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ROBUST H_2-CONTROLLER DESIGN UNDER STRUCTURED NOISE-UNCERTAINTY

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Abstract: We consider the problem of minimizing the worst asymptotic output variance of a control system that is affected by a structurally constrained class of noise. We discuss how to impose direct size-restrictions on the noise's covariance matrices or multiple lower and upper bounds on its spectral density over various frequency intervals. Technically all these scenarios are translated into parametric-dynamic optimization problems which have been recently shown to admit efficient numerical solutions in terms of linear matrix inequalities.

Keywords: Stochastic systems, uncertain dynamic systems, robust control, output feedback, convex optimization.

1. INTRODUCTION

Let us consider a system that is described as

$$x_{t+1} = Ax_t + Bw_t, \quad z_t = Cx_t + Dw_t \qquad (1)$$

where A has all its eigenvalues in the open unit disk and where the disturbance w_t is a vector-valued wide-sense stationary zero-mean stochastic process with d components. We assume that the first $m+1$ of all covariance matrices

$$R_k = E[w_t w_{t+k}^T], \quad k = 0, 1, ..., m$$

are uncertain whereas all higher-lag covariance matrices vanish. The goal in this paper is to derive a computationally verifiable criterion which guarantees that the asymptotic output variance $\lim_{t\to\infty} E[z_t^T z_t]$ is bounded by some number γ, uniformly for all possible noise processes acting on the system. Moreover, we aim at a characterization which is amenable to efficient controller design algorithms. We also point out various important extensions and relations to recently suggested optimization techniques over positive polynomials (Genin *et al.*, 2000*a*; Genin *et al.*, 2000*b*; Alkire and Vandenberghe, 2001). The hypothesis on the noise input can be viewed as the opposite extreme of the case considered in (Gusev, 1995*a*; Gusev, 1995*b*; Gusev, 1996; Pransath and Nagpal, 2000; Scherer, 2000*b*) where it was assumed that the coefficients R_k for $j = m+1, m+2, ...$ are not constrained. Clearly our scenario can be viewed as a generalization of the H_2-criterion which corresponds to $R_0 = I$, $R_k = 0$ for $k \geq 1$, whereas the problem with unconstrained tail is rather related to the H_∞-case which corresponds to noise characterized as $\mathrm{Tr}(R_0) \leq 1$, R_k free for $k \geq 1$.

The notation is standard. We stress that a transfer matrix $T(z)$ is called (strictly) positive real if all its poles are contained in the open unit disk, and if $T(z) + T(z)^* \geq 0$ (> 0) for all $|z| = 1$. For any matrix tuple $M = (M_0, M_1, \ldots, M_m)$ we use the abbreviations

$$\mathcal{T}(M) = \begin{pmatrix} M_0 & M_1^T & \cdots & M_m^T \\ M_1 & M_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & M_1^T \\ M_m & \cdots & M_1 & M_0 \end{pmatrix}$$

and

$$\mathcal{F}(M,z) = \frac{1}{2}M_0 + \frac{1}{z}M_1 + \cdots \frac{1}{z^m}M_m.$$

2. ANALYSIS PROBLEM

Let us assume that the restrictions on the first $m+1$ covariance coefficients of the disturbance w_t are modeled as

$$\big(E[w_t w_t^T], ..., E[w_t w_{t+m}^T]\big) \in \mathcal{R} \tag{2}$$

where $\mathcal{R}$ denotes some subset of $(\mathbb{R}^{d\times d})^{m+1}$. We only need to consider those elements in $R \in \mathcal{R}$ which do actually correspond to the Toeplitz matrix of covariance coefficients of a wss noise process. This is guaranteed if the transfer matrix $\mathcal{F}(R)$ is positive real which is supposed to be true for all $R \in \mathcal{R}$. As two technical properties we assume that

$$\exists R \in \mathcal{R}: \ \mathcal{F}(R) \ \text{ is strictly positive real} \tag{3}$$

and

$$\{R_0 : \ (R_0, R_1, ..., R_m) \in \mathcal{R}\} \ \text{ is bounded.} \tag{4}$$

At this point there is no need to make special hypothesis on the description of further restrictions imposed through $\mathcal{R}$, but for computational purposes this collection will be assumed to be represented by the solution set of a linear matrix inequality (LMI).

