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PRINCIPAL COMPONENT GPC WITH TERMINAL EQUALITY CONSTRAINT [1]

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Abstract: This paper presents a modification to the generalized predictive control algorithm which guarantees closed-loop stability. The GPC controller is designed using a terminal equality constraint. The available degrees of freedom are presented to the designer as parameters called principal components. This components can be left or removed from the solution to get different performances. Two methods to select the degrees of freedom are presented based on percentage of index minimized and control effort applied to the process respectively. This methods can be an alternative to the empirical selection of the weighting control factor λ.

Keywords: Predictive Control, closed-loop stability, terminal constraint

1. INTRODUCTION

Generalized predictive control (GPC) (Clarke *et al.*, 1987*a*) has been shown to be an effective way of controlling single-input single-output discrete processes. The strategy proposed by GPC is simple to understand and makes good practical sense: predict the behaviour of the output as a function of future control increments and minimize over these incrementes a cost index. This cost includes the errors between predicted and desired outputs and the control effort. Despite its advantages, GPC is deficient in that it does not offer a general stability result. Indeed, stability is only guaranteed in some special cases (infinite horizons). Several publications proposed modifications to the generalized predictive control algorithm which guarantee closed-loop stability:

- *Constrained receding-horizon predictive control (CRHPC) (Clarke and Scattolini, 1991) , Stabilizing I/O receding horizon control (SIORHC) (Mosca* et al.*, 1990)*: Optimize a quadratic function over a prediction horizon subject to the condition that the output matches the reference value over a further constraint range.
- *Stable Generalized Predictive Control (SGPC) (Kouvaritakis* et al.*, 1992)* applies the GPC algorithm to the system after it has been stabilized by means of an internally stabilizing controller.
- *Infinite Horizon GPC (GPC$^\infty$) (Scokaert and Clarke, 1993)* : where an infinite prediction horizon is used but the control horizon is reduced to a finite value.

It is shown in (Kouvaritakis *et al.*, 1992), that theoretically all approaches are equivalent. Stability results for some of the algorithms have traditionally been derived in the state space using the properties of the solution of the Ricatti equation associated with the control law. Others, by forcing the objective function

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to be monotonically decreasing with respect to time, yield stable control-loop systems In this paper, another algorithm is presented, sharing the terminal constraint philosophy, but with advantages in terms of numerical stability.

2. GENERALIZED PREDICTIVE CONTROL

The GPC formulation with quadratic cost index has been extensively developed in (Clarke *et al.*, 1987*a*), (Clarke *et al.*, 1987*b*) and (Clarke and Mohtadi, 1989). Such formulation uses the following CARIMA stochastic model:

$$y(k) = \frac{B(z^{-1})}{A(z^{-1})}u(k-1) + \frac{T(z^{-1})}{\Delta A(z^{-1})}\xi(k)$$

Where $y(k)$ is the system output, $u(k)$ is the control action, $\xi(k)$ represents disturbance (white noise), $T(z^{-1})$ is a disturbance filtration polynomial,$B(z^{-1})$ and $A(z^{-1})$ are numerator and denominator of the discrete transfer function of the process and Δ is the difference operator $(1-z^{-1})$.

A GPC controller is obtained by minimizing the following cost index

$$J(\Delta u) = \sum_{i=N_1}^{N_2} \alpha_i[y(k+i) - w(k+i)]^2 + \\ + \sum_{j=1}^{N_u} \lambda_j[\Delta u(k+j-1)]^2 \quad (1)$$

where $N = N_2 - N_1 + 1$ is the prediction horizon, N_u is the control horizon, Δu is the future vector of control increments, α_i is the prediction error weighting factor, λ_j is the control weighting factor and $w(k+i)$ is the future setpoint vector.

For simplicity's sake, $\alpha_i = \alpha', \forall i$ and $\lambda_j = \lambda', \forall j$.

