15th Triennial World Congress, Barcelona, Spain

# CONSTRUCTION OF SINGULAR SURFACES IN MULTIPLE INTEGRAL VARIATIONAL PROBLEM

**Arik Melikyan** [1]

*Institute for Problems in Mechanics, Russian Academy of Sciences;*
*Vernadsky Ave 101-1, Moscow 117526, Russia*
*E-mail: melik@ipmnet.ru; Fax: +7 (095) 9382048*

Abstract: The classical method of characteristics is a powerful tool for construction of smooth solutions to nonlinear first order PDEs. Certain generalization of this approach (method of singular characteristics (MSC)) is useful for the construction of the surfaces where the solution is non-smooth. In this paper it is shown that the MSC can be used for the construction of singular surfaces (weak waves) in some second order PDEs – Euler-Lagrange equation for multiple integral variational problem. A two dimensional variational wave equation is considered as an example. The phenomenon of bifurcation of the weak waves (singular lines) is found using analytical and numerical methods.

Keywords: variational problem, singular surface, bifurcation

## 1. REGULAR AND SINGULAR CHARACTERISTICS

Some problems in nonlinear PDEs of the first or second order geometrically are equivalent to the construction of the integral surfaces $\Sigma$ of the 1-form $\alpha = du - p\,dx$, or their projections $\Gamma$ onto the subspace $R^n_x$, (Courant, 1962), (Arnold, 1988). The surfaces $\Sigma$ and $\Gamma$ may have the dimension $n, n-1, ..., 1$. The Cauchy problem for the first order equation, $H(x,u,p) = 0$, is formulated in terms of a $n$-dimensional surface $\Sigma_0$, while the initial conditions define a $(n-1)$-dimensional surface $\Sigma_1 \subset \Sigma_0$ (initial strip). The construction of $\Sigma_0$, together with a smooth solution of the equation $H(x,u,p) = 0$, is known to be reduced to the integration of the following system of regular (classical) characteristics:

$$\dot{x} = H_p, \quad \dot{u} = \langle p, H_p \rangle, \quad \dot{p} = -H_x - pH_u \qquad (1)$$

with the initial conditions on the manifold $\Sigma_1$.

The system (1) defines a one-dimensional (characteristic) subspace of the tangent space for the even-dimensional surface $W_1$ defined by the equation $H(x,u,p) = 0$ in the $(2n+1)$-dimensional space of $(x,u,p)$. Using similar geometry one can define such a tangent field for even-dimensional surfaces of codimension $3, 5, \ldots$. The corresponding ODE system is called the system of singular characteristics.

Singular characteristics allow to construct the surfaces $\Sigma$ of lower dimension. In the case of $(n-1)$-dimensional surface $\Sigma_1$ and the initial $(n-2)$-dimensional surface $\Sigma_2$ one must have a submanifold $W_3 \subset R^{2n+1}$ of codimension 3 ($z = (x,u,p)$):

$$W_3: \quad F_0(z) = 0,\ F_1(z) = 0,\ F_{-1}(z) = 0 \qquad (2)$$

where the functions $F_i(x,u,p)$ are defined by the conditions of the problem. The modified characteristic system has the same form (1) with the so-called singular Hamiltonian $H^\sigma$, instead of $H$:

$$\mu H^\sigma = \{F_1 F_0\}F_{-1} + \{F_0 F_{-1}\}F_1 + \{F_{-1}F_1\}F_0 \qquad (3)$$

Here $\mu$ is a nonzero homogeneity multiplier chosen by the convenience reasoning, and $\{FG\}$ is the Jacobi (Poisson) bracket

$$\{FG\} = \langle F_x + pF_u, G_p \rangle - \langle G_x + pG_u, F_p \rangle$$

The restriction of such system to the manifold $W_3$ is a tangent field which actually is used in constructions.

[1] Partially supported by the Russian Foundation for Basic Research, grant N 01-01-00376.

A complete formulation of a theorem guaranteeing the local existence of the surface $\Sigma$ searched for one can find in (Melikyan, 1998).

Singular characteristics correspond to some singular paths in nonlinear and optimal control, differential games (Subbotin, 1995), (Isidori, 1996), (Bardi and Dolcetta, 1997), (Melikyan, 1998).

