_{22}(w) \end{bmatrix}, T_1B(w) = \begin{bmatrix} B_1(w) \\ B_2(w) \end{bmatrix},$$

$$T_1E(w) = \begin{bmatrix} E_1(w) \\ E_2(w) \end{bmatrix}, C(w)T_2 = \begin{bmatrix} C_1(w) & C_2(w) \end{bmatrix},$$

$$T_2^{-1}x = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

with $A_{11}(w) \in \Re^{n_E\times n_E}, B_1(w) \in \Re^{n_E\times m}, E_1(w) \in \Re^{n_E\times q}, C_1(w) \in \Re^{p\times n_E}, \bar{x}_1 \in \Re^{n_E}$, and all other matrices have proper dimensions.

In terms of $\bar{x}_1$ and $\bar{x}_2$, the first equation of (3.9) becomes

$$\begin{aligned} &\dot{\bar{x}}_1 = \bar{A}_{11}(w)\bar{x}_1 + \bar{A}_{12}(w)\bar{x}_2 + \bar{B}_1(w)K_1 z \\ &+\bar{E}_1(w)v + o(x,z,v,w) \\ &0 = \bar{A}_{21}(w)\bar{x}_1 + \bar{A}_{22}(w)\bar{x}_2 + \bar{B}_2(w)K_1 z \\ &+\bar{E}_2(w)v + o(x,z,v,w) \end{aligned} \tag{3.10}$$

where

$$\bar{A}_{ij}(w) = A_{ij}(w) + B_i(w)K_2C_j(w)$$
$$\bar{B}_i(w) = B_i(w)$$
$$\bar{E}_i(w) = E_i(w) + B_i(w)K_2F(w), i,j \in \{1,2\}$$

Note that the strong stability of (3.7) implies that $det\left(\bar{A}_{22}\right) \neq 0$. Thus the implicit function theorem guarantees the existence of a unique sufficiently smooth solution of the second equation of (3.10), and this solution has the form

$$\begin{aligned} \bar{x}_2 = \gamma(\bar{x}_1,z,v,w) &= -\bar{A}_{22}^{-1}\bar{A}_{21}\bar{x}_1 - \bar{A}_{22}^{-1}\bar{B}_2K_1 z \\ &- \bar{A}_{22}^{-1}\bar{E}_2 v + o(\bar{x}_1,z,v,w) \end{aligned} \tag{3.11}$$

Substituting (3.11) into the first equation of (3.10) and the second and third equations of (3.9) gives a reduced order normal system as follows

$$\begin{aligned} &\dot{\bar{x}}_1 = \bar{f}_{1c}(\bar{x}_1,z,v,w) = \hat{A}\bar{x}_1 + \hat{B}z + \hat{E}v \\ &+o(\bar{x}_1,z,v,w) \\ &\dot{z} = g_1 z + g_2 y \\ &y = \bar{h}_c(\bar{x}_1,z,v,w) = \hat{C}\bar{x}_1 + \hat{D}z + \hat{F}v \\ &+o(\bar{x}_1,z,v,w) \end{aligned}$$

where

$$\hat{A} = \left(\bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21}\right), \hat{B} = \left(\bar{B}_1 - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{B}_2\right)K_1,$$
$$\hat{C} = \left(C_1 - C_2\bar{A}_{22}^{-1}\bar{A}_{21}\right), \hat{D} = \left(-C_2\bar{A}_{22}^{-1}\bar{B}_2\right)K_1,$$
$$\hat{E} = \left(\bar{E}_1 - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{E}_2\right), \hat{F} = \left(F - C_2\bar{A}_{22}^{-1}\bar{E}_2\right)$$

Again if follows from the strong stability of (3.7) that the matrix

$$\begin{bmatrix} \hat{A} & \hat{B} \\ g_2\hat{C} & g_1 + g_2\hat{D} \end{bmatrix} \tag{3.12}$$

is Hurwitz. Thus, from the center manifold theorem (Carr, 1981), there exists sufficiently smooth functions $\bar{\mathbf{x}}_1^{(k)}(v,w)$ and $\mathbf{z}^{(k)}(v,w)$ with $\bar{\mathbf{x}}_1^{(k)}(0,0) = 0$, and $\mathbf{z}^{(k)}(0,0) = 0$ satisfying

$$\begin{aligned} &\frac{\partial \bar{\mathbf{x}}_1^{(k)}(v,w)}{\partial v} a(v) = \bar{f}_{1c}(\bar{\mathbf{x}}_1^{(k)}(v,w), \mathbf{z}^{(k)}(v,w), v, w) \\ &\frac{\partial \mathbf{z}^{(k)}(v,w)}{\partial v} a(v) = g_1\mathbf{z}^{(k)}(v,w) + g_2\mathbf{y}(v,w) \\ &\mathbf{y}(v,w) = \bar{h}_c(\bar{\mathbf{x}}_1^{(k)}(v,w), \mathbf{z}^{(k)}(v,w), v, w) \end{aligned} \tag{3.13}$$

