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THE H_∞ CONTROL PROBLEM FOR NEUTRAL SYSTEMS WITH MULTIPLE DELAYS

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Abstract: This paper presents the H_∞ control problem for linear neutral systems with unknown constant multiple delays, in delay independent case. A sufficient condition for the existence of an H_∞ controller of any order is given in terms of three linear matrix inequalities, when the coefficient D_{12} of the input in the controlled output is zero.

Keywords: Neutral systems, output feedback, H_∞-control.

1. INTRODUCTION

In this paper we consider the H_∞ control problem for linear neutral systems with unknown constant multiple delays in delay independent case. H_∞ control problem is defined as finding a controller such that the H_∞-norm of the closed-loop transfer function is strictly less than an arbitrarily given real $\gamma > 0$. This problem is examined mainly by two approaches: the algebraic Riccati equations (AREs) and the linear matrix inequalities (LMIs). In the literature, various related works for linear systems have been reported, see (e.g. Zhou and Khagonekar (1988); Doyle et. al. (1989), for ARE and Iwasaki and Skelton (1994); Gahinet and Apkarian (1994), for LMI). H_∞ control problem for systems with time-delay has rarely been considered. Recently, the state feedback H_∞-control problem, for linear neutral systems is examined in Mahmoud (2000a,b). The output feedback H_∞ controller design for linear time-delay systems by LMI approach is also achieved in Choi and Chung (1997). But, at the knowledge of the author no paper treats output feedback H_∞-control problem for linear neutral systems.

Consider the n^{th} order linear time-invariant generalized neutral systems Σ described by the following equation:

$$\dot{x}(t) - E\dot{x}(t-\tau) = Ax(t) + \qquad (1)$$

$$\sum_{i=1}^{k} A_{d_i} x(t-d_i) + B_1 w(t) + B_2 u(t)$$

$$z(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t) \qquad (2)$$

$$y(t) = C_2 x(t) + D_{21} w(t) + D_{22} u(t) \qquad (3)$$

$$x(t_0+\theta) = \phi(\theta), \quad \forall \theta \in [-\max(\tau, d_i), 0], \qquad (4)$$

where $i \in \{1, 2, ..., k\}$, $x \in \mathbf{R}^n$ is the plant state, $w \in \mathbf{R}^q$ is any exogenous input, including plant disturbances, measurement noise, etc., $u \in \mathbf{R}^m$ is the control input, $z \in \mathbf{R}^p$ is the regulated output and $y \in \mathbf{R}^k$ is the measured output, A, A_{d_i}, B_1, B_2 C_1, C_2 and D_{ij}, for $i, j = 1, 2$ are known real constant matrices of the apropriate dimensions. $\tau > 0$ and all $d_i > 0$'s are unknown constant delays. $\phi \in \mathcal{C}_{\tau,n}$, where $\mathcal{C}_{\tau,n} = \mathcal{C}([-\tau, 0], \mathbf{R}^n)$ be the space of continuous functions taking $[-\tau, 0]$ into $\mathbf{R}^n$. It is assumed that $D_{22} = 0$. It should be noted that this assumption involve no loss of generality, while considerably simplifying algebraic manipulations, (Gahinet and Apkarian, 1994; Iwasaki and Skelton 1994). We assume also that

Assumption 1.1. The triple (A, B_2, C_2) is stabilizable and detectable.

Assumption 1.2. $\lambda \mid E \mid < 1$.

We remark that Σ is a continuous-time model for which Assumption 1 is quite standard. However, Assumption 2 gives a condition in the discrete-time sense and its role will be clarified in the subsequent analysis.

Consider the ${n_c}^{th}$ order linear time-invariant dynamic $(n_c > 0)$ and static $(n_c = 0)$ controllers

$$\dot{x}_c(t) = K_{21}y(t) + K_{22}x_c(t) \quad (5)$$
$$u(t) = K_{11}y(t) + K_{12}x_c(t) \quad (6)$$

where $x_c \in \mathbf{R}^{n_c}$ is the controller state, K_{11}, K_{12}, K_{21} and K_{22} have appropriate dimensions. We shall denote the class of controllers by Σ_c.