## 2. MULTIPLE INTEGRAL VARIATIONAL PROBLEM

### 2.1 *First variation formula*

Consider the following variational problem with the unknown scalar function $u(x)$, $x \in G \subset R^n$, subject to some boundary conditions:

$$J = \int\limits_G F(x, u(x), p(x))dx \ \rightarrow \ extr \qquad (4)$$

$$(p = \partial u/\partial x) \quad B[u(x)]\Big|_{x\in\partial G} = 0$$

More exact formulation of the boundary conditions $B[u(x)] = 0$ is not essential for the sequel. The functional (4) is considered on the set

$$U = \{u^*(x), G_*\} \qquad (5)$$

consisting of the pairs $(u^*(x), G_*)$, where the continuous function $u^*(x)$ is defined in its own domain of definition $G_*$ and is piecewise twice differentiable there. Thus, a variational problem with variable (not fixed) boundary is considered. The Lagrangian $F$ is supposed to be smooth enough.

A twice differentiable solution of the problem (4) is known to satisfy the Euler-Lagrange equation – a second-order quasilinear partial differential equation:

$$F_u - divF_p = 0, \quad x \in G \qquad (6)$$

$$\left( divF_p = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F_{p_i} \right)$$

Generally, a nonsmooth function from the class (5) can also solve the variational problem (4). For such functions the Euler equation (6) is fulfilled only for the points of smoothness.

Fix two elements of $U$: $(u(x), G)$, $(h(x), G)$ with smooth $u(x), h(x)$ and a smooth vector function $\phi(x) = (\phi_1, \ldots, \phi_n)$. Define one-parameter family of admissible functions as:

$$\bar{u}(x, \varepsilon) = u(\bar{x}) + \varepsilon h(\bar{x}) + \ldots, \quad x \in G_\varepsilon \qquad (7)$$

$$\bar{x} = x + \varepsilon\phi(x) \in G$$

where $G_\varepsilon$ is preemage of $G$. For $\varepsilon = 0$ one has $G_\varepsilon = G$ and $\bar{u}(x, 0) = u(x)$ since $\bar{x} = x$.

Substituting the family (7) into the functional (4) and differentiating with respect to $\varepsilon$ at $\varepsilon = 0$ one can get the following first variation formula:

$$\delta J = \int\limits_G (F_u - divF_p)\bar{h}(x)dx \qquad (8)$$

$$+ \int\limits_{\partial G} \left\langle \bar{h}(x)F_p + F\phi(x), n(x) \right\rangle d\sigma$$

where $\bar{h}(x) \equiv h(x) - < \partial u(x)/\partial x, \phi >$, and $n(x)$ is an outward normal to the boundary of the domain $G$ at the point $x \in \partial G$, a normal to the surface element $d\sigma$. Here the function $u(x)$ is assumed to be twice differentiable and the surface $\partial G$ to be piecewise smooth. The formula (8) shows that for the first variation the values of $\phi(x)$ are actually important only at the points of the boundary $\partial G$.

Using in (8) the variations with fixed boundary and boundary values, when the function $\phi(x)$ vanishes in the whole domain $G$ and the function $h(x)$ vanishes on the boundary $\partial G$ only one can get from $\delta J = 0$ the Euler-Lagrange equation (6).

### 2.2 *Necessary conditions for singular surface*

Let a pair $(u(x), G)$ be the solution of the problem (4), and let a smooth surface $\Gamma \subset G$ divide the domain $G$ into two open subdomains $G^-$, $G^+$: $G = G^- + \Gamma + G^+$. Suppose that the function $u(x)$ is continuous in $G$ and twice differentiable in either domain $G^-$, $G^+$, while its gradient has a jump at $\Gamma$. The restrictions of the solution $u(x)$ to the domains $G^-, G^+$ will be denoted as $u^+(x), u^-(x)$ and their gradients as:

$$p = \frac{\partial u^-(x)}{\partial x}, \quad x \in G^- \qquad (9)$$

$$q = \frac{\partial u^+(x)}{\partial x}, \quad x \in G^+$$

Thus, $u^-(x) \in C^2(G^-)$, $u^+(x) \in C^2(G^+)$, while the vectors $p(x)$, $q(x)$ have, by assumption, continuous extensions up to the surface $\Gamma$ from the domains $G^-$, $G^+$.