In terms of notation $v^{[l]}$, $\bar{\mathbf{x}}_1^{(k)}(v,w)$, $\mathbf{z}^{(k)}(v,w)$ and $\mathbf{y}(v,w)$ can be uniquely expressed as

$$\begin{aligned} \bar{\mathbf{x}}_1^{(k)}(v,w) &= \sum_{l=1}^{k} \phi_{lw} v^{[l]} + O\left(v^{k+1}\right) \\ \mathbf{z}^{(k)}(v,w) &= \sum_{l=1}^{k} \theta_{lw} v^{[l]} + O\left(v^{k+1}\right) \\ \mathbf{y}(v,w) &= \sum_{l=1}^{k} Y_{lw} v^{[l]} + O\left(v^{k+1}\right) \end{aligned} \tag{3.14}$$

Substituting (3.14) into (3.13), expanding (3.13) as power series in $v^{[l]}, l = 1, \cdots, k$, yield the following

$$\begin{array}{l} \phi_{lw} A_l = \hat{A}\phi_{lw} + \hat{B}\theta_{lw} + \hat{U}_{lw} \\ \theta_{lw} A_l = g_1 \theta_{lw} + g_2 (\hat{C}\phi_{lw} + \hat{D}\theta_{lw} + \hat{V}_{lw}) \\ Y_{lw} = \hat{C}\phi_{lw} + \hat{D}\theta_{lw} + \hat{V}_{lw} \end{array} \quad (3.15)$$

where $\left(\hat{U}_{1w}, \hat{V}_{1w}\right) = \left(\hat{E}, \hat{F}\right)$, and for $l = 2, 3, \cdots$, $\left(\hat{U}_{lw}, \hat{V}_{lw}\right)$ depends only on $\phi_{1w}, \cdots, \phi_{(l-1)w}$ and $\theta_{1w}, \cdots, \theta_{(l-1)w}$.

(3.15) is the Sylvester equation described in Lemma 3.1, and satisfies all conditions of Lemma 3.1. Therefore, it has a unique solution for any $\left(\hat{U}_{lw}, \hat{V}_{lw}\right)$ and the solution satisfies

$$Y_{lw} = 0, l = 1, \cdots, k \quad (3.16)$$

Thus

$$\begin{aligned} \mathbf{y}(v,w) &= \overline{h}_c(\overline{\mathbf{x}}_1^{(k)}(v,w), \mathbf{z}^{(k)}(v,w), v, w) \\ &= O\left(v^{k+1}\right) \end{aligned} \quad (3.17)$$

Finally define

$$\begin{array}{l} \overline{\mathbf{x}}_2^{(k)}(v,w) = \gamma(\overline{\mathbf{x}}_1^{(k)}(v,w), \mathbf{z}^{(k)}(v,w), w) \\ \mathbf{x}^{(k)}(v,w) = T_2[\overline{\mathbf{x}}_1^{(k)}(v,w)^T, \overline{\mathbf{x}}_2^{(k)}(v,w)^T]^T \\ \mathbf{x}_c(v,w) = [\mathbf{x}^{(k)}(v,w)^T, \mathbf{z}^{(k)}(v,w)^T]^T \end{array} \quad (3.18)$$

Then it is ready to verify that $\mathbf{x}_c(v,w)$ satisfies (3.8) upon using (3.13), (3.17), and (3.18).

Remark 3.3: It has been shown in (Dai, 1989) that, given a linear singular system of the form

$$\begin{array}{l} \mathbf{E}\dot{x} = Ax + Bu + Ev, \\ y = Cx + Fv \\ \dot{v} = \hat{A}_1 v \end{array} \quad (3.19)$$

with $x \in \Re^n, u \in \Re^m, y \in \Re^p$, which satisfies assumptions **A2** and **A3**, and

A5:

$$rank \begin{bmatrix} A - \mathbf{E}\lambda & B \\ C & 0 \end{bmatrix} = n + p, \forall \lambda \in \sigma\left(\hat{A}_1\right)$$

and matrices g_1 and g_2 that are constructed according to Lemma 3.1 with $r = p$, there exist two matrices K_1 and K_2 such that the pair

$$\left(\begin{bmatrix} \mathbf{E} & 0 \\ 0 & I_{n_c} \end{bmatrix}, \begin{bmatrix} A + BK_2C & BK_1 \\ g_2C & g_1 \end{bmatrix} \right) \quad (3.20)$$

is strongly stable. As a result, the following result is obtained.

