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CHEAP CONTROL PERFORMANCE LIMITATIONS OF INPUT CONSTRAINED LINEAR SYSTEMS

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Abstract: Limited authority of actuators implies that real control signals are always constrained, and in almost all cases, this produces a degradation in the performance of the system. It is thus of practical importance to understand the fundamental aspects of this performance degradation. In this paper, we take an initial step by proposing a way to characterize the performance limitations that arise in closed-loop stable linear systems due to the constraints on the magnitude of the control signal. Specifically, we evaluate the cost associated with the constraints via the L_2 norm of the tracking error of a constrained limiting optimal compensator.

Keywords: Performance limitations, cheap control, constrained control systems.

1. INTRODUCTION

The issue of limitations in control has given rise to on-going interest since Bode's original work in the 1940s (Bode, 1945). This issue is central to any feedback system since it reveals what can and cannot be achieved given the system's structural and dynamical characteristics.

The tools for analyzing the limits of performance for unconstrained linear systems include logarithmic sensitivity integrals and limiting linear quadratic optimal control. The first approach has been extensively developed and reviewed, for example, in (Serón *et al.*, 1997), while the second has been studied, for example, by (Qiu and Davison, 1993). These two apparently independent tools have also recently been shown to be intimately related (Middleton and Braslavsky, 2000).

In this paper, we propose a way to characterize the performance limitations in closed-loop stable linear systems that arise due to constraints on the magnitude of the control signal. In our approach, we evaluate the cost associated with the constraints using the value of the L_2 norm of the tracking error of a constrained limiting optimal compensator when a unit step reference is applied. To solve this problem, we exploit the relationship between limiting linear quadratic optimal control problems and open loop inverse control. We focus on SISO linear systems.

The rest of the paper is organized as follows. In section 2, we review the so called "cheap" limiting optimal control problem and the use of the optimal cost as a measure of performance limitations. In section 3, we utilize this key idea to evaluate the L_2 norm of the tracking error for a step input in a constrained cheap control compensator. In section 4, we present examples to illustrate the method. Finally, in section 5, we draw conclusions.

2. CHEAP CONTROL FUNDAMENTAL LIMITATIONS

Allowing arbitrarily large control signals is obviously impractical. However, the fact that, even under these conditions, the cost function cannot be reduced to zero exposes the presence of fundamental limitations that are related to the structure and dynamics of a system. Thus, by not restricting the control effort (Cheap control), the cost obtained is a benchmark against which other more realistic scenarios can be judged. In this section we review properties of cheap optimal control for linear systems. We will use this as a basis for the analysis of the constrained case in section 3.

Consider a linear time-invariant system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \; x \in \mathbb{R}^n , \; u \in \mathbb{R} \\ y(t) &= Cx(t) , \qquad\qquad y \in \mathbb{R} , \; x(0) = x_0, \end{aligned} \tag{1}$$

which is assumed to be stabilizable and detectable, and the cost functional

$$J_\varepsilon = \int_0^\infty \Big(y^{\mathrm{T}}(t)y(t) + \varepsilon^2 u^{\mathrm{T}}(t)u(t) \Big) dt \,. \tag{2}$$

The *cheap control problem* seeks the state feedback control u that minimizes J_ε as ε tends to zero.

The structure of the solution to the cheap control problem is better appreciated if we make a change of variables $x \to [\eta^{\mathrm{T}} \quad z^{\mathrm{T}}]^{\mathrm{T}}$ taking the system into the normal form or "zero dynamics" form (Isidori, 1995):

$$\begin{aligned} \dot{\eta} &= A_0\eta + B_0 z_1 \\ \dot{z} &= A_1\eta + A_2 z + B_1 u \\ y &= z_1 \,. \end{aligned} \tag{3}$$

In (3), $\eta \in \mathbb{R}^m$, $z = [z_1, z_2, \ldots, z_{n-m}]^{\mathrm{T}} \in \mathbb{R}^{n-m}$, and the eigenvalues of A_0 are the m zeros of the system transfer function $H(s) = C(sI - A)^{-1}B$. The η-subsystem is called the *zero-dynamics subsystem*. With the system matrices in the form (3), and considering $\varepsilon > 0$, the solution $P(\varepsilon) > 0$ can be computed, in the form of a series in ε, using the Riccati equation associated with (2):

$$A^{\mathrm{T}}P(\varepsilon) + P(\varepsilon)A + C^{\mathrm{T}}C - \frac{1}{\varepsilon^2}P(\varepsilon)BB^{\mathrm{T}}P(\varepsilon) = 0 \,. \tag{4}$$

Once $P(\varepsilon)$ is obtained, taking the limit as $\varepsilon \to 0$, the cheap control cost is found to be (Serón *et al.*, 1999)

$$J \triangleq \lim_{\varepsilon \to 0} J_\varepsilon = \int_0^\infty y^2(t)dt = \eta(0)^{\mathrm{T}} P_0 \eta(0) \,, \tag{5}$$

where P_0 is a positive semi-definite solution of

$$A_0^{\mathrm{T}}P_0 + P_0A_0 = P_0B_0B_0^{\mathrm{T}}P_0 \,. \tag{6}$$

For simplicity, we assume that the zeros of (3) (*i.e.,* the eigenvalues of A_0) are either all minimum phase or all non-minimum phase. If the zeros are all minimum phase, the solution of (6) is $P_0 = 0$, and therefore the cost (5) is zero. On the other hand, if the zeros are all non-minimum phase then $-A_0$ is Hurwitz and (6) has a unique positive definite solution $P_0 > 0$ (note that (A_0, B_0) is controllable, because the system (3) is stabilizable and $-A_0$ is Hurwitz). Thus, the cheap control cost (5) is nonzero in this case if the initial zero-dynamics state is different from zero.