To derive the necessary optimality conditions for the surface $\Gamma$, let us represent the functional $J$ as a sum of two functionals $J^-$, $J^+$, defined in the domains $G^-, G^+$, correspondingly. One has $\delta J = \delta J^- + \delta J^+$, while the variations $\delta J^-$, $\delta J^+$ are due to variations:

$$\delta u^-(x) = (h^-(x), \phi(x)), \qquad (10)$$

$$\delta u^+(x) = (h^+(x), \phi(x))$$

$$h^-(x) = h^+(x) = h(x), \quad x \in \Gamma$$

The last condition here follows from the continuity of the solution $u(x)$ in the domain $G$. Suppose that the boundary $\partial G$ of the original domain is fixed, the functions $u^-(x), u^+(x)$ satisfy the Euler equation in the domains $G^-, G^+$, their values on $\partial G$ are fixed, while the common part $\Gamma$ of the boundaries of the domains $G^-, G^+$ is subject to variation together with the common values of these functions on $\Gamma$. The first variation of the functional which vanishes for all admissible variations (10), takes the form:

$$\delta J = \int_\Gamma \Big[h^- \langle F_p, n\rangle - h^+ \langle F_q, n\rangle + \tag{11}$$
$$+\big\langle (F(x,u,p) - F(x,u,q))n$$
$$- \langle F_p, n\rangle p + \langle F_q, n\rangle q, \phi\big\rangle\Big] d\sigma$$

Due to the main lemma of Variation Calculus from the condition $\delta J = 0$ it follows that the scalar multiplier at $h(x)$ and the vector multiplier at $\phi(x)$ vanish on $\Gamma$:

$$\langle F_p - F_q, n\rangle = 0 \tag{12}$$
$$(F(x,u,p) - F(x,u,q))n - \langle F_p, n\rangle p + \langle F_q, n\rangle q = 0$$

Due to continuity of $u(x)$, the vector $p - q$ is a normal to $\Gamma$, i.e. $n = \lambda(p - q)$ for some scalar $\lambda$. One has $\langle F_q, n\rangle (p-q) = \langle F_q, p - q\rangle n$. This equation together with the first equality in (12) allows to reduce the second equality in (12) to the form: $[F(x,u,p) - F(x,u,q) - \langle F_q, p - q\rangle]n = 0$, which means that the left hand side of the second equality in (12) also is colinear to the vector $n$. Since $n$ is a nonzero (unit) vector, the scalar multiplier at $n$ must vanish. Thus, a scalar and a vector equalities in (12) are equivalent to the following two scalar equations which are fulfilled on the surface $\Gamma$:

$$F(x,u,p) - F(x,u,q) - \langle F_q, p - q\rangle = 0, \tag{13}$$
$$\langle F_p - F_q, p - q\rangle = 0$$

The equations (13) are generalizations of the Weierstrass-Erdmann corner conditions known in the scalar integral variational problem.

## 3. METHOD OF SINGULAR CHARACTERISTICS

The following two $(n-1)$-dimensional surfaces in the space $R^{2n+1}$ of vectors $(x,u,p)$ are associated with the surface $\Gamma$:

$$\Sigma^- = \{(x,u,p) \in R^{2n+1} : \tag{14}$$
$$u = u^-(x),\ p = \frac{\partial u^-(x)}{\partial x},\ x \in \Gamma\},$$
$$\Sigma^+ = \{(x,u,p) \in R^{2n+1} :$$
$$u = u^+(x),\ p = \frac{\partial u^+(x)}{\partial x},\ x \in \Gamma\}$$

By construction, the surfaces $\Sigma^\pm$ are projected into the surface $\Gamma$ and are the integral surfaces of the 1-form $\alpha = du - p\,dx$, i.e. tangent vectors of $\Sigma^\pm$ are zeros of the form $\alpha$. Such surfaces can be constructed using the method of singular characteristics (Melikyan, 1998).