The cost index (1) expressed in matrix form results in

$$J(\Delta u) = (Y-W)^T\alpha(Y-W) + \Delta u^T\lambda\Delta u \quad (2)$$

where $\alpha = \alpha' I_{N\times N}$ and $\lambda = \lambda' I_{N_u\times N_u}$ are diagonal matrices, and $Y_{N\times 1}$ and $W_{N\times 1}$ are the output prediction and projected setpoint vectors respectively.

This cost index, on minimization, gives the unconstrained control move vector

$$\Delta u = (G^T\alpha G + \lambda)^{-1}G^T\alpha(W - \Gamma\Delta u^f - Fy^f) \quad (3)$$

where G, Γ, F are matrices from the prediction model, Δu^f and y^f are past control moves, and past outputs respectively filtered by polynomial T.

3. GPC WITH TERMINAL EQUALITY CONSTRAINT (CRHPC)

In this algorithm a future control sequence Δu_{opt} is chosen minimizing the cost function (1) subject to the following equality constraints (figure 1) :

$$y(t+N+i) = w(t+N+i) \quad i \in [1\ldots m] \quad (4)$$
$$\Delta u(t+N_u+j) = 0 \quad j > 0$$

where $N_u = N + 2 - d$, and $m \leq N_u$ is the terminal constraint horizon.

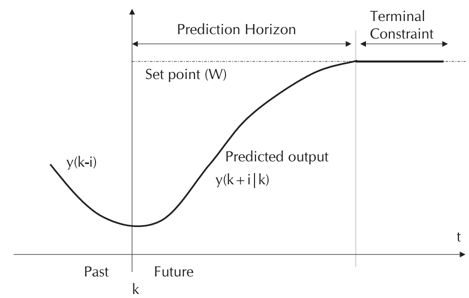

Fig. 1. CRHPC imposes an equality constraint.

Theorem: The closed loop under CRHPC control is asymptotically stable, in the undisturbed case, if

i) $\alpha_N \geq \cdots \geq \alpha_1 \geq 0$
ii) $\lambda_{N_u} \geq \cdots \lambda_1 > 0$
iii) $N \geq n+d+2$, where $n = max[deg(A), deg(B)]$
iv) $N_u = N+1-d$, where d is the process delay
v) $m = n+1$

Proof: See (Clarke and Scattolini, 1991) and (Scokaert and Clarke, 1994).

4. CALCULATION OF GPC USING SVD

In this analysis, no weighting factor is assumed ($\lambda = 0$), so the GPC control law (3), can be written as

$$\Delta u = ((QG)^T QG)^{-1}(QG)^T QE \quad (5)$$

where

$$\alpha = Q^T Q$$
$$E = (W - \Gamma\Delta u^f - FY^f)$$

Applying the SVD to the $QG_{N\times N_u}$ matrix results in $QG = U\Sigma V^T$ where

$$\Sigma = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix};\; S = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \sigma_r \end{bmatrix}$$

and substituting it in (5), the equation can be expressed as

$$\Delta u = ((U\Sigma V^T)^T U\Sigma V^T)^{-1}(U\Sigma V^T)^T QE$$

And profiting from the orthogonality of matrices U and V, the calculation results in

$$\Delta u = V\Sigma^+ U^T QE = (QG)^+ QE \qquad (6)$$

where $(QG)^+$ is the pseudoinverse matrix of QG.

Solutions (5) and (6) are equivalent if the matrix QG is of full column rank. However, when matrix QG is closed to lose its rank (its condition number is large), the problem is ill conditioned. This means that small changes in matrix QG can result in large changes in the elements of the solution Δu. This is the case where the solution calculated can be significantly different from the *real* solution, mainly due to this ill conditioning.

Therefore, controllers that are calculated with expressions (5) or (6) usually render large and unacceptable control moves because as N and N_u horizons become large, matrix QG becomes worse conditioned (Meyer, 2000).

An alternative to manage this problem is the use of the control weighting factor λ, that reduces the magnitude of control increments. However, this factor must be chosen empirically with a few guidelines available to aid in its selection.