Let $x_e(t) = [x^T(t) \;\; x_c^T(t)]^T$. Then, the closed-loop system, Σ_{cl} is the following;

$$\dot{x}_e(t) - \bar{E}F\dot{x}_e(t-\tau) = \quad (7)$$

$$\bar{A}x_e(t) + \sum_{i=1}^{k} \bar{A}_{d_i}Fx_e(t-d_i) + \bar{B}w(t)$$

$$z(t) = \bar{C}x_e(t) + \bar{D}w(t) \quad (8)$$

where

$$\bar{A} = \hat{A} + \hat{B}_2K\hat{C}_2,\ \bar{B} = \hat{B}_1 + \hat{B}_2K\hat{D}_{21}, \quad (9)$$
$$\bar{C} = \hat{C}_1 + \hat{D}_{12}K\hat{C}_2, \bar{D} = D_{11} + \hat{D}_{12}K\hat{D}_{21}$$

$$F^T = \begin{bmatrix} I \\ 0 \end{bmatrix}, \bar{E} = \begin{bmatrix} E \\ 0 \end{bmatrix}, \hat{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix},$$

$$\bar{A}_{d_i} = \begin{bmatrix} A_{d_i} \\ 0 \end{bmatrix}, \hat{B}_1 = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \hat{B}_2 = \begin{bmatrix} B_2 & 0 \\ 0 & I \end{bmatrix},$$

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}, \hat{C}_2 = \begin{bmatrix} C_2 & 0 \\ 0 & I \end{bmatrix}, \hat{D}_{21} = \begin{bmatrix} D_{21} \\ 0 \end{bmatrix},$$

$$\hat{C}_1 = \begin{bmatrix} C_1 & 0 \end{bmatrix}, \hat{D}_{12} = \begin{bmatrix} D_{12} & 0 \end{bmatrix} \quad (10)$$

The closed-loop transfer matrix $T_{zw}(s)$ from w to z is given by

$$T_{zw}(s) = \bar{D} + \quad (11)$$

$$\bar{C}\left[s(I - \bar{E}Fe^{-s\tau}) - \bar{A} - \sum_{i=1}^{k} \bar{A}_{d_i}Fe^{-sd_i} \right]^{-1} \bar{B}$$

Definition 1.3. Given a scalar $\gamma > 0$. The controller Σ_c is said to be an H_∞-controller if the following two conditions hold:

(i) $\bar{A}$ is asymptotically stable,
(ii) $\parallel T_{zw} \parallel_\infty < \gamma$.

Lemma 1.4. (Schur complement). Given constant matrices Ω_1, Ω_2 and Ω_3 where $0 < \Omega_1 = \Omega_1^T$ and $0 < \Omega_2 = \Omega_2^T$ then $\Omega_1 + \Omega_3^T\Omega_2^{-1}\Omega_3 < 0$ if and only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^T \\ \Omega_3 & -\Omega_2 \end{bmatrix} < 0.$$

Lemma 1.5. Given a symmetric matrix Ω and two matrices Γ and Σ with appropriate dimensions. The inequality

$$\Omega + \Sigma K\Gamma + (\Sigma K\Gamma)^T < 0 \quad (12)$$

is solvable for K if and only if

$$\bar{\Gamma}^T\Omega\bar{\Gamma} < 0, \quad \bar{\Sigma}\Omega\bar{\Sigma}^T < 0 \quad (13)$$

where $\tilde{\Gamma}$ and $\tilde{\Sigma}$ denote the orthogonal complements of Γ and Σ, respectively.

Proof 1.6. See Gahinet and Apkarian (1994) and Iwasaki and Skelton (1994).

2. THE MAIN RESULTS

Define

$$W := \bar{A}^TP + P\bar{A} + \bar{Q} + \sum_{i=1}^{k} \bar{S}_i + \bar{C}^T\bar{C} \quad (14)$$
$$+ (P\bar{B} + \bar{C}^T\bar{D})\Phi^{-1}(P\bar{B} + \bar{C}^T\bar{D})^T$$
$$+ \Psi\bar{E}R^{-1}\bar{E}^T\Psi^T + \sum_{i=1}^{k} P\bar{A}_{d_i}S_i^{-1}\bar{A}_{d_i}^TP$$

$$\Phi := \gamma^2I - \bar{D}^T\bar{D} \quad (15)$$

$$R := Q - \bar{E}^T(\bar{C}^T\bar{C} + \bar{Q} + \sum_{i=1}^{k} \bar{S}_i + \quad (16)$$
$$\bar{C}^T\bar{D}\Phi^{-1}\bar{D}^T\bar{C})\bar{E},$$

$$\Psi := P\bar{A} + \bar{Q} + \sum_{i=1}^{k} \bar{S}_i + \bar{C}^T\bar{C} \quad (17)$$
$$+ (P\bar{B} + \bar{C}^T\bar{D})\Phi^{-1}\bar{D}^T\bar{C}$$

where $\bar{S}_i = F^TS_iF$ and $\bar{Q} = F^TQF$.