Modify, first, the Weierstrass-Erdmann conditions by simplifying notation. The conditions (13) are quite symmetric with respect to both smooth solution branches $u^\pm(x)$. Note, that in a construction procedure (numerical or analytical) one of the branches could be found prior to the construction of the surface $\Gamma$, while the construction of the second branch requires the knowledge of $\Gamma$. For the branch $u^-(x)$ we omit the superscript and denote it simply as $u(x)$, the branch $u^+(x)$ will be denoted as $v(x)$, while for the gradients the same notation $p, q$ will be used.

For definiteness, we assume that the branch $v(x)$, $q(x)$, or more precisely, a certain its smooth extension to the domain $G$, is known. Substitute the values $v(x), q(x)$ into the left hand sides of the equalities (13), and consider these expressions as the functions of $(x,u,p)$, denoted, correspondingly, by $H(x,u,p)$, $R(x,u,p)$. The surface $\Sigma^+$ in (14) is considered as a searched for one. Thus, the following three necessary optimality conditions are fulfilled on that surface:

$$H(x,u,p) = F(x,u,p) - F(x,v(x),q(x)) -$$
$$- \langle F_q(x,v(x),q(x)), p - q(x)\rangle = 0,$$
$$R(x,u,p) = \langle F_p(x,u,p) - \tag{15}$$
$$-F_q(x,v(x),q(x)), p - q(x)\rangle = 0,$$
$$F_1(x,u) = u - v(x) = 0$$

The first two equations here represent the modified Weierstrass-Erdmann conditions, while the last one means simply the continuity condition of the solution to the problem (4) on the surface $\Gamma$. The latter condition looks trivial but it is a necessary addition to the Weierstrass-Erdmann conditions for the implementation of the method of singular characteristic.

Another useful observation is that the function $R$ in the second Weierstrass-Erdmann condition can be expressed through the first condition as the following Jacobi bracket:

$$R(x,u,p) = \{F_1 H\} = \langle H_p(x,u,p), p - q\rangle \tag{16}$$
$$(H_p = F_p - F_q)$$

The Jacobi bracket turns to be the Poisson bracket if there is no dependence on $u$. It should be mentioned that such a dependence always exists in the continuity condition $F_1(x,u) = 0$, even if the Lagrangian $F$ in (4) does not depend on $u$.

The relations (15),(16) suggest an invariant interpretation of the Weierstrass-Erdmann conditions.

Thus, the conditions (15) define in the space $R^{2n+1}$ the following manifold $W_3$, generally, of codimension 3:

$$W_3 : \quad H(x,u,p) = 0, \tag{17}$$
$$R(x,u,p) = \{F_1 H\} = 0, \quad F_1(x,u) = 0$$

This manifold is one of the necessary components for the construction of the surface $\Gamma$.

Using the functions (17) in (2) and (3), taking $\mu = \{\{F_1 F\} F_1\}$ and writing the system (1) in terms of the corresponding $H^\sigma$, one can get the following system of singular characteristics:

$$\dot{x} = H_p, \quad \dot{u} = \langle p, H_p\rangle, \tag{18}$$
$$\dot{p} = -H_x - pH_u - \frac{\{\{HF_1\}H\}}{\{\{F_1 H\}F_1\}}(p - q(x))$$

As an initial manifold $\Sigma_2$ for the system (18) may serve, for instance, some shifting over a submanifold

$\Gamma_2 \subset \partial G$, $dim\Gamma_2 = n-2$, on which the boundary value is nonsmooth.

The system (18) describes one of the types of singular characteristics associated with a nonlinear first order partial differential equation $H(x,u,p) = 0$. The role of the system (18) for the second order PDE (6) is that it describes the propagation of the disturbances (nonsmoothness) of the solution. Using the system (18) one can find, in particular, a subdomain of the boundary $\partial G$ which affects on the value of the solution at a given point of the domain $G$.

In the theory of differential games the system (18) represents a certain type of singular characteristics of the Bellman-Isaacs equation describing so-called equivocal singular paths.