A different way to handle poorly conditioned problems is called *Principal Components Analysis (PCA)* (D.E. Seborg P.R. Maurath and Mellichamp, 1988). This approach uses a singular value decomposition (SVD) of matrix QG. By means of a lower rank approximation to this matrix, a solution can be determined which results in only a slightly larger residual cost (poorer control) with a solution vector of smaller norm (smaller control increments but better robustness).

5. PRINCIPAL COMPONENTS ANALYSIS

The problem with minimizing index (1) [2] for the calculation of the control law can be written with the Euclidean norm ($\| \cdot \|_2$) as

$$J(\Delta u) = \| QG\Delta u - QE \|_2^2$$

Orthogonal transformations do not modify the Euclidean norm, so if U is orthogonal,

$$\| Ux \|_2 = \| x \|_2$$

then the optimization problem can be transformed through the SVD of matrix QG as

[2] Henceforth, no weighting factor λ is assumed.

$$J(\Delta u) = \| U\Sigma V^T \Delta u - QE \|_2^2$$
$$J(p) = \| \Sigma p - g \|_2^2 \qquad (7)$$

where $p = V^T\Delta u$ and $g = U^T QE$

The solution to this least-squares problem is trivial

$$p = \Sigma^+ g \qquad (8)$$

The components of vector p are known as *as principal components of the solution that minimizes the quadratic index*. The final solution, Δu, can be calculated from the former expressions:

$$\Delta u = Vp = V\Sigma^+ U^T QE = (QG)^+ QE$$

Each principal component p_i can be easily calculated through (8) and the cost index (7) can be written as

$$J(p) = (\sigma_1 p_1 - g_1)^2 + \cdots + (\sigma_r p_r - g_r)^2 + C \qquad (9)$$

where r is the rank of matrix Σ (with a maximum rank N_u), and C is a constant that appears if $r < N$. Nevertheless, this constant is neglected, as it does not affect the optimum.

From (9), it is deduced that every principal component p_i, contributes to improve the solution. If the i-th component is excluded from the solution ($p_i = 0$), the residue is increased in g_i^2. On the contrary, if component p_i is included, then the solution is improved, as the residue is decreased exactly g_i^2.

Furthermore, since matrix V is orthogonal and $\Delta u = Vp$, vectors p and Δu have the same Euclidean norm. If a component p_i increases the magnitude of vector p in a quantity corresponding to p_i^2, the magnitude of the control increments will also be increased the same. So, components that correspond to the smallest singular values only decrease the residue in a very small quantity (g_i are small), but on the other hand they significantly increase the magnitude of vector p (p_i^2 is large). Therefore, the suppression of such components would be desirable in order to conditioning the problem, yielding the same effect than using the weighting factor λ.

6. PRINCIPAL COMPONENT GPC WITH TERMINAL EQUALITY CONSTRAINT

Using the predictions from CARIMA model:

$$y(k+i|k) = G_i'\Delta u(k+i-1) + \underbrace{\Gamma_i \Delta u^f(k-1) + F_i y^f(k)}_{f(k+1|k)}$$

With G_i', Γ_i and F_i polynomials recursively calculated as in (Clarke *et al.*, 1987*a*) for $i = 1..N$, the output response for the prediction horizon can be obtained:

$$\begin{bmatrix} y(k+1) \\ \dots \\ y(k+N) \end{bmatrix} = \begin{bmatrix} G_1' \\ \dots \\ G_N' \end{bmatrix} \begin{bmatrix} \Delta u(k) \\ \dots \\ \Delta u(k+N_u-1) \end{bmatrix} + \\ + \begin{bmatrix} f(k+1) \\ \dots \\ f(k+N) \end{bmatrix}$$