The class of disturbances satisfying (2) together with $E[w_t w_{t+k}^T] = 0$ for $k > m$ is denoted as $W_{\mathcal{R}}$. The goal of this section is to obtain a computationally efficient algorithm which allows to compute the worst asymptotic output variance

$$\sup_{w\in W_{\mathcal{R}}} \lim_{t\to\infty} E[z_t^T z_t]. \tag{5}$$

Let us assume that $x_0 = 0$. With the impulse response sequence $T_0 = D$ and $T_\nu = CA^\nu B$, $\nu = 1, 2, \ldots$ one easily shows

$$z_t = \begin{pmatrix} T_0 & \cdots & T_t \end{pmatrix} \begin{pmatrix} w_t \\ \vdots \\ w_0 \end{pmatrix}$$

which implies

$$E[z_t z_t^T] = \begin{pmatrix} T_0 & \cdots & T_t \end{pmatrix} \begin{pmatrix} R_0 & \cdots & R_t^T \\ \vdots & \ddots & \vdots \\ R_t & \cdots & R_0 \end{pmatrix} \begin{pmatrix} T_0^T \\ \vdots \\ T_t^T \end{pmatrix}.$$

Since T_t converges exponentially to zero for $t \to \infty$ and since the sequence R_t is bounded, it follows from this formula that the limit of $E[z_t z_t^T]$ exists and that it equals

$$\sum_{\tau=0}^{\infty} T_\tau R_0 T_\tau^T + \sum_{\nu=1}^{\infty}\sum_{\tau=0}^{\infty} \left(T_{\tau+\nu} R_\nu T_\tau^T + T_\tau R_\nu^T T_{\tau+\nu}^T\right).$$

For noise with $R_\nu = 0$, $\nu > m$, and with the abbreviation

$$H_\nu = \sum_{\tau=0}^{\infty} T_{\tau+\nu}^T T_\tau, \quad \nu \geq 0$$

we infer that

$$\lim_{t\to\infty} E[z_t^T z_t] = \mathrm{Tr}(R_0 H_0) + 2\sum_{\nu=1}^{m} \mathrm{Tr}(R_\nu H_\nu^T). \tag{6}$$

To proceed with our reasoning it is convenient to rewrite this as

$$\lim_{t\to\infty} E[z_t^T z_t] = \langle \mathcal{T}(M \circ R), \mathcal{T}(H)\rangle$$

where $H = (H_0, ..., H_m)$, $M = (\frac{1}{m+1}I, \frac{1}{m}I, ..., I)$ (with I of size d), $A \circ B$ denotes the Schur product (element-wise product) of A and B and $\langle A, B\rangle$ is the standard matrix inner product $\mathrm{Tr}(A^T B)$.

We conclude that the worst output variance is given by

$$\sup_{R\in\mathcal{R}} \langle \mathcal{T}(M \circ R), \mathcal{T}(H)\rangle. \tag{7}$$

The rather explicit formula (7) for the worst-case output variance has two major disadvantages, in particular for the problem of controller synthesis. First it is formulated in terms of the impulse response coefficients of the underlying system (1) and, second, $\langle \mathcal{T}(M \circ R), \mathcal{T}(H)\rangle$ is only bilinear as a function of both R and H together. It will be of crucial importance to have a re-formulation in which the noise's covariance matrices and the system-related impulse response sequence H enter affinely. This is achieved by observing that (7) equals

$$\inf_{S \text{ with } \mathcal{T}(H) < \mathcal{T}(S)} \ \sup_{R\in\mathcal{R}} \langle \mathcal{T}(M \circ R), \mathcal{T}(S)\rangle. \tag{8}$$

The proof relies on the following lemma which is a consequence of Caratheodory's well-known extension theorem and spectral factorization.