Theorem 2.1. Subject to Assumptions 1 and 2 the closed-loop neutral systems Σ_{cl} with multiple delay is asymptotically stable independent of delay and the H_∞ performance bound constraint $\parallel T_{zw} \parallel_\infty < \gamma$ holds for a given $\gamma > 0$, if there exist matrices $0 < P^T = P$,$0 < Q^T = Q$ and $0 < {S_i}^T = S_i$, for $i = 1, 2, ..., k$ satisfying

$$W < 0$$

while

$$\Phi > 0,\ R > 0$$

Proof 2.2. Let a Lyapunov- Krasovskii functional $V(x_t)$ of the form

$$V(x_t) = [x_e(t) - \bar{E}Fx_e(t-\tau)]^T P [x_e(t) - \bar{E}Fx_e(t-\tau)] + \int_{-\tau}^{0} x_e^T(t+\theta)\bar{Q}x_e(t+\theta)d\theta + \sum_{i=1}^{k}\int_{-d_i}^{0} x_e^T(t+\theta)\bar{S}_i x_e(t+\theta)d\theta \quad (18)$$

Observe that $V(x_t)$ satisfies

$\lambda_m(P)r^2 \leq V(r) \leq [\lambda_M(P)+\tau^\star\lambda_M(\bar{Q},\bar{S}_1,,,\bar{S}_k)]r^2$

for some r, where $\tau^\star = max(\tau, d_1, ..., d_k)$. In order to show that the closed-loop system (7) is asymptotically stable with disturbance attenuation γ, it is required that the associated Hamiltonian $H(x_t, w, t)$ satisfies

$H(x_t, w, t) = \dot{V}(x_t)+z^T(t)z(t)-\gamma^2 w^T(t)w(t) < 0,$

where $V(x_t)$ is given by (18), Zhou (1998). By differentiating (18) along the trajectories x_t and using the difference operator $\mathcal{M}(x_t) := x_e(t) - \bar{E}Fx_e(t-\tau)$ the result follows.

Remark 2.3. The Lyapunov- Krasovskii functional $V(x_t)$ in (18) is of the form given in Verriest and Niculescu (1997), except that the term with $\bar{Q}$. If we removed this term we would derive the condition $R := -\bar{E}^T(\bar{C}^T\bar{C} + \sum_{i=1}^{k}\bar{S}_i + \bar{C}^T\bar{D}\Phi^{-1}\bar{D}^T\bar{C})\bar{E} > 0$. It is clear that this inequality is not solvable.

Now, let

$$V := \bar{A}^T P + P\bar{A} + \bar{Q} + \sum_{i=1}^{k}\bar{S}_i + \frac{1}{\gamma}\bar{C}^T\bar{C} + \gamma(P\bar{B} + \frac{1}{\gamma}\bar{C}^T\bar{D})\Phi^{-1}(\bar{B}^T P + \frac{1}{\gamma}\bar{D}^T\bar{C}) + \sum_{i=1}^{k} P\bar{A}_{d_i}S_i^{-1}\bar{A}_{d_i}^T P + \Psi\bar{E}R^{-1}\bar{E}^T\Psi^T \; < \; 0 \quad (19)$$

W is equivalent to V, where $\gamma = 1$, $i = 1$ and $d_1 = \tau$.

Theorem 2.4. Subject to Assumptions 1 and 2 the closed-loop neutral systems Σ_{cl} with multiple delay is asymptotically stable independent of delay and the H_∞ performance bound constraint $\| T_{zw} \|_\infty < \gamma$ holds for a given $\gamma > 0$, if there exist matrices $0 < P^T = P$,$0 < Q^T = Q$ and $0 < {S_i}^T = S_i$, for $i = 1, 2, ..., k$ satisfying

$$V < 0$$

while

$$\Phi > 0, \; R > 0$$

Proof 2.5. The proof is omitted.