The above set of predictions can be written in a matrix form as follows:

$$Y_{N\times 1} = G_{N\times Nu}\Delta u + F_{N\times 1}$$

This prediction can be extended, over the terminal constraint, from $N+1$ to $N+m$:

$$\begin{bmatrix} y(k+N+1) \\ \dots \\ y(k+N+m) \end{bmatrix} = \begin{bmatrix} G_{N+1}' \\ \dots \\ G_{N+m}' \end{bmatrix} \begin{bmatrix} \Delta u(k) \\ \dots \\ \Delta u(k+N_u-1) \end{bmatrix} + \\ + \begin{bmatrix} f(k+N+1) \\ \dots \\ f(k+N+m) \end{bmatrix}$$

which in matrix form is:

$$\overline{Y}_{m\times 1} = \overline{G}_{m\times Nu}\Delta u + \overline{F}_{m\times 1}$$

The cost index (1) and the equality constraint (4) can be rewritten as:

$$J(\Delta u) = (G\Delta u + F - W)^T \alpha (G\Delta u + F - W) + \\ + \Delta u^T \lambda \Delta u \quad (10)$$

$$\overline{G}\Delta u = (\overline{W} - \overline{F}) \quad (11)$$

where $\overline{W}$ is the future set-point from $k+N$ to $k+N+m$.

The minimization of (10) subject to (11) is presented in (Clarke and Scattolini, 1991) via Lagrange multipliers. The implementation of this algorithm requires the inversion of two matrices which, though symmetric, may nevertheless be very badly conditioned. An alternative approach to this problem can be to obtain an expression for the general solution of the underdetermined linear system (11) and minimize (10) over the remaining degrees of freedom in this general solution.

Reducing $[\overline{G}|\overline{W} - \overline{F}]$ [3] to a row echelon form using Gaussian elimination and then solving for the basic variables in terms of the free variables leads to the general solution:

$$\Delta u = \{\Delta u_p + z \mid z \in \mathcal{N}(\overline{G})\} \quad (12)$$

where

- Δu_p is a particular solution of the nonhomogeneous system
- z is the general solution of the associated homogeneous system $\overline{G}z = 0$:

  $$z = h_1 \Delta u_{f_1} + \ldots + h_{N_u-m}\Delta u_{f_{N_u-m}} \\ = H_{Nu\times(N_u-m)}\Delta u_{f_{(N_u-m)\times 1}}$$

  where Δ_{f_i} are the free variables and the set of vectors $\{h_1, \ldots h_{N_u-m}\}$ is a basis for the null space of $\overline{G}$, say $\mathcal{N}(\overline{G})$

One particular solution is the least-norm solution, which is very suitable for control purposes:

$$\Delta u_p = \overline{G}^+(\overline{W} - \overline{F})$$

where $\overline{G}^+$ is the pseudo-inverse of $\overline{G}$

As z characterizes available choices in the final solution, it must be obtained so that the cost index (10) is minimized. Using (12), eqn. (10) can be rewritten as:

$$J(\Delta u_f) = \| QG(\Delta u_p + H\Delta u_f) - Q(W-F) \|_2^2 + \\ + \lambda \| \Delta u_p + H\Delta u_f \|_2^2$$

If no weighting factor for the control increments is used, that index can be written as

$$J(\Delta u_f) = \| QGH\Delta u_f - \widetilde{E} \|_2^2 \quad (13)$$

where $\widetilde{E} = Q(W - F - G\Delta u_p)$.

Solution to this least squares problem can be calculated, using the SVD of matrix QGH:

$$QGH = U \begin{bmatrix} S \\ 0 \end{bmatrix} V^T$$

and defining the partitioned vector $U^T\widetilde{E}$ as

$$\begin{bmatrix} \widetilde{E}_{1m\times 1} \\ \widetilde{E}_{2(N_u-m)\times 1} \end{bmatrix} = U^T\widetilde{E}$$

the cost index (13) is transformed as:

$$J(\Delta u_f) = \| U \begin{bmatrix} S \\ 0 \end{bmatrix} V^T - \widetilde{E} \|_2^2$$

$$J(p) = \| Sp - \widetilde{E_1} \|_2^2 + \| \widetilde{E_2} \|_2^2$$

and the principal components can be obtained as

$$p = S^{-1}\widetilde{E_1}$$

These principal components can be seen as the available degrees of freedom. Finally, the optimal solution is given by

$$\Delta u = \Delta u_p + HVp$$

Theorem: The closed loop is asymptotically stable, in the undisturbed case, if

- $\alpha > 0$

[3] $[\overline{G}|\overline{W} - \overline{F}]$ is the augmented matrix for the nonhomogeneous system in which $rank(\overline{G}) = m$