When $\varepsilon > 0$ is small, the optimal control solution has high gain and induces a two-time-scale decomposition in the closed-loop system. To simplify the analysis, suppose, for the moment, that the system (3) has relative degree one, *i.e.,* $z = z_1 = y$ and $B_1 \neq 0$ is a scalar in (3). Then the optimal control is

$$u_\varepsilon^{\mathrm{OPT}} = -\frac{1}{\varepsilon}\left[y + B_0^{\mathrm{T}}P_0\eta + O(\varepsilon)\right] \tag{7}$$

, where $\lim_{\varepsilon \to 0} \dfrac{O(\varepsilon)}{\varepsilon}$ is a nonzero finite real number. The closed-loop system is then in the standard singular perturbation form:

$$\begin{aligned} \dot{\eta} &= A_0\eta + B_0 y \,, \\ \varepsilon\dot{y} &= -B_1(y + B_0^{\mathrm{T}}P_0\eta) + O(\varepsilon) \,. \end{aligned} \tag{8}$$

For a slow-fast analysis (Kokotović *et al.*, 1986) we set $\varepsilon = 0$ in the second equation of (8) and obtain the output as a function of the zero-dynamics state, *i.e.,* $y = -B_0^{\mathrm{T}}P_0\eta$. Then, substituting in the η-equation, we find that the slow subsystem of (8) is the "optimal zero-dynamics subsystem"

$$\dot{\eta} = -P_0^{-1}A_0^{\mathrm{T}}P_0\eta \,. \tag{9}$$

These slow dynamics evolve in the *singular hyperplane*

$$y + B_0^{\mathrm{T}}P_0\eta = 0 \,,$$

while the fast dynamics represent the convergence of $y + B_0^{\mathrm{T}}P_0\eta$ to zero.

In the limit as $\varepsilon \to 0$, no matter what the initial condition $(\eta(0), z(0))$ is, the state $(\eta(0^+), z(0^+))$ is on this singular hyperplane, and evolves inside this subspace thereafter. The initial fast response of the state is singular and so is the control that takes the state from the initial condition into the hyperplane (Francis, 1979). Once the state is on the hyperplane, it presents a slow evolution given by the dynamics of (9) with $y = -B_0^{\mathrm{T}}P_0\eta$.

For systems with higher relative degree, a similar analysis shows that the singular hyperplane is given by (Saberi and Sannuti, 1987)

$$[B_0^{\mathrm{T}}P_0 \quad 1 \quad 0\ldots0]\,[\eta^{\mathrm{T}} \quad z^{\mathrm{T}}]^{\mathrm{T}} = 0 \,. \tag{10}$$

For the minimum phase case this singular hyperplane is simply the origin.

Cheap Tracking Performance

Consider now the problem of regulating the output to a constant setpoint r starting from zero initial state. Define the error variables $e(t) = y(t) - r$, $\tilde{\eta}(t) = \eta(t) - \bar{\eta}$, $\tilde{z}_i(t) = z_i(t) - \bar{z}_i$, $i = 2, \ldots, n-m$, $v(t) = u(t) - \bar{u}$, where the "bars" on the variables denote their steady state values corresponding to the setpoint r. Then the cheap constant setpoint tracking problem is equivalent to the cheap regulator problem described above with the error variables replacing the original variables and with cost

$$J \triangleq \lim_{\varepsilon \to 0} \int_0^\infty (e^2(t) + \varepsilon^2 v^2(t))dt \,.$$

From (5) we know that the optimal value of J is

$$J = \tilde{\eta}(0)^{\mathrm{T}} P_0 \tilde{\eta}(0)\,, \qquad (11)$$

where P_0 is the solution of (6). Suppose that all the zeros are nonminimum phase. Then A_0 is nonsingular and the initial condition for the zero-dynamics subsystem is $\tilde{\eta}(0) = A_0^{-1} B_0 r$. Using this value in (11), and assuming $r = 1$, we obtain

$$J = 2\, trace A_0^{-1} = 2 \sum_{i=1}^{m} \frac{1}{q_i}\,. \qquad (12)$$

where q_i, $i = 1, \ldots, m$, are the nonminimum phase zeros of the system. The above result, obtained by Qiu and Davison (1993), says that the smallest achievable L_2 norm for the tracking error (*i.e.,* in the limit as $\varepsilon \to 0$) is larger the closer the nonminimum phase zeros are to the imaginary axis.

We end this section with a review of the asymptotic input-output properties of the cheap tracking controller (Kwakernaak and Sivan, 1972). Let the transfer function of the system (1) be

$$G(s) = \alpha \frac{\prod_{i=1}^{m}(s - q_i)}{\prod_{i=1}^{n}(s - p_i)}, \quad \alpha \neq 0