*Quadratic Lagrangian.* In some problems of mathematical physics the Lagrangian is a quadratic function of the vector $p$:

$$F(x,u,p) = \frac{1}{2}\langle A(x,u)p, p\rangle \tag{19}$$

Here $A$ is a square symmetric matrix, $A = A^T$, with elements $a_{ij}$ depending, generally, on $x, u$. Computations show that the Hamiltonian $H(x,u,p)$, the function $R(x,u,p)$ and the Jacobi brackets in the relations (15), (18) for the case of quadratic Lagrangian take the form:

$$\begin{aligned} H(x,u,p) &\equiv \frac{1}{2}\langle A(x,u)(p-q(x)), p-q(x)\rangle \\ &\equiv F(x,u,p-q(x)), \\ R(x,u,p) &\equiv \{F_1 H\} \equiv 2H(x,u,p) \\ \{\{F_1 H\}F_1\} &\equiv -4H, \quad \{\{HF_1\}H\} \equiv 0 \end{aligned} \tag{20}$$

In this case two of three functions $F_i(x,u,p)$ in (17) coincide, the manifold $W_3$ has the codimension less than three, and thus, the uniqueness conditions for the surface $\Sigma_1$ are violated. Indeed, one can choose arbitrarily the missed third condition in (17) and obtain, generally, different surfaces $\Sigma_1$. One can show, that in case of the quadratic Lagrangian the projection $\Gamma_1$ of the surface $\Sigma_1$ will be the same for all the choices of the missed function, and for the constructions one can use the system of regular characteristics (1) with the Hamiltonian (20). The latter system can be simplified to the form (($\xi = p - q$)):

$$\dot{x} = F_\xi, \quad \dot{\xi} = -F_x - qF_u, \quad \dot{u} = \langle q, F_\xi\rangle \tag{21}$$

where $q = q(x)$ is regarded as a known function. Since the solution is continuous on $\Gamma$ the last equation is decoupled from the first two equations by substituting $u = v(x)$.

## 4. EXAMPLE

### 4.1 *Problem formulation*

Consider a two-dimensional problem (4) with the quadratic Lagrangian of the particular form:

$$F(x,u,p) = \frac{1}{2}(-\alpha(u)p_1^2 + p_2^2) = \frac{1}{2}\langle A(u)p, p\rangle \tag{22}$$

The matrix $A$ here is diagonal with the entries satisfying the conditions:

$$\begin{aligned} \alpha_{11} = det(A(u)) = -\alpha(u) < 0, \\ \alpha_{12} = \alpha_{21} = 0, \quad \alpha_{22} = 1 \end{aligned} \tag{23}$$

Thus, the function $\alpha(u)$ is positive for all $u$.

Introduce the componentwise notations: $x = x_1$, $y = x_2$. The domain $G$ is a rectangular lying in the half-plane $y > 0$, the bottom side lies on the abscissa axis $y = 0$, while its midpoint coincides with the origin of the coordinate system. One does not need a more precise description of the domain because the considerations below carry a local character and involve the vicinity of the origin. The Euler equation (6) using (22) and corresponding initial conditions have the form:

$$\begin{aligned} \frac{\partial^2 u}{\partial y^2} = \alpha(u)\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\alpha'(u)\Big(\frac{\partial u}{\partial x}\Big)^2 \\ u(x,0) = w(x), \quad \frac{\partial u(x,0)}{\partial y} = \psi(x) \end{aligned} \tag{24}$$

The functions $w(x), \psi(x)$ are smooth enough everywhere except for the origin, $x = 0$, where $w(x)$ may be nonsmooth (being continuous); the function $\psi(x)$ may have also a finite jump.

The second order terms of the equation (24) are the same as in a quasilinear wave equation with the wave speed $a(u) = \sqrt{\alpha(u)}$ depending upon the solution $u$, but the first order term is different. Such an equation is called the variational wave equation. The use of $\alpha$ instead of $a^2$ happens to be more convenient for the computations in the sequel.

### 4.2 *Initial conditions*

The irregularities of the functions $w(x), \psi(x)$ at the origin may cause a nonsmoothness in the solution, i.e. generate several weak waves propagating from the point $(0,0)$ into the domain $G$.