3. H_∞-CONTROLLER DESIGN

Now, we will concentrate on the H_∞-controller design. For this aim, first consider the following LMI:

$$\begin{bmatrix} \bar{\Theta} & P\bar{B} & \bar{C}^T & \Psi\bar{E} & P\bar{A}_d \\ \bar{B}^T P & -\gamma I & \bar{D}^T & 0 & 0 \\ \bar{C} & \bar{D} & -\gamma I & 0 & 0 \\ \bar{E}^T\Psi^T & 0 & 0 & -R & 0 \\ \bar{A}_d^T P & 0 & 0 & 0 & -\Delta_s \end{bmatrix} < 0, \quad (20)$$

where $\bar{\Theta} := \bar{A}^T P + P\bar{A} + \bar{Q} + \sum_{i=1}^{k}\bar{S}_i$, $\bar{A}_d := [\,\bar{A}_{d_1}\,\bar{A}_{d_2}\,....\bar{A}_{d_k}]$ and $\Delta_s := \text{diag}\,\{S_1\,,\; S_2,\;\ldots,\; S_k\}$.

In terms of lemma 1.4, it can be shown that the LMI in (20) is equivalent to the inequality $V \; < \; 0$.

Now, let $D_{12} = 0$. By using the expressions (9), (10) we can rewrite (20) as follows:

$$\Omega + \Sigma K\Gamma + (\Sigma K\Gamma)^T < 0 \quad (21)$$

where

$$\Omega := \begin{bmatrix} \hat{\Theta} & P\hat{B}_1 & \hat{C}_1^T & \hat{\Psi}\bar{E} & P\bar{A}_d \\ \hat{B}_1^T P & -\gamma I & D_{11}^T & 0 & 0 \\ \hat{C}_1 & D_{11} & -\gamma I & 0 & 0 \\ \bar{E}^T\hat{\Psi}^T & 0 & 0 & -R & 0 \\ \bar{A}_d^T P & 0 & 0 & 0 & -\Delta_s \end{bmatrix} \quad (22)$$

$$\hat{\Theta} := \hat{A}^T P + P\hat{A} + \bar{Q} + \sum_{i=1}^{k}\bar{S}_i$$

$$\Sigma^T := \begin{bmatrix} \hat{B}_2^T P & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Gamma := \begin{bmatrix} \hat{C}_2 & \hat{D}_{21} & 0 & (\hat{C}_2 + \hat{D}_{21}\Phi^{-1}D_{11}^T\hat{C}_1^T)\bar{E} & 0 \end{bmatrix}$$

and

$\hat{\Psi} := P\hat{A}+\bar{Q}+\sum_{i=1}^{k}\bar{S}_i+\hat{C}_1^T\hat{C}_1+(P\hat{B}_1+\hat{C}_1^T D_{11})\Phi^{-1}D_{11}^T\hat{C}_1$

By lemma (1.5), the inequality (21) is equivalent to (13).

Now, let us partition P and P^{-1} as

$$P =: \begin{bmatrix} Y & M \\ M^T & * \end{bmatrix} \quad , \quad P^{-1} =: \begin{bmatrix} X & N \\ N^T & * \end{bmatrix} \quad (23)$$

where Y and X are the $n \times n$ positive matrices. Define Ω_Y and Ω_X as follows:

$$\Omega_Y = \begin{bmatrix} \Theta_Y & YB_1 & C_1^T & \Psi_Y E & YA_d \\ B_1^T Y & -\gamma I & D_{11}^T & 0 & 0 \\ C_1 & D_{11} & -\gamma I & 0 & 0 \\ E^T\Psi_Y^T & 0 & 0 & -R & 0 \\ A_d^T Y & 0 & 0 & 0 & -\Delta_s \end{bmatrix} \quad (24)$$

