15th Triennial World Congress, Barcelona, Spain

A MULTIOBJECTIVE $\mathcal{H}_\infty$ CONTROL PROBLEM: MODEL MATCHING AND DISTURBANCE REJECTION

Leyla GÖREN and Murat AKIN

Istanbul Technical University,
Department of Electrical Engineering,
80626, Maslak, Istanbul/TURKEY.
goren@elk.itu.edu.tr , murakin@elk.itu.edu.tr
Fax:(+90) 212 285 67 00

Abstract In this paper, the multiobjective $\mathcal{H}_\infty$ control problem of model matching and disturbance rejection by dynamic state feedback and disturbance measurement feedforward is studied via Linear Matrix Inequality (LMI) approach. To solve the problem, first, the multiobjective $\mathcal{H}_\infty$ control problem is defined and shown that this problem can be reduced in two independent $\mathcal{H}_\infty$ Model Matching Problems (MMP) and then the LMI-based solution of the $\mathcal{H}_\infty$ MMP is derived by using the solution of the $\mathcal{H}_\infty$ Optimal Control Problem (OCP) in the formulation of the LMI. The LMI-based solvability conditions for the multiobjective $\mathcal{H}_\infty$ control problem and a design procedure for the controller are given.
Keywords: Model Matching, Disturbance Rejection, Linear Matrix Inequalities, $\mathcal{H}_\infty$ Control.

1. INTRODUCTION

The MMP is a fundamental problem of linear control theory. It resembles the problem of dynamics assignment, but the emphasis here is on achieving a desired input-output behaviour, rather than a desired natural free behaviour.

The $\mathcal{H}_\infty$ norm of a transfer matrix is the maximum value over all frequencies of its largest singular value. The $\mathcal{H}_\infty$ MMP is to find a controller transfer matrix $R(s)$ as a precompensator, with property of stable and causal rational matrix, that is $R(s) \in \mathcal{RH}_\infty$, that minimizes the $\mathcal{H}_\infty$ norm of $T_m(s) - T(s)R(s)$ such that the stable and proper rational matrices $T_m(s)$ and $T(s)$ are given as the model and the system transfer matrices, respectively. This means that the closed-loop performance of the controlled system approximates the desired performance as given in $T_m(s)$ in the sense of the following criterion,

$$\gamma_{opt} = \inf_{R(s)\in\mathcal{RH}_\infty} \|T_m(s) - T(s)R(s)\|_\infty \qquad (1)$$

In the literature, there are some results on the $\mathcal{H}_\infty$ MMP. Two of them are based on Nevanlinna-Pick Problem (NPP) (Doyle *et al.*, 1992) and Nehari Problem (NP) (Francis, 1987; Francis and Doyle, 1987). A state-space solution of the $\mathcal{H}_\infty$ MMP that is based on canonical spectral factorizations and solutions of the algebraic Riccati Equations (ARE), is given in Hung (1989).

In all these studies, the $\mathcal{H}_\infty$ MMP has been solved for the systems that have only references inputs. However, the systems to be controlled are also faced to external disturbances, in many situations. In our formulation, the reference inputs and the disturbance to be attenuated are assumed as accessible or measurable. This type assumption was used for a 2DOF control structure in an $\mathcal{H}_2$ context (Mosca, *et al.*, 1989) and for the optimal regulation with disturbance measurement feedforward (Sternard and Söderstrom, 1989) for SISO case and for MIMO one (Hunt and Sebek, 1989).

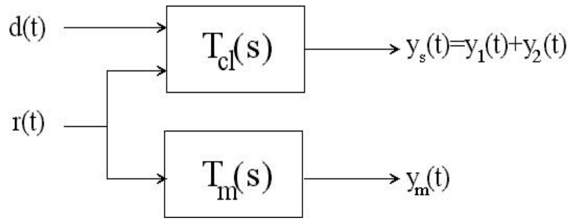

Figure 1

In Figure 1, $T_{cl}(s)$ and $T_m(s)$ are the closed-loop system and the model system transfer matrices, respectively, and $y_1(t)$ and $y_2(t)$ are the output components of related references, namely $r(t)$, and disturbance, namely $d(t)$, respectively. The multiobjective problem of $\mathcal{H}_\infty$ MMP and DRP can be defined as to find a control law for the system to be controlled that minimizes the following cost functions simultaneously,

$$J_1 = \sup_{\|r(t)\|_2 \le 1} \|y_m(t) - y_1(t)\|_2 \tag{2}$$

$$J_2 = \sup_{\|d(t)\|_2 \le 1} \|y_2(t)\|_2 \tag{3}$$

In this study, the multiobjective $\mathcal{H}_\infty$ control problem as defined above is considered and in the case measurable disturbances an LMI-based solution of the problem is given by dynamic state feedback.