- $N \geq n+d+2$, where n is the system order
- $N_u = N+1-d$, where d is the process delay
- $m = n+1$
- $p < r$, where p is the number of components included in the solution and $r = rank(QGH)$ is the number of singular values.

Proof: See (Scokaert and Clarke, 1993), and set the weighting factor $\lambda = 0$.

7. SIMULATED EXAMPLES

7.1 *Example: SISO process*

In order to illustrate the application of this design, the following non-minimum phase underdamped process is considered (Scokaert and Clarke, 1993):

$$G(z^{-1}) = \frac{z^{-4}(-z^{-1}+2z^{-2})}{1-1.5z^{-1}+0.7z^{-2}}$$

The parameters for the GPC design were chosen to be

N_1	N_2	m	N_u	$T(z^{-1})$	α	λ
1	15	3	12	1	1	0

(14)

There are $N_u - m = 13$ degrees of freedom available, therefore some criteria to select the components can be defined. For example, the percentage of cost index that is minimized if i components are included in the solution, can be calculated as

$$r_i = 100 \cdot \sum_{k=1}^{i} \frac{g_k^2}{g^T g} \%$$

Using the criterion that $r_i \leq 95\%$, 4 components are selected as it is shown in fig. 2. In figure 3, the closed loop response and control increments are shown. Finally, fig. 4 demonstrates how the objective function is monotonically decreasing with respect to time.

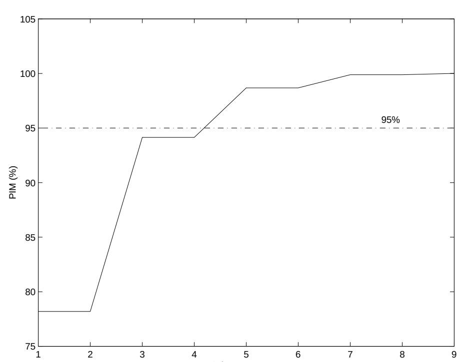

Fig. 2. Percentage of minimized index if each component is included.

7.2 *Example: MIMO process*

The ideas exposed above can be generalized straightforwardly to multivariable systems. Only the matrices size increases according to the number of inputs and outputs of the system. For example, Ogunnaik and Ray (Luyben, 1990) give the following transfer function matrix for an industrial distillation column:

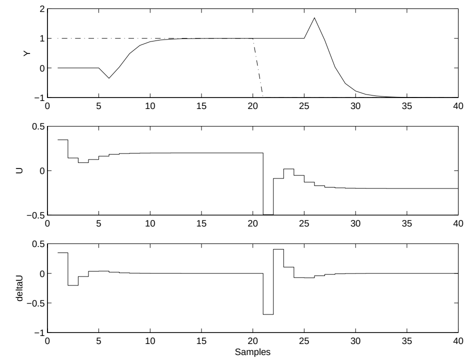

Fig. 3. Closed loop responses using 4 components.