Lemma 1. Let $R = (R_0, ..., R_m)$ be the covariance matrices of a wss stochastic process and define M as above. If H satisfies $\mathcal{T}(H) > 0$ then

$$\langle \mathcal{T}(M \circ R), \mathcal{T}(H)\rangle \geq 0.$$

Remark. Note that $\mathcal{T}(R)$ is positive semi-definite and $\mathcal{T}(M)$ is indefinite. Hence $\mathcal{T}(M \circ R) = \mathcal{T}(M) \circ \mathcal{T}(R)$ cannot be expected to be semi-definite in general. However, since $\mathcal{F}(R)$ is positive real, $\mathcal{T}(R)$ actually satisfies stronger hypotheses which might counteract the effect of the element-wise multiplication with the fixed matrix $\mathcal{T}(M)$. Nevertheless, it is pretty simple to numerically construct examples for which $\mathcal{T}(M) \circ \mathcal{T}(R)$ is indeed not positive semi-definite. This implies, by Féjer's theorem on the dual of the positive semi-definite cone, that $\langle \mathcal{T}(M \circ R), A\rangle$ will not be nonnegative for arbitrary $A \geq 0$ without any structure which highlights the non-triviality of Lemma 1.

Proof. Let us choose any H with $\mathcal{T}(H) > 0$. This implies that $H_0, ..., H_m$ can be extended with $H_{m+1}, H_{m+2}, ...$ such that

$$f(z) = \frac{1}{2}H_0 + \frac{1}{z}H_1 + \cdots$$

is a proper rational function which has all its poles in the unit disk and which is positive real. (This result is classical. The existence of a *rational* extension can be easily proved using LMI-based positive real design arguments.) Therefore there exists a proper rational stable function g with proper and stable inverse such that

$$g(e^{i\tau})^* g(e^{i\tau}) = f(e^{i\tau})^* + f(e^{i\tau}) \quad \text{for} \quad \tau \in [0, 2\pi].$$

If we denote the impulse response coefficients of g by $T_0, T_1, \ldots$, we observe that they are related to H as in the above construction. This guarantees that $\langle \mathcal{T}(M \circ R), \mathcal{T}(H) \rangle$ just equals the asymptotic output variance of the filter defined by g if driven with a noise input w whose covariance coefficients are given by $R_0, ..., R_m$. We can thus conclude $\langle \mathcal{T}(M \circ R), \mathcal{T}(H) \rangle \geq 0$. ■

Lemma 2. The values (7) and (8) are identical.

Proof. Let S satisfy $\mathcal{T}(H) < \mathcal{T}(S)$. According to Lemma 1, $\langle \mathcal{T}(M \circ R), \mathcal{T}(S) \rangle \geq \langle \mathcal{T}(M \circ R), \mathcal{T}(H) \rangle$. This implies (8) $\geq$ (7). Conversely, let us choose S_ϵ with $\mathcal{T}(S_\epsilon) = \mathcal{T}(H) + \epsilon I$, $\epsilon > 0$. This implies

$$\sup_{R \in \mathcal{R}} \langle \mathcal{T}(M \circ R), \mathcal{T}(S_\epsilon) \rangle = \\ = \sup_{R \in \mathcal{R}} \langle \mathcal{T}(M \circ R), \mathcal{T}(H) \rangle + \epsilon \sup_{R \in \mathcal{R}} \mathrm{Tr}(R_0).$$

By hypothesis (4), the limit for $\epsilon \to 0$ thus equals $\sup_{R \in \mathcal{R}} \langle \mathcal{T}(M \circ R), \mathcal{T}(H) \rangle$ which proves that (8) and (7) actually coincide. ■

Let us finally overcome the problem that all the derived conditions are formulated in terms of the impulse response coefficients. This last step is instrumental for an elegant state-space synthesis procedure.