Corollary 3.4: Under assumptions **A1** to **A3**, assume the plant composed of (2.1) and (2.2) satisfies **A5** with $\hat{A}_1 = diag\{A_1, \cdots, A_k\}$, then there exists a controller of the form (3.6) such that property P1 holds and equation (3.8) is satisfied.

Remark 3.5: It can be shown using the center manifold theory (Carr, 1981; Isidori and Byrnes, 1990) that if the closed-loop system satisfies Property P1 and equation (3.8), then for sufficiently small $\left(x_{c0}, v_0\right)$, the solution of the closed-loop system exists for all $t \geq 0$, and is bounded, and

$$\limsup_{t\to\infty} y(t) = O\left(v^{k+1}(t)\right), \ \forall w \in W$$

For this reason, a controller that renders the closed-loop system these two properties is called k^{th}-order robust regulator.

4. SOLVABILITY OF THE PROBLEM

The k^{th}-order robust regulator is interesting in its own since it guarantees the steady state tracking error of the closed-loop system is in the order of $k+1$ of the exogenous signal regardless of the small variation of the uncertain parameter w. Moreover, under some additional assumptions, the k^{th}-order robust regulator actually solves the robust output regulation problem.

A6: The exosystem is linear, that is, $a(v) = A_1 v$.

Theorem 4.1: Under assumptions **A1** to **A6**, and suppose $\mathbf{u}(v,w)$ is k^{th} degree polynomial in v, then the same controller that solves the k^{th}-order robust output regulation also solves the robust output regulation problem.

Proof: By assumptions, there exist a linear control law of the form (3.6) that solves the k^{th}-order robust regulation problem. Clearly, the closed-loop system composed of the plant and this control law satisfies condition (i) of Theorem 2.2. We need to show that the closed-loop system also satisfies condition (ii) of Theorem 2.2. To this end, consider the following system

$$\begin{array}{l} \mathbf{E}\dot{x} = f\left(x, u + K_2 y, v, w\right) \\ y = h\left(x, v, w\right) \end{array} \quad (4.1)$$

Performing the same coordinate transformation on (4.1) as what was done for the closed-loop system (3.9) in the proof of Lemma 3.2 gives

$$\begin{array}{l} \dot{\overline{x}}_1 = \overline{f}_1\left(\overline{x}_1, u, v, w\right) \\ y = \overline{h}\left(\overline{x}_1, u, v, w\right) \end{array} \quad (4.2)$$

where $\overline{x}_1$ is such that $T_2^{-1} x = \left(\overline{x}_1^T, \overline{x}_2^T\right)^T$, and $\overline{f}_1\left(\overline{x}_1, u, v, w\right)$, and $\overline{h}\left(\overline{x}_1, u, v, w\right)$ is such that

$$\begin{array}{l} \overline{f}_1\left(\overline{x}_1, K_1 z, v, w\right) = \overline{f}_{1c}\left(\overline{x}_1, z, v, w\right) \\ \overline{h}\left(\overline{x}_1, K_1 z, v, w\right) = \overline{h}_c\left(\overline{x}_1, z, v, w\right) \end{array} \quad (4.3)$$

Next, let $\mathbf{x}(v,w)$ and $\mathbf{u}(v,w)$ be the solution of (2.8). Then clearly, $\mathbf{x}(v,w)$ and $\mathbf{u}(v,w)$ also satisfy the following

$$\begin{array}{l} \mathbf{E}\dfrac{\partial \mathbf{x}(v,w)}{\partial v} A_1 v = f(\mathbf{x}(v,w), \mathbf{u}(v,w) \\ +K_2 h(\mathbf{x}(v,w), v, w), v, w) \\ h(\mathbf{x}(v,w), v, w) = 0 \end{array} \quad (4.4)$$

Let $T_2^{-1}\mathbf{x}(v,w) = (\overline{\mathbf{x}}_1^T(v,w), \overline{\mathbf{x}}_2^T(v,w))^T$. Then (4.2) and (4.4) imply

$$\frac{\partial \bar{\mathbf{x}}_1(v,w)}{\partial v}A_1 v = \bar{f}_1\left(\bar{\mathbf{x}}_1(v,w),\mathbf{u}(v,w),v,w\right) \quad (4.5)$$
$$\bar{h}(\bar{\mathbf{x}}_1(v,w),\mathbf{u}(v,w),v,w) = 0$$