2. PROBLEM STATEMENT

Consider a causal continuous Linear Time Invariant (LTI) system to be controlled described by the transfer matrix $T(s) = \left[\, T_1(s) \ T_2(s) \,\right]$ and the controller in the form of precompensator transfer matrix $\left[\, R_1(s) \ R_2(s) \,\right]$, that can easily be implemented by dynamic state feedback. Figure 2 illustrates these considerations:

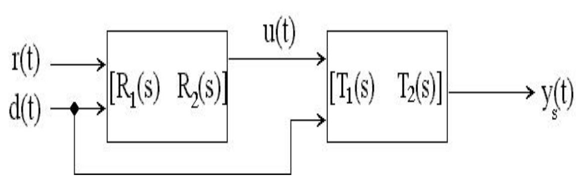

Figure 2

The outputs of the systems to be controlled and the controller can be written as follows,

$$Y_s(s) = T_1(s)U(s) + T_2(s)D(s) \tag{4}$$

$$U(s) = R_1(s)R(s) + R_2(s)D(s) \tag{5}$$

In addition, the output of the closed-loop system is obtained as,

$$\begin{aligned} Y_s(s) &= T_1(s)R_1(s)R(s) + (T_1(s)R_2(s) \\ &\quad + T_2(s))D(s) = Y_1(s) + Y_2(s) \end{aligned} \tag{6}$$

The cost functions defined in (2) and (3), that must simultaneously be minimized to solve the multiobjective $\mathcal{H}_\infty$ control problem, can be described as,

$$\begin{aligned} J_1 &= \sup_{\|r(t)\|_2 \le 1} \|y_m(t) - y_1(t)\|_2 \\ &= \|T_m(s) - T_1(s)R_1(s)\|_\infty \end{aligned} \tag{7}$$

$$J_2 = \sup_{\|d(t)\|_2 \le 1} \|y_2(t)\|_2 = \|T_2(s) + T_1(s)R_2(s)\|_\infty \tag{8}$$

Thus, it has been shown that the problem of simultaneous $\mathcal{H}_\infty$ MMP and DRP can be regarded as two numbers of the $\mathcal{H}_\infty$ MMP's. Namely, to solve the multiobjective $\mathcal{H}_\infty$ control problem, $T_1(s)$ must be matched to $T_m(s)$ and $-T_2(s)$ separately in the sense of $\mathcal{H}_\infty$ optimality criterion through the controller $R_1(s)$ and $R_2(s)$, respectively. As a result of above analysis, the following Remark can be given:

Remark 2.1 *To solve the multiobjective $\mathcal{H}_\infty$ control problem of MMP and DRP for the model and the system given as $T_m(s)$ and $T(s) = \left[\, T_1(s) \ T_2(s) \,\right]$, respectively, is equivalent to solve two different $\mathcal{H}_\infty$ MMP's, separately. They can be described the pairs of $(T_1(s), T_m(s))$ and $(T_1(s), -T_2(s))$. This idea is explained in Figure 3:*

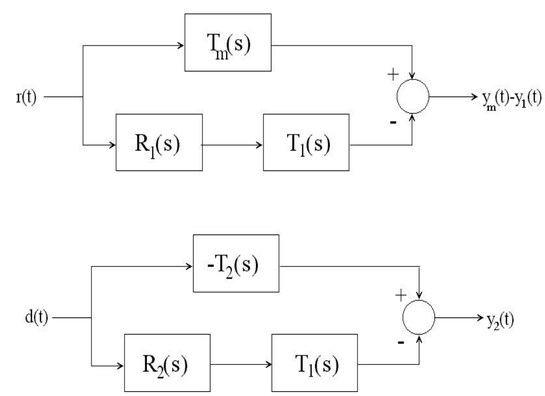

Figure 3

It should be noted that the controller , that solves the multiobjective $\mathcal{H}_\infty$ control problem, is composed by $R_1(s)$ and $R_2(s)$. In order to avoid of the trivial case, we assume that $T_1^{-1}(s)T_m(s)$ and $T_1^{-1}(s)T_2(s)$ are not stable or causal transfer matrices.