Lemma 3. The inequality $\mathcal{T}(S) > \mathcal{T}(H)$ holds iff there exists a proper rational stable $Q(z)$ such that

$$[\mathcal{F}(S, z) + \frac{1}{z^{m+1}} Q(z)] + [\mathcal{F}(S, z) + \frac{1}{z^{m+1}} Q(z)]^* - \\ - T(z)^* T(z) > 0 \quad \text{for all} \quad |z| = 1. \quad (9)$$

Proof. By comparing coefficients, (9) can be seen to be equivalent to strict positive realness of

$$\frac{1}{2}(S_0 - H_0) + \frac{1}{z}(S_1 - H_1) + \cdots + \frac{1}{z^m}(S_m - H_m) + \\ + \frac{1}{z^{m+1}}(Q_0 - H_{m+1}) + \frac{1}{z^{m+2}}(Q_1 - H_{m+2}) + \cdots$$

where Q_j denote the impulse response coefficients of $Q(z)$. Therefore the result follows from the rational version of the Caratheodory extension theorem. ■

We arrive at the following main analysis result which is an immediate consequence of Lemmas 2-3 and the equivalence of (9) and (11) (Schur).

Theorem 4. The asymptotic worst-case output variance (5) of system (1) is strictly smaller than γ if and only if there exists a strictly proper stable Q and some $S = (S_0, ..., S_m)$ with

$$\sup_{R \in \mathcal{R}} \langle \mathcal{T}(M \circ R), \mathcal{T}(S) \rangle < \gamma \quad (10)$$

that renders

$$\begin{pmatrix} \frac{1}{2}I & T(z) \\ 0 & \mathcal{F}(S, z) + \frac{1}{z^{m+1}} Q(z) \end{pmatrix} \quad (11)$$

strictly positive real.

3. CONTROLLER ANALYSIS AND SYNTHESIS ALGORITHMS

3.1 *Support Functionals*

Let us recall that

$$\langle \mathcal{T}(M \circ R), \mathcal{T}(S) \rangle = \\ = \mathrm{Tr}(R_0 S_0) + 2 \sum_{\nu=1}^{m} \mathrm{Tr}(R_\nu S_\nu^T).$$

Hence (10) means, by definition (see Section A), nothing but $(S_0, 2S_1, ..., 2S_m)$ belonging to the sub-level set of the support functional of $\mathcal{R}$ (where one identifies linear functionals with space elements as usual in Hilbert spaces).

Let us now sketch various typical practical constraints on the set $\mathcal{R}$ and how they translate into LMI constraints for the coefficients R_0,...,R_m. For any combination of such requirements it is routine, according to Appendix A, to derive the corresponding LMI constraints for the coefficients $S_0, ..., S_m$.

• First recall that each element R of $\mathcal{R}$ needs to render $\mathcal{F}(R)$ positive real. Since we actually require (3), it suffices to constrain the coefficients such that $\mathcal{F}(R)$ is strictly positive real (because including or neglecting accumulation points is irrelevant for computing support functionals). With the realization

$$\left[\begin{array}{c|c} A_p & B_p \\ \hline C_p(R) & D_p(R) \end{array}\right] = \left[\begin{array}{cccc|c} 0 & I & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & I & 0 \\ 0 & 0 & \cdots & 0 & I \\ \hline R_m & R_{m-1} & \cdots & R_1 & 0.5R_0 \end{array}\right]$$

we can apply the positive real lemma (since A_p is stable) to infer the equivalence of strict positive realness of $\mathcal{F}(R)$ with the existence of P such that

$$\begin{pmatrix} A_p^T P A_p - P & A_p^T P B_p + C_p(R)^T \\ B_p^T P A_p + C_p(R) & B_p^T P B_p + D_p(R) + D_p(R)^T \end{pmatrix} < 0.$$

For the direct implementation of this constraint it is instructive to consult (Alkire and Vandenberghe, 2001).

- One can model the fact that noise components are uncorrelated by imposing structural constraints on the coefficients. For some fixed R^0, this can be expressed as $X = R - R^0$ belonging to some suitably chosen subspace $\mathcal{X}$. It then simply suffices to perform the dualization as described in Appendix A with respect to this subspace $\mathcal{X}$.