Next we will show that there exists a sufficiently smooth function $\mathbf{z}(v,w)$ such that

$$\begin{aligned}\mathbf{u}(v,w) &= K_1\mathbf{z}(v,w)\\ \frac{\partial \mathbf{z}(v,w)}{\partial v}A_1 v &= g_1\mathbf{z}(v,w)\end{aligned} \quad (4.6)$$

To this end, let $\bar{\mathbf{x}}_1^{(k)}(v,w)$, and $\mathbf{z}^{(k)}(v,w)$ be as defined in (3.13). Then by (4.3) and (3.17), $\bar{\mathbf{x}}_1^{(k)}(v,w)$, and $\mathbf{z}^{(k)}(v,w)$ also satisfy the following

$$\frac{\partial \bar{\mathbf{x}}_1^{(k)}(v,w)}{\partial v}A_1 v = \bar{f}_1\left(\bar{\mathbf{x}}_1^{(k)}(v,w),K_1\mathbf{z}^{(k)}(v,w),v,w\right)$$
$$\bar{h}(\bar{\mathbf{x}}_1^{(k)}(v,w),K_1\mathbf{z}^{(k)}(v,w),v,w) = O(v^{k+1}) \quad (4.7)$$

Since $\bar{\mathbf{x}}_1^{(k)}(v,w),\mathbf{z}^{(k)}(v,w)$ take the form given by (3.14), and ϕ_{lw} and θ_{lw} satisfy (3.15) and (3.16), comparing (4.5) with (4.7) shows that there exist sufficiently smooth functions $\bar{\mathbf{x}}_{1k}(v,w) = O\left(v^{(k+1)}\right)$ and $\mathbf{u}_k(v,w) = O\left(v^{(k+1)}\right)$ such that

$$\bar{\mathbf{x}}_1(v,w) = \sum_{l=1}^{k}\phi_{lw}v^{[l]} + \bar{\mathbf{x}}_{1k}(v,w)$$
$$\mathbf{u}(v,w) = \sum_{l=1}^{k}K_1\theta_{lw}v^{[l]} + \mathbf{u}_k(v,w)$$

But since $\mathbf{u}(v,w)$ is assumed to be k^{th} degree polynomial in v, it must hold that $\mathbf{u}(v,w) = \sum_{l=1}^{k}K_1\theta_{lw}v^{[l]}$. Let $\mathbf{z}(v,w) = \sum_{l=1}^{k}\theta_{lw}v^{[l]}$. Clearly the first equation of (4.6) is satisfied. Now using (3.15) and (3.16) gives $\theta_{lw}A_l = g_1\theta_{lw}, l = 1,\cdots,k$. Hence

$$\sum_{1}^{k}\theta_{lw}A_l v^{[l]} = \sum_{1}^{k}g_1\theta_{lw}v^{[l]} \quad (4.8)$$

Using $\frac{\partial v^{[l]}}{\partial v}A_1 v = \dot{v}^{[l]} = A_l v^{[l]}$ in (4.8) gives

$$\frac{\partial \mathbf{z}(v,w)}{\partial v}A_1 v = \sum_{l=1}^{k}\theta_{lw}\frac{\partial v^{[l]}}{\partial v}A_1 v = g_1\sum_{l=1}^{k}\theta_{lw}v^{[l]}$$

Thus, the second equation of (4.6) is satisfied.

Finally letting $\mathbf{x}_c(v,w) = (\mathbf{x}^T(v,w),\mathbf{z}^T(v,w))^T$ and substituting $\mathbf{x}_c(v,w)$ and $\mathbf{u}(v,w)$ into (2.5) gives

$$h_c(\mathbf{x}_c(v,w),v,w) = h(\mathbf{x}(v,w),v,w) = 0$$

and, additionally using (2.8) and (4.6) gives

$$\mathbf{E}_c\frac{\partial \mathbf{x}_c(v,w)}{\partial v}a(v) = \begin{bmatrix} f(\mathbf{x}(v,w),\mathbf{u}(v,w),v,w)\\ g_1\mathbf{z}(v,w)\end{bmatrix}$$

But (2.5) gives

$$f_c(\mathbf{x}_c(v,w),v,w) = \begin{bmatrix} f(\mathbf{x}(v,w),\mathbf{u}(v,w),v,w)\\ g_1\mathbf{z}(v,w)\end{bmatrix}$$

Thus condition (ii) of Theorem 2.2 is also satisfied.

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