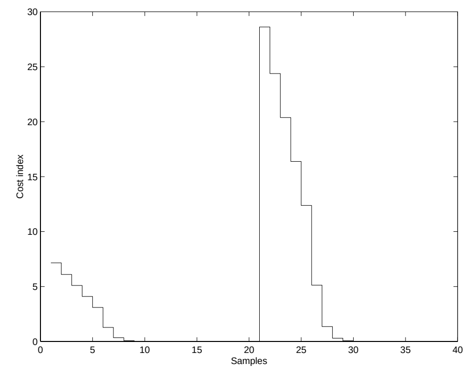

Fig. 4. Cost index values using 4 components.

$$G(s) = \begin{bmatrix} \frac{0.66e^{-2.6s}}{(6.7s+1)} & \frac{-0.61e^{-3.5s}}{(8.64s+1)} & \frac{-0.0049e^{-s}}{(9.06s+1)} \\ \frac{1.11e^{-6.5s}}{(3.25s+1)} & \frac{-2.36e^{-3s}}{(5s+1)} & \frac{-0.012e^{-1.2s}}{(7.09s+1)} \\ \frac{-34.68e^{-9.2s}}{(8.15s+1)} & \frac{46.2e^{-9.4s}}{(10.9s+1)} & \frac{-0.87(11.61s+1)e^{-s}}{(3.89s+1)(18.8s+1)} \end{bmatrix}$$

Using a sample time of 2.5 min. the parameters for the multivariable GPC were chosen to be

	N_1	N_2	N_m	N_u	$T(z^{-1})$	α	λ
Y_1	1	25	2	26	1	1	0
Y_2	1	25	2	26	1	1	0
Y_3	1	50	3	51	1	10	0

(15)

The total number of principal components are 103. Other guideline to select a subset of components is based on the magnitude of the future control increments calculated. The Euclidean norm of the control vector if i components are included in the solution can be written as

$$n_i = \sum_{k=1}^{i} \left(\frac{g_k}{\sigma_k} \right)^2$$

Using the criterion $n_i \leq \beta$, the components can be selected. In fig. 5, the values for n_i are shown and setting $\beta = 3$, at the most 45 components could be selected.

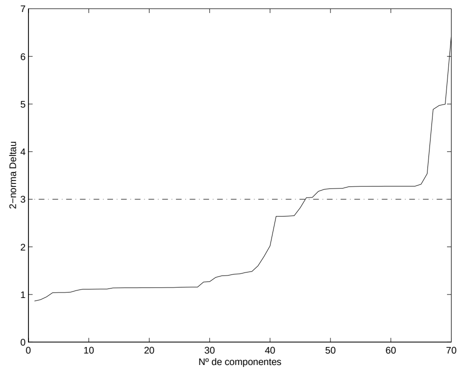

Fig. 5. Control vector norm for the first 70 components.

In fig. 6 the closed loop responses are shown whereas in fig. 7, the decreasing property of the cost index is depicted.

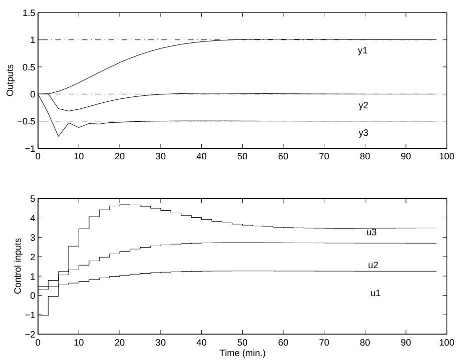

Fig. 6. Closed loop responses for 45 components.

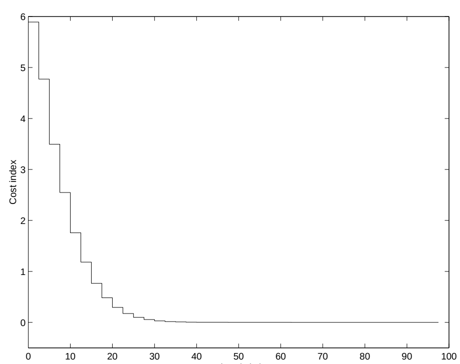

Fig. 7. Cost index when 45 components are used.

8. CONCLUSIONS

In this work a new algorithm for GPC design is presented based in principal component analysis. Furthermore, the inclusion of a terminal equality constraint ensures the stability of the closed loop. Some criteria for principal component selection are discussed. This criteria can be an alternative for the empirical selection of factor λ. Two illustrative examples for SISO and MIMO processes have shown the good behaviour of the proposed methodology.

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