- Let us assume that $\mathcal{R}$ is finitely generated as $\mathcal{R} = \text{convex hull}\{R^1, \ldots, R^N\}$. By linearity (10) is guaranteed iff $\langle \mathcal{T}(M \circ R_j), \mathcal{T}(S) \rangle < \gamma$ for all $j = 1, \ldots, N$, where the latter is just a finite set of affine inequalities.

- If imposing element-by-element constraints on matrices as in the previous item the corresponding system of affine inequalities can be pretty large. This can be circumvented if relying on matrix norm-bounds. Since the inequalities

$$\sum_{j=1}^{m} \|R_j\| < 1 \quad \text{or} \quad \max_{j=1}^{m} \|R_j\| < 1$$

(where $\|.\|$ either denotes the spectral or the Frobenius norm) have LMI representations, the same is true of the corresponding support functionals, and one can systematically derive a suitable description by dualization as in Appendix A.

- Colored noise is modeled as the output of some filter $H(z)$ whose input is white. Noise property variations can then be seen as resulting from a whole family of filters, which are e.g. parameterized by real structured uncertainties. In practice it is often more realistic to describe such filter families by imposing different lower and upper bounds on the filter's frequency response magnitude $|H(e^{i\tau})|$ (for SISO filters) over different τ-intervals, thus directly imposing bounds on the output noise's spectral density over these intervals. For MIMO filters such bounds are imposed as $\alpha_j < \lambda_{\min}(H(e^{i\tau})^* H(e^{i\tau}))$ and/or $\lambda_{\max}(H(e^{i\tau})^* H(e^{i\tau})) < \beta_j$ for all $\tau \in [\underline{\tau}_j, \overline{\tau}_j]$. Such constraints are not directly convex in the coefficients describing H. However, after the re-parameterization

$$\mathcal{F}(R, e^{i\tau}) + \mathcal{F}(R, e^{i\tau})^* = H(e^{i\tau})^* H(e^{i\tau}),$$

they read as

$$\alpha_j I < \mathcal{F}(R, e^{i\tau}) + \mathcal{F}(R, e^{i\tau})^* < I\beta_j$$

for all $\tau \in [\underline{\tau}_j, \overline{\tau}_j]$. Such a family of constraints then do admit LMI formulations in the coefficients

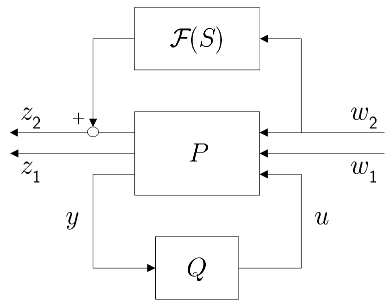

Fig. 1. Analysis Configuration

R as made explicit in (Wu *et al.*, 1998; Genin *et al.*, 2000*b*; Alkire and Vandenberghe, 2001). At this point we would like to stress that this insight is also a rather immediate consequence of older results in robust control. Indeed by a Cayley transformation positivity constraints on unit circle segments can be written as positivity constraints of a rational function in one real variable confined to some interval. The abstract version of the full-block S-procedure (Scherer, 2001) applies in order to to translate such a robust positivity constraint in *one* real variable into an LMI-constraint, and the results of (Meinsma *et al.*, 1997; Iwasaki *et al.*, 2000) reveal that the resulting characterization is indeed lossless.

3.2 *Analysis*

If defining

$$P(z) = \left(\begin{array}{cc|c} 0.5I & T(z) & 0 \\ 0 & 0 & z^{-m-1} \\ \hline 0 & I & 0 \end{array}\right),$$

we observe that (11) is the interconnection of

$$\begin{pmatrix} z_1 \\ z_2 \\ y \end{pmatrix} = P \begin{pmatrix} w_1 \\ w_2 \\ u \end{pmatrix} + \begin{pmatrix} 0 \\ \mathcal{F}(S) \\ 0 \end{pmatrix} w_2$$

with $u = Qy$. Therefore, the *analysis* problem amounts to characterizing the existence of a proper stable dynamic Q and parameters S subject to LMI constraints which render this latter interconnection strictly positive real. Let us observe that the interconnection can be depicted as in Figure 1, and that $\mathcal{F}(S)$ actually depends affinely on S. Therefore, this latter problem nicely fits into the general parametric model-matching *synthesis* framework as formulated in (Scherer, 2000*a*). This leads us to efficient LMI-algorithms for computing the worst-case asymptotic output variance for system (1) against pretty general classes of (structured) uncertain stochastic signals.