3. PRELIMINARIES AND LMI-BASED SOLUTION OF THE $\mathcal{H}_\infty$ MMP

In order to solve the multiobjective $\mathcal{H}_\infty$ control problem of model matching and disturbance rejection via LMI approach, first the $\mathcal{H}_\infty$ MMP should be solved using the solution of the standard $\mathcal{H}_\infty$ OCP via LMI approach by Gahinet and Apkarian (1994). For this aim, we shall consider a minimal realizations (A, B, C, D) of $T(s)$, namely controlled system, and (F,G,H,J) of $T_m(s)$, namely model system, so the state-space equations of these systems can be given as follows:

$$T(s): \quad \begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y_c(t) &= Cx(t) + Du(t) \end{aligned} \tag{9}$$

$$T_m(s): \quad \begin{aligned} \dot{q}(t) &= Fq(t) + Gw(t) \\ y_m(t) &= Hq(t) + Jw(t) \end{aligned} \tag{10}$$

and the control input $u(t)$ is generated by the reference input $w(t)$ via the precompensator $R(s)$ such that,

$$U(s) = R(s)W(s) \tag{11}$$

where $x(t) \in \mathcal{R}^n$, $q(t) \in \mathcal{R}^{n_m}$, $u(t) \in \mathcal{R}^m$, $r(t) \in \mathcal{R}^{m_r}$, $y_c(t) \in \mathcal{R}^p$ and $y_m(t) \in \mathcal{R}^{p_m}$.
Moreover, if one can consider a plant $P(s)$ described by,

$$\begin{bmatrix} \dot{x}(t) \\ \dot{q}(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} \begin{bmatrix} x(t) \\ q(t) \end{bmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix} w(t) + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t) \tag{12}$$

$$z(t) = y_m(t) - y_c(t) = \begin{bmatrix} -C & H \end{bmatrix} \begin{bmatrix} x(t) \\ q(t) \end{bmatrix} + Jw(t) - Du(t) \tag{13}$$

$$y(t) = w(t) \tag{14}$$

and from Figure 4, the closed-loop transfer matrix $T_{zw}(s)$ is obtained as,

$$T_{zw}(s) = T_m(s) - T(s)R(s) \tag{15}$$

by using the following relation,

$$P(s) = \begin{bmatrix} T_m(s) & -T(s) \\ I & 0 \end{bmatrix} \tag{16}$$

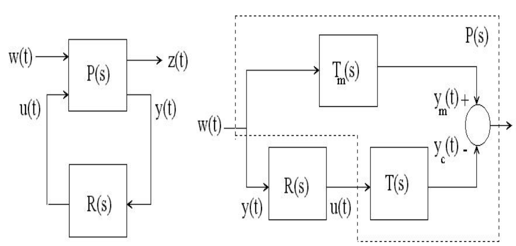

Figure 4

Since the $\mathcal{H}_\infty$ OCP is to find all admissible controllers $R(s)$ such that $\|T_{zw}(s)\|_\infty$ is minimized, the synthesis theorem for the $\mathcal{H}_\infty$ OCP in formulation of LMIs (Gahinet and Apkarian, 1994) can be arranged for the $\mathcal{H}_\infty$ MMP as the following Lemma:

Lemma 3.1 *A controller $R(s)$ with order $n_R \geq (dim(A) + dim(F))$ which holds $\|T_{zw}(s)\|_\infty \leq \gamma$ exists for the plant described by (12), (13), (14), and closed-loop system is internally stable for $\mathcal{H}_\infty$ Optimal Control Problem if and only if there exist matrices $X > 0$ and $Y > 0$ such that,*

$$\begin{bmatrix} N_o & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix}^* X + X \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix} & X \begin{pmatrix} 0 \\ G \end{pmatrix} & \begin{pmatrix} -C^* \\ H^* \end{pmatrix} \\ \begin{pmatrix} 0 & G^* \end{pmatrix} X & -\gamma I & J^* \\ \begin{pmatrix} -C & H \end{pmatrix} & J & -\gamma I \end{bmatrix} \begin{bmatrix} N_o & 0 \\ 0 & I \end{bmatrix} < 0 \tag{17}$$