As shown in (Melikyan, 1998) the number of waves in generic case is 2. These waves divide the upper half-plane into 3 sectors. By assumption, the solution is twice differentiable in each sector and in linear approximation has the form:

$$\begin{aligned} u_i(x,y) = a_i x + b_i y + c, \quad i = 1,2,3 \\ a_i = \frac{\partial u_i(0,0)}{\partial x}, \quad b_i = \frac{\partial u_i(0,0)}{\partial y}, \\ c = u_i(0,0) = w(0) \end{aligned} \tag{25}$$

The constant $c = w(0)$ here is the common value of all the branches at the origin. Let $k_i$ be the slope of the tangent line to the $i$-th wave at the origin. Due to continuity of the solution two neighboring branches (25) equal each other at the common curve of the weak jump. It follows from here that the parameters $k_i$ and $a_i, b_i$ satisfy the relations:

$$a_{i+1} - a_i + k_i(b_{i+1} - b_i) = 0, \quad i = 1, 2 \quad (26)$$

From the total number of 9 parameters: $c, k_1, k_2, a_1, a_2, a_3, b_1, b_2, b_3$, the following 5 parameters are given due to the initial conditions:

$$c = w(0), \quad a_1 = \frac{\partial w(+0)}{\partial x}, \quad b_1 = \psi(+0),$$

$$a_3 = \frac{\partial w(-0)}{\partial x}, \quad b_3 = \psi(-0)$$

The two coinciding Weierstrass-Erdmann conditions: $R = 2H = 0$, see (20), give the following quadratic equation with respect to $k$:

$$\alpha_{11}k^2 - 2\alpha_{12}k + \alpha_{22} = 0 \quad \left(k = -\frac{a_i - a_{i+1}}{b_i - b_{i+1}}\right)$$

$$k_1 = \frac{\alpha_{12} - \sqrt{\alpha_{12}^2 - \alpha_{11}\alpha_{22}}}{\alpha_{11}},$$

$$k_2 = \frac{\alpha_{12} + \sqrt{\alpha_{12}^2 - \alpha_{11}\alpha_{22}}}{\alpha_{11}}$$

The entries of the matrix $A$ should be taken at the origin: $\alpha_{ij} = \alpha_{ij}(0, 0, c)$. As soon as the values $k_1 > k_2$ are known, the coefficients $a_2, b_2$ can be found from the equations:

$$k_1 = -\frac{a_2 - a_1}{b_2 - b_1}, \quad k_2 = -\frac{a_3 - a_2}{b_3 - b_2} \quad (27)$$

One has:

$$a_2 = \frac{k_1 a_3 - k_2 a_1}{k_1 - k_2} + \frac{k_1 k_2}{k_1 - k_2}(b_3 - b_1)$$

$$b_2 = \frac{a_1 - a_3}{k_1 - k_2} + \frac{k_1 b_1 + k_2 b_3}{k_1 - k_2}$$

Substituting the entries of the matrix (23) into the above formulas, one can get the following expressions:

$$k_1 = \frac{1}{\sqrt{\alpha_0}}, \quad k_2 = -k_1 = -\frac{1}{\sqrt{\alpha_0}} \quad (\alpha_0 = \alpha(c))$$

$$a_2 = \frac{a_3 + a_1}{2} - \frac{1}{2\sqrt{\alpha_0}}(b_3 - b_1) \quad (28)$$

$$b_2 = \frac{b_3 + b_1}{2} - \frac{\sqrt{\alpha_0}}{2}(a_3 - a_1)$$

### 4.3 *Equations of singular characteristics*

For definiteness we will consider in the sequel one of two shock waves $\Gamma$ corresponding to $k_1$, and the sector $G^+$ (one of the three sectors) in the half-plane $y \geq 0$, bounded by $\Gamma$ and by positive half-axis $y = 0$, $x \geq 0$. The smooth branch of the solution $u(x, y)$ restricted to that sector will be denoted by $v(x, y)$. The Hamiltonian (20) for the considered problem has the form:

$$H(x, y, u, p_1, p_2) = F(x, y, u, \xi_1, \xi_2)$$
$$= (1/2)(-\alpha(u)\xi_1^2 + \xi_2^2),$$
$$\xi_1 = p_1 - q_1(x, y), \quad \xi_2 = p_2 - q_2(x, y),$$

$$q_1 = \partial v/\partial x, \quad q_2 = \partial v/\partial y$$

Using the componentwise notations $\xi = \xi_1$, $\gamma = \xi_2$, one can write the following equations of singular characteristics, defining the curve $\Gamma$, in the form:

$$\dot{x} = -\alpha(u)\xi, \ \dot{y} = \gamma, \ \dot{u} = -q_1\alpha(u)\xi + q_2\gamma \quad (29)$$

$$\dot{\xi} = \frac{1}{2}q_1\alpha'(u)\xi^2, \quad \dot{\gamma} = \frac{1}{2}q_2\alpha'(u)\xi^2$$

The initial conditions for the system (29) on the base of (28) take the form:

$$x(0) = 0, \quad y(0) = 0, \quad u(0) = c \quad (30)$$

$$\xi(0) = \xi_0 = a_2 - a_1 = \frac{a_3 - a_1}{2} - \frac{1}{2\sqrt{\alpha_0}}(b_3 - b_1)$$

$$\gamma(0) = \gamma_0 = b_2 - b_1 = \frac{b_3 - b_1}{2} - \frac{\sqrt{\alpha_0}}{2}(a_3 - a_1)$$
$$= -\xi_0\sqrt{\alpha_0}$$

To integrate the system (29) one has to find in advance and substitute into the system the gradient

$$q_1(x, y) = \frac{\partial v}{\partial x}, \quad q_2(x, y) = \frac{\partial v}{\partial y}$$

of the smooth branch of the solution $v(x, y)$, defined in some neighborhood of the sector $G^+$.

### 4.4 *Asymptotics in the vicinity of the origin*

In the vicinity of the origin the curve $\Gamma$ is the graph of some function $y = g_1(x)$, whose Taylor expansion up to several first terms can be written as:

$$y = Y_1 x + Y_2\frac{x^2}{2} + Y_3\frac{x^3}{6} \quad (31)$$

By definition of the slope $k_1$ one has for the first coefficient in (31): $Y_1 = k_1 = 1/\sqrt{\alpha_0}$. The aim of this section is to find the second coefficient $Y_2$.

Consider the expansions up to the cubic terms for the boundary functions:

$$w(x) = c + a_1 x + A_1 x^2/2 + D_1 x^3/6 \quad (32)$$
$$\psi(x) = b_1 + B_1 x + E_1 x^2/2$$

for the primary solution $v(x, y)$:

$$v(x, y) = c + a_1 x + b_1 y + (A_1 x^2 + 2B_1 xy + C_1 y^2)/2 \quad (33)$$
$$+ (D_1 x^3 + 3E_1 x^2 y + 3F_1 xy^2 + G_1 y^3)/6$$

and for the function $\alpha(u) = \alpha(v)$ (since on $\Gamma$ one has $u = v$):

$$\alpha(v) = \alpha_0 + \alpha_1(v - c) + \alpha_2\frac{(v - c)^2}{2} + \alpha_3\frac{(v - c)^3}{6} \quad (34)$$

The same parameters $c, a_1, b_1, A_1, B_1, D_1, E_1$ are used in different expansions here to meet the boundary conditions. The remaining coefficients $C_1, F_1, G_1$ one can find by substitution of the series (32) – (34) into the Euler equation (24).

Substitute into the system (29) the following expansions using for $u$ the value $u = v$:

$$x = x_1 t + x_2 \frac{t^2}{2} + x_3 \frac{t^3}{6}, y = y_1 t + y_2 \frac{t^2}{2} + y_3 \frac{t^3}{6} \quad (35)$$

$$\xi = \xi_0 + \xi_1 t + \xi_2 \frac{t^2}{2} + \xi_3 \frac{t^3}{6}, \gamma = \gamma_0 + \gamma_1 t + \gamma_2 \frac{t^2}{2} + \gamma_3 \frac{t^3}{6}$$