3.3 *Output Feedback Controller Design*

Let us view (1) as the state-space realization of the interconnection of

$$\begin{pmatrix} z \\ y_1 \end{pmatrix} = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} w \\ u_1 \end{pmatrix} \tag{12}$$

with an internally stabilizing output-feedback controller $u_1 = Ky_1$. If we now define

$$P_e(z) = \left(\begin{array}{cc|cc} 0.5I & P_{11}(z) & P_{12}(z) & 0 \\ 0 & 0 & 0 & z^{-m-1} \\ \hline 0 & P_{21}(z) & P_{22}(z) & 0 \\ 0 & I & 0 & 0 \end{array}\right)$$

then (11) can be seen as

$$\begin{pmatrix} z_1 \\ z_2 \\ y_1 \\ y_2 \end{pmatrix} = P_e \begin{pmatrix} w_1 \\ w_2 \\ u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \mathcal{F}(S) \\ 0 \\ 0 \end{pmatrix} w_2 \tag{13}$$

interconnected with the *structured* controller

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} K & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

We arrive at yet another motivation for investigating structured controller design problems. A general direct LMI solution for structured synthesis is not known. Instead one can parameterize $Q(z)$ as $Q_0 + Q_1/z + \cdots Q_\nu/z^\nu$ with matrix coefficients Q_j. Due to the specific structure of $(u_1, u_2) \rightarrow (y_1, y_2)$ in (13), the coefficients $Q_0, ..., Q_\nu$ enter the interconnection affinely, similarly as $S_0, ..., S_m$. Hence we can again apply the algorithm in (Scherer, 2000a) in order to compute both these parameters and the dynamic controller K to find the optimal achievable worst-case output variance bound γ_ν for this particular Q-parameterization. Clearly, for $\nu \rightarrow \infty$ the sequence γ_ν converges from above to the actually achievable bound γ_* for general Q. Let us observe that the techniques described in (Scherer, 1999) do allow to compute another sequence of lower bounds of γ_* which converges to this value. This gives rise to efficient stopping criteria since one can numerically estimate the conservatism caused by introducing the FIR-expansion of Q.

Let us conclude this section with the following interesting observation which allows a non-conservative direct LMI-solution of the problem discussed in this paper. After introducing the Youla parameterization for (12), all controlled transfer matrices can be written as $T_1 + T_2RT_3$ with proper and stable R. If facing a two-block problem, translating into $T_3 = I$ without any further hypothesis on T_2, one can circumvent the FIR-parameterization of Q and arrive at a full direct problem solution. One just needs to observe that (11) can now be described as the interconnection of

$$\begin{pmatrix} z_1 \\ z_2 \\ y \end{pmatrix} = \left(\begin{array}{cc|cc} 0.5I & T_1(z) & T_2(z) & 0 \\ 0 & \mathcal{F}(S,z) & 0 & z^{-m-1} \\ \hline 0 & I & 0 & 0 \end{array}\right) \begin{pmatrix} w_1 \\ w_2 \\ u_1 \\ u_2 \end{pmatrix}$$

with

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} R \\ Q \end{pmatrix} y.$$

Similarly as for analysis, we now arrive at the problem of parametric dynamic optimization with an *unstructured* controller and this is amenable to the techniques of (Scherer, 2000a). This guarantees an a priori bound on the McMillan degree of R and Q, and hence on that of the to-be-designed controller K.

4. EXTENSIONS AND CONCLUSIONS

The main purpose of this paper was to complement the results of (Gusev, 1995a; Gusev, 1995b; Gusev, 1996; Pransath and Nagpal, 2000; Scherer, 2000b) to an extension of robust H_2-control against structured classes of noise with a variety of practically relevant interpretations. Both controller analysis and synthesis could be reduced to LMI-computations. For general plants, approaching optimality will generally lead to an explosion of the controller's McMillan degree, whereas for two-block-plants it could be shown that this explosion can be fully avoided.