$$\begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix} Y + Y \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix}^* & Y \begin{pmatrix} -C^* \\ H^* \end{pmatrix} & \begin{pmatrix} 0 \\ G \end{pmatrix} \\ \begin{pmatrix} -C & H \end{pmatrix} Y & -\gamma I & J \\ \begin{pmatrix} 0 & G^* \end{pmatrix} & J^* & -\gamma I \end{bmatrix} \begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix} < 0 \tag{18}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0 \tag{19}$$

where N_o and N_c are full rank matrices whose images satisfy

$$ImN_o = Ker \begin{bmatrix} 0 & I \end{bmatrix} \tag{20}$$

$$ImN_c = Ker \begin{bmatrix} B^* & 0 & -D^* \end{bmatrix} \tag{21}$$

in which (A,B,C,D) and (F,G,H,J) are the state-space description of $T(s) \in \mathcal{RH}_\infty$ and $T_m(s) \in \mathcal{RH}_\infty$, respectiveley.

Proof: The claims of the Lemma are same as those of the synthesis theorem for the $\mathcal{H}_\infty$ OCP in formulation of LMIs given in Gahinet and Apkarian (1994). □

However, it may be seen that the above Lemma can be simplified. For this aim, the following Lemmas are given:

Lemma 3.2 *Suppose A and Q are square matrices and $Q > 0$. Then A is Hurwitz if and only if there exists the unique solution*

$$X = \int_0^\infty e^{A^* t} Q e^{At} dt > 0 \tag{22}$$

to the Lyapunov equation $A^* X + XA + Q = 0$.

Proof: See Dullerud and Paganini (2000). □

Lemma 3.3 *The block matrix*

$$\begin{bmatrix} P & M \\ M^* & N \end{bmatrix} < 0 \tag{23}$$

if and only if $N < 0$ and $P - MN^{-1}M^* < 0$. In the sequel, $P - MN^{-1}M^*$ will be referred to as the **Schur Complement** *of N.*

Proof: See Boyd *et al.* (1994). □

Lemma 3.4 *Suppose A, C, X and Y are square matrices and $\gamma \in R$. If the matrix A is Hurwitz, then for every pair of $\gamma > 0$ and $Y > 0$, there always exists a matrix $X > 0$ such that holds the following inequalities,*

$$A^*X + XA + \frac{1}{\gamma}C^*C < 0 \qquad (24)$$

$$X - Y^{-1} \geq 0. \qquad (25)$$

Proof: It is easy to see that there always exists a matrix $X > 0$ satisfying (24) for every $\gamma > 0$ since A is Hurwitz. First, consider the Lyapunov equation with the matrix $Q > 0$ and $\epsilon \in \mathcal{R}^+$ as,

$$A^*X + XA + \frac{1}{\gamma}C^*C + \epsilon Q = 0 \qquad (26)$$

Since $(\frac{1}{\gamma}C^*C + \epsilon Q) > 0$, the unique solution of the equation (26) can be found from Lemma 3.2 as in the following form,

$$\begin{aligned} X &= \int_0^\infty e^{A^*t}(\frac{1}{\gamma}C^*C + \epsilon Q)e^{At}dt \\ &= \frac{1}{\gamma}L_o + \int_0^\infty e^{A^*t}\epsilon Q e^{At}dt > 0 \end{aligned} \qquad (27)$$

where the matrix L_o is **Observability Gramian** of (A, C) defined as follows,

$$L_o = \int_0^\infty e^{A^*t}C^*Ce^{At}dt \geq 0 \qquad (28)$$

To complete the proof, it will be sufficient to show that there exists a matrix $X > 0$ satisfying the condition $X \geq Y^{-1}$ i.e., (25) and also a solution the Lyapunov equation (26), since the every matrix X derived by (27) also satisfies the Lyapunov inequality of (24) as long as the matrix $Q > 0$. In that respect, consider the matrix P_0 be defined by,

$$P_0 = \int_0^\infty e^{A^*t}Qe^{At}dt \qquad (29)$$

then we can say from Lemma 3.2 that P_0 is a solution of the equation $A^*P_0 + P_0A + Q = 0$ and $P_0 > 0$ since A is Hurwitz. The matrix P_0 is positive definite if and only if $P_0 = P^*P$ and P is nonsingular. So

$$\begin{aligned} X = \frac{1}{\gamma}L_0 + \epsilon P_0 \geq Y^{-1} \qquad &\Longleftrightarrow \\ \epsilon \geq \lambda_{max}[(P^{-1})^*(Y^{-1} - \frac{1}{\gamma}L_0)P^{-1}] \end{aligned} \qquad (30)$$