Since the first two equalities in (35) give the parametric representation of the curve (31), one has:

$$Y_1 = \frac{y_1}{x_1}, \quad Y_2 = \frac{x_1 y_2 - x_2 y_1}{x_1^3}, \quad (36)$$

$$Y_3 = \frac{x_1 y_3 - x_3 y_1}{x_1^4} - 3Y_2 \frac{x_2}{x_1^2}$$

Using the expansions (32) – (35) in the system (29) one can find the following expressions for the coefficients of the expansions (35):

$$x_1 = -\xi_0 \alpha_0, \quad x_2 = -\alpha_0 \xi_1 - \alpha_1 \xi_0 (a_1 x_1 + b_1 y_1)$$

$$y_1 = \gamma_0, \ y_2 = \gamma_1, \ \xi_1 = \frac{1}{2} a_1 \alpha_1 \xi_0^2, \ \gamma_1 = \frac{1}{2} b_1 \alpha_1 \xi_0^2$$

in therms of which the formula for $Y_2$ in (36) takes the form:

$$Y_2 = -\frac{1}{2} \frac{\alpha_1 (b_1 + a_1 \sqrt{\alpha_0})}{\alpha_0^2} \quad (37)$$

Depending upon the sign of $Y_2$ the curve $\Gamma$ may be either convex or concave in the vicinity of the origin. A global analysis of this curve requires numerical computations.

### 4.5 *Bifurcation and smoothening of weak waves*

Two weak waves may intersect at some point P. Asymptotic analysis in the vicinity of P is similar to that of the origin. Let the functions $y = g_i(x)$ for $i = 1, 3$ represent the right wave and for $i = 2, 4$ the left wave passing through P. One can find the following expressions for the second derivatives of these functions ($A = \alpha_1 / 2\alpha_0^2$):

$$Y_{21} = -A(b_1 + a_1 \sqrt{\alpha_0}), \quad Y_{23} = -A(b_3 + a_3 \sqrt{\alpha_0})$$

$$Y_{22} = -A(b_3 - a_3 \sqrt{\alpha_0}), \quad Y_{24} = -A(b_1 - a_1 \sqrt{\alpha_0})$$

This follows from (28) and a symmetry of the equation (24) with respect to $y \to -y$. Thus second derivative of a wave jumps at a point of wave intersection. These expressions and the formula (28) show that if through a point P (possibly initial) passes only one wave (say, left), then the second derivative of that wave must be continuous:

$$b_3 - a_3 \sqrt{\alpha_0} = b_1 - a_1 \sqrt{\alpha_0}$$

The second derivative remains continuous when the second (right) wave has zero jump of the gradient at the point P. In that case the formula (27) for $k_1 = \dot{y} / \dot{x}$ has an uncertainty removable by the L'Hospital rule:

$$-\sqrt{\alpha} = \lim \frac{\gamma}{\xi} = \lim \frac{\dot{\gamma}}{\dot{\xi}} = \frac{q_2}{q_1}$$

Taking $a_1 = q_1$, $b_1 = q_2$ one gets:

$$b_1 + a_1 \sqrt{\alpha} = 0$$

which means that the right wave, generally, has an inflection point.

Using the results of this subsection, numerical computations were carried out by the author and V.A.Korneev. The bifurcation of the weak waves was constructed. Using appropriate initial conditions the situations were found when only one wave runs generated by an initial singularity.

## REFERENCES

Courant, R. (1962). *Partial Differential Equations.* InterScience, New York, London.

Giaquinta, M., and Hildebrandt, S. (1996). *Calculus of Variations I, II.* Springer, Berlin.

Arnold, V.I. (1988). *Geometrical Methods in the Theory of Ordinary Differential Equations.* Springer, New York.

Subbotin, A.I. (1995). *Generalized Solutions of First Order PDEs: the Dynamical Optimization Perspective*. Birkhäuser, Boston.

Isidori, A. (1996). *Nonlinear Control Systems.* Springer, London, New York.

Bardi, M. and Dolcetta, I.C. (1997). *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*. Birkhäuser, Boston.

Melikyan A.A. (1998). *Generalized Characteristics of First Order PDEs: Applications in Optimal Control and Differential Games.* Birkhäuser, Boston.