The approach in this paper circumvents some of the technical delicacies of previous work and hence admits a rather immediate extension to systems described in continuous time. Moreover, it is not difficult to extend the results to imposing asymptotic output variance bounds on more than one output signals as required in multi-objective control. The consequence of our results for H_2-control of linear parameter-varying systems is presently under investigation.

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Appendix A. SUB-LEVEL SETS OF SUPPORT FUNCTIONS

Let $A : X \to Y$ be a bounded linear mapping from the normed space X into the normed space Y which is equipped with a positive cone with nonempty interior (Luenberger, 1969). For a fixed $y_0 \in Y$ consider the convex set

$$\mathcal{X} := \{x \in X : \ y_0 + Ax \leq 0\}$$

which clearly is a generalization of the solution set set of a finite-dimensional linear matrix inequality. Let us assume that

$$\exists x \in X : \ y_0 + Ax < 0. \tag{A.1}$$

We are specifically interested in the support function of this set, which is defined on the dual space X^* as

$$s_{\mathcal{X}}(x^*) := \sup_{x \in \mathcal{X}} x^* x.$$

Our intention is to show that the sub-level sets of $s_{\mathcal{X}}$ can be described as the solution set of affine inequalities in Y^*, defined in a natural fashion by cone duality. (If P is the positive cone in Y, then P^* is defined as $P^* := \{y^* : \ y^* y \geq 0 \ \text{ for all } \ y \in P\}$.) A simple Lagrange duality argument is at the heart of the reasoning.

Indeed suppose that x_0^* satisfies $s_{\mathcal{X}}(x_0^*) < \gamma$ or

$$\inf_{x \in X,\, y_0 + Ax \leq 0} -x_0^* x > -\gamma.$$

Due to the constraint qualification (A.1) we can apply a general theorem on Lagrange duality (Luenberger, 1969) to infer

$$\max_{y^* \geq 0} \inf_{x \in X} y^*(y_0 + Ax) - x_0^* x > -\gamma.$$

By using the dual mapping $A^* : Y^* \to X^*$, we can rearrange the functional to 'sort for the variable x' which leads to

$$\max_{y^* \geq 0} \inf_{x \in X} y^* y_0 + (A^* y^* - x_0^*)x > -\gamma.$$

The inner infimum can only be finite if $A^* y^* = x_0^*$, and then its value just equals $y^* y_0$. We infer

$$\max_{y^* \geq 0,\, A^* y^* = x_0^*} y^* y_0 > -\gamma.$$

This dualization argument lets us conclude that

$$\{x^* \in X^* : \ s_{\mathcal{X}}(x^*) < \gamma\} =$$
$$= \{x^* \in X^* : \ \exists y^* \geq 0,\, A^* y^* = x^*, y^* y_0 + \gamma > 0\}.$$

Hence sub-level sets of the support functional $s_{\mathcal{X}}$ are indeed described by affine equation and inequality constraints defined by the two linear mappings A^* and y_0 (viewed as acting on Y^*) .

Let us finally make this description explicit for the standard form linear matrix inequality

$$F_0 + x_1 F_1 + \cdots x_N F_N \leq 0 \tag{A.2}$$

(with $X = \mathbb{R}^N$ and Y the space of real symmetric matrices of a fixed dimension, equipped with the positive semi-definite cone) which is assumed to be strictly feasible. Hence a vector $x_0 = (x_{01}, ..., x_{0N}) \in \mathbb{R}^N$ satisfies

$$x_0^T x < \gamma \ \text{ for all solutions } x \text{ of (A.2)}$$

iff

$$\exists Z \geq 0 : \ \text{Tr}(ZF_j) = x_{0j}, \ \text{Tr}(ZF_0) + \gamma > 0.$$

In summary, the sub-level sets of a support functional has an LMI description if the underlying set itself does admit an LMI description with strict feasibility.