Therefore, the proof is completed, since there also exist some $\epsilon \in \mathcal{R}^+$ satisfying (30). □

As consequence of the previous results, the following theorem can be given on LMI-based solution of the $\mathcal{H}_\infty$ MMP as a simplified version of Lemma 3.1:

Theorem 3.5 *A controller $R(s)$ with order $n_R \geq (dim(A)+dim(F))$ which holds $\|T_m(s)-T(s)R(s)\|_\infty < \gamma$, exists for the plant described by (12), (13), (14) and the closed-loop system is internally stable, i.e., there exists a solution the $\mathcal{H}_\infty$ MMP if and only if there exists a matrix $Y > 0$ such that,*

$$\begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix} Y + Y \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix}^* & Y \begin{pmatrix} -C^* \\ H^* \end{pmatrix} & \begin{pmatrix} 0 \\ G \end{pmatrix} \\ \begin{pmatrix} -C & H \end{pmatrix} Y & -\gamma I & J \\ \begin{pmatrix} 0 & G^* \end{pmatrix} & J^* & -\gamma I \end{bmatrix} \begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix} < 0 \qquad (31)$$

where N_c is a full rank matrix with

$$ImN_c = Ker \begin{bmatrix} B^* & 0 & -D^* \end{bmatrix} \qquad (32)$$

in which (A, B, C, D) and (F, G, H, J) are the state-space description of $T(s) \in \mathcal{RH}_\infty$ and $T_m(s) \in \mathcal{RH}_\infty$, respectively.

Proof: It is easily seen that the claim of the Theorem is the same as the condition (18) of Lemma 3.1. In addition, the condition (17) in Lemma 3.1 can be rewritten as follows:

$$\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix}^* X + X \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix} & X \begin{pmatrix} 0 \\ G \end{pmatrix} & \begin{pmatrix} -C^* \\ H^* \end{pmatrix} \\ \begin{pmatrix} 0 & G^* \end{pmatrix} X & -\gamma I & J^* \\ \begin{pmatrix} -C & H \end{pmatrix} & J & -\gamma I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} < 0 \qquad (33)$$

since $ImN_o = Ker \begin{bmatrix} 0 & I \end{bmatrix}$ so $N_o = \begin{bmatrix} I \\ 0 \end{bmatrix}$. The reduced form of above inequality is a Lyapunov inequality as follows:

$$\begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix}^* X + X \begin{bmatrix} A & 0 \\ 0 & F \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} -C^* \\ H^* \end{bmatrix} \begin{bmatrix} -C & H \end{bmatrix} < 0 \qquad (34)$$

Also, the condition (19) in Lemma 3.1 is equivalent to $X - Y^{-1} \geq 0$ due to Schur Complement of the matrix appeared in (19) as given in Lemma 3.3. Therefore, Lemma 3.4 proves both necessity and sufficiency. □

In order to determine the iteration starting point of γ in using The LMI Control Toolbox (Gahinet *et al.*, (1995)) for solving (31), the following Lemma and Remark can be given:

Lemma 3.6 *Consider a continuous-time transfer function $T(s)$ of (not necessarily minimal) realization $T(s) = D + C(sI - A)^{-1}B$. The following statements are equivalent:*
i) *$\|D + C(sI - A)^{-1}B\|_\infty < \gamma$ and A is Hurwitz;*
ii) *there exists a solution $Y > 0$ to the LMI:*

$$\begin{bmatrix} AY + YA^* & YC^* & B \\ CY & -\gamma I & D \\ B^* & D^* & -\gamma I \end{bmatrix} < 0. \qquad (35)$$

Proof: This theorem is dual form of **The Bounded Real Lemma** whose proof is in Dullerud and Paganini, (2000). □

Remark 3.7 *The following in equality always holds*

$$\gamma_{opt} = \inf_{R(s)\in\mathcal{RH}_\infty} \|T_m(s)-T(s)R(s)\|_\infty \leq \|T_m(s)\|_\infty. \tag{36}$$