$$\Omega_X = \quad (25)$$

$$\begin{bmatrix} \Theta_X & B_1 & XC_1^T & \Psi_X E & A_d & X_{sq} \\ B_1^T & -\gamma I & D_{11}^T & 0 & 0 & 0 \\ C_1 X & D_{11} & -\gamma I & 0 & 0 & 0 \\ E^T \Psi_X & 0 & 0 & -R & 0 & \\ A_d^T & 0 & 0 & 0 & -\Delta_s & 0 \\ X_{sq}^T & 0 & 0 & 0 & 0 & \Delta_{sq}^{-1} \end{bmatrix}$$

where $\Theta_Y := A^T Y + YA + Q + \sum_{i=1}^k S_i$, $\Theta_X := XA^T + AX$, $\Psi_Y := YA + Q + \sum_{i=1}^k S_i + {C_1}^T C_1 + (YB_1 + C_1^T D_{11})\Phi^{-1} D_{11}^T C_1$, $\Psi_X := A + B_1 \Phi^{-1} D_{11}^T C_1 + X(Q + \sum_{i=1}^k S_i + {C_1}^T C_1 + C_1^T D_{11} \Phi^{-1} D_{11}^T C_1)$, $A_d := [\ A_{d_1}\ A_{d_2}, ., A_{d_k}]$, $X_{sq} := [\ X, ., X\ ]$ and $\Delta_{sq}^{-1} := \mathrm{diag}\,(Q^{-1}, S_1^{-1}, ., S_k^{-1})$.

Along similar lines to Gahinet and Apkarian (1994), The inequality (21) is equivalent to

$$\tilde{\Gamma}\Omega_Y \tilde{\Gamma}^T < 0, \quad \tilde{\Sigma}^T \Omega_X \tilde{\Sigma} < 0 \tag{26}$$

and

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \geq 0. \tag{27}$$

where $\tilde{\Gamma} := \begin{bmatrix} V_1^T & V_2^T & 0 & V_3^T & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}$, $\quad \tilde{\Sigma} := \begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix}$.
$\begin{bmatrix} V_1^T & V_2^T & V_3^T \end{bmatrix}^T$ and W denote any basis of the null spaces of
$\begin{bmatrix} C_2 & D_{21} & (C_2 + D_{21}\Phi^{-1} D_{11}^T C_1)E \end{bmatrix}$ and B_2^T, respectively.

Remark 3.1. In summary, we can say that there exist a positive definite matrix P and a control gain matrix K, satisfying (20) if and only if there exist symmetric matrices X and Y satisfying (26) and (27). So, the solution depends on the existence of X and Y. Moreover, if rank $(I - XY) = k < n$ for solution matrices X and Y then there exist a reduced order H_∞-controller of order k.

In order to construct an H_∞- controller, we first compute some solution (X, Y) of the LMI's (26) and (27) by using a convex optimization algorithm for some γ and the positive matrices Q, R, S_i's. As it is noted in Choi and Chung (1997) that If $k = \mathrm{rank}\,(I - XY) = 0$ then we set $P = Y$. Otherwise, using the matrices M and N wich are of full column rank such that $MN^T = I - XY$, we obtain the unique solution P to the equation

$$\begin{bmatrix} Y & I \\ N^T & 0 \end{bmatrix} = P \begin{bmatrix} I & X \\ 0 & M^T \end{bmatrix}. \tag{28}$$

An explicit description of all solutions of LMI in (21) can be given as follows in state space:

$$K = -\rho \Sigma^T \Xi \Gamma^T (\Gamma \Xi \Gamma^T)^{-1} + U^{\frac{1}{2}} L (\Gamma \Xi \Gamma^T)^{-\frac{1}{2}}$$

where ρ and L are free parameters subject to

$$\Xi := (\Sigma\Sigma^T - \frac{1}{\rho}\Omega)^{-1} > 0, \quad \|L\| \leq \rho$$

and the matrix U is defined by

$$U := I - \Sigma^T [\Xi - \Xi\Gamma^T (\Gamma\Xi\Gamma^T)^{-1} \Gamma\Xi]\Sigma.$$

4. CONCLUSIONS

The problem of designing output feedback H_∞ controllers for linear neutral systems with multiple time-delay has ben considered in delay independent case based on the linear matrix inequality (LMI) approach. A necessary and sufficient condition for the existence of H_∞ controllers of any order is given in terms of three LMIs , when the coefficient D_{12} of the input in the controlled output is zero. Output feedback H_∞-control problem for the same systems in delay dependent case is the subject of further research.

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