Proof: It is obvious that if there exists a solution $Y > 0$ to the following LMI,

$$\begin{bmatrix} \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix} Y + Y \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix}^* & Y \begin{pmatrix} -C^* \\ H^* \end{pmatrix} & \begin{pmatrix} 0 \\ G \end{pmatrix} \\ \begin{pmatrix} -C & H \end{pmatrix} Y & -\gamma I & J \\ \begin{pmatrix} 0 & G^* \end{pmatrix} & J^* & -\gamma I \end{bmatrix} < 0 \tag{37}$$

for some value of γ, then there exists a solution of the LMI (31) given in Theorem 3.5 and so we can say $\gamma_{opt} \leq \gamma$. Moreover, from Lemma 3.6, the inequality (37) is equivalent to

$$\|J + \begin{bmatrix} -C & H \end{bmatrix} \begin{bmatrix} (sI-A)^{-1} & 0 \\ 0 & (sI-F)^{-1} \end{bmatrix} \begin{bmatrix} 0 \\ G \end{bmatrix} \|_\infty = \|T_m(s)\|_\infty < \gamma \tag{38}$$

Therefore, the inequality $\gamma_{opt} \leq \|T_m(s)\|_\infty$ always holds. □

We can say that the iteration in solving LMI (31) should be started at $\|T_m(s)\|_\infty = \gamma_m$, because of Remark 3.7. Suppose the matrix $Y > 0$ and $\gamma_{opt} \in \mathcal{R}^+$ are found as a solution of (31) by using The LMI Control Toolbox, then there always exists a matrix $X > 0$ such that $X - Y^{-1} \geq 0$ and the inequality (34) holds. Therefore, a matrix $X > 0$ is easily taken through the solution of inequality (34) by using the results of Lemma 3.4. Finally, the controller transfer matrix, which achieves model matching in the sense of $\mathcal{H}_\infty$ for the system and the model described in (9) and (10), respectively, is obtained as,

$$R(s) = D_K + C_K(sI - A_K)^{-1}B_K \tag{39}$$

through the controller reconstruction procedure given in Gahinet and Apkarian (1994) by using the matrices X and Y.

4. MAIN RESULT

In this chapter, an LMI-based solution of the multiobjective $\mathcal{H}_\infty$ control problem will be presented by using the results presented in previous chapters. For this aim, we shall consider a minimal realization (A, B, C, D, E_1, E_2) of $T(s)$, i.e., the system to be controlled, and (F, G, H, J) of $T_m(s)$, i.e., the model system, so the state-space equations of these systems can be written as follows,

$$T(s): \quad \begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + E_1 d(t) \\ y(t) &= Cx(t) + Du(t) + E_2 d(t) \end{aligned} \tag{40}$$

$$T_m(s): \quad \begin{aligned} \dot{q}(t) &= Fq(t) + Gr(t) \\ y_m(t) &= Hq(t) + Jr(t) \end{aligned} \tag{41}$$

Since $T_1(s)$ and $T_2(s)$ are defined as the transfer matrices input/output and disturbance/output, respectively, the state equations of these systems are obtained as,

$$T_1(s): \quad \begin{aligned} \dot{x_1}(t) &= Ax_1(t) + Bu(t) \\ y_1(t) &= Cx_1(t) + Du(t) \end{aligned} \tag{42}$$

$$T_2(s): \quad \begin{aligned} \dot{x_2}(t) &= Ax_2(t) + E_1 d(t) \\ y_2(t) &= Cx_2(t) + E_2 d(t) \end{aligned} \tag{43}$$

where $x(t), x_1(t), x_2(t) \in \mathcal{R}^n$, $q(t) \in \mathcal{R}^{n_m}$, $u(t) \in \mathcal{R}^m$, $r(t) \in \mathcal{R}^{m_r}$, $y(t), y_1(t), y_2(t) \in \mathcal{R}^p$ and $y_m(t) \in \mathcal{R}^{p_m}$.

As a result of Theorem 3.5 and Remark 2.1, the following Theorem can be presented on the solution of the simultaneous $\mathcal{H}_\infty$ MMP and $\mathcal{H}_\infty$ DRP with disturbance measurement and dynamic state feedback:

Theorem 4.1 *There exists a solution of the multiobjective $\mathcal{H}_\infty$ control problem of MMP and DRP for the system and the model given as $T(s)$ and $T_m(s)$, respectively, if and only if there exist matrices $Y_1 > 0$ and $Y_2 > 0$ such that,*

$$\begin{bmatrix} N_c & 0 \\ 0 & I_{m_r} \end{bmatrix}^* \begin{bmatrix} \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix} Y_1 + Y_1 \begin{pmatrix} A & 0 \\ 0 & F \end{pmatrix}^* & Y_1 \begin{pmatrix} -C^* \\ H^* \end{pmatrix} & \begin{pmatrix} 0 \\ G \end{pmatrix} \\ \begin{pmatrix} -C & H \end{pmatrix} Y_1 & -\gamma_1 I & J \\ \begin{pmatrix} 0 & G^* \end{pmatrix} & J^* & -\gamma_1 I \end{bmatrix} \begin{bmatrix} N_c & 0 \\ 0 & I_{m_r} \end{bmatrix} < 0 \tag{44}$$

$$\begin{bmatrix} N_c & 0 \\ 0 & I_{m_d} \end{bmatrix}^* \begin{bmatrix} \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} Y_2 + Y_2 \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}^* & -Y_2 \begin{pmatrix} I \\ I \end{pmatrix} C^* & \begin{pmatrix} 0 \\ E_1 \end{pmatrix} \\ -C \begin{pmatrix} I & I \end{pmatrix} Y_2 & -\gamma_2 I & -E_2 \\ \begin{pmatrix} 0 & E_1^* \end{pmatrix} & -E_2^* & -\gamma_2 I \end{bmatrix} \begin{bmatrix} N_c & 0 \\ 0 & I_{m_d} \end{bmatrix} < 0 \tag{45}$$

where N_c is a full rank matrix with

$$ImN_c = Ker \begin{bmatrix} B^* & 0 & -D^* \end{bmatrix} \tag{46}$$

in which (A, B, C, D, E_1, E_2) and (F, G, H, J) are the state-space description of $T(s) \in \mathcal{RH}_\infty$ and $T_m(s) \in \mathcal{RH}_\infty$, respectively.

Proof: Remark 2.1 and Theorem 3.5 prove both necessity and sufficiency. □

5. CONTROLLER CONSTRUCTION AND IMPLEMENTATION

Although the Theorem 4.1 given in previous chapter is about the solvability conditions of the multiobjective $\mathcal{H}_\infty$ control problem of MMP and DRP with dynamic state feedback and disturbance measurements, it also provides a procedure for the construction and implementation of the controller in feedback configuration:

Step 1: Find the matrix Y_1 and Y_2 for the optimal value of γ_1 and γ_2 with satisfying the LMI given in (44) and (45) respectively by using The LMI Control Toolbox.

Step 2: For the values Y_1, Y_2 and γ_1, γ_2 found in Step 1, find the matrix X_1 and X_2 respectively via the way explained in Chapter 3.

Step 3: Construct by using controller reconstruction procedure given in Gahinet and Apkarian, (1994) and then find $R_1(s) = D_{K_1} + C_{K_1}(sI - A_{K_1})^{-1}B_{K_1}$ and $R_2(s) = D_{K_2} + C_{K_2}(sI - A_{K_2})^{-1}B_{K_2}$.

Step 4: Construct the controller transfer matrix and then find its realization $R(s) = \begin{bmatrix} R_1(s) & R_2(s) \end{bmatrix}$ as (A_K, B_K, C_K, D_K) which achieves the optimal model matching and disturbance rejection in the sense of $\mathcal{H}_\infty$.

Step 5: Implement the controller in feedback configuration by using The Theorem 6.1 and Figure 6.2 in Kucera (1991).

It is easily seen that the optimal values of γ_1 and γ_2 are equal to J_1 and J_2 given in (7) and (8), respectively. Moreover, because of Remark 3.7 the iterations should be started from $\|T_m(s)\|_\infty$ and $\|T_2(s)\|_\infty$ respectively for solving (44) and (45).

6. CONCLUSIONS

In this study, a multiobjective $\mathcal{H}_\infty$ control problem has been formulated and solved by LMI approach. The problem has been defined as simultaneous $\mathcal{H}_\infty$ model matching and $\mathcal{H}_\infty$ disturbance rejection by dynamic state feedback and disturbance measurements. The existence conditions of the multiobjective $\mathcal{H}_\infty$ control problem have been presented and a construction procedure and implementation scheme has been provided. Therefore, a well-known combined problem called simultaneous model matching and disturbance rejection (Gören and Çamlıbel, 1997), in linear control theory, has been carried in $\mathcal{H}_\infty$ control theory. Finally, we note that the multiobjective $\mathcal{H}_\infty$ control problem of MMP and DRP studied in this paper is not completely general, in that the feedback and the disturbances are restricted to be dynamic state feedback and to be measurable or accessible, respectively. This subject is a current topic of our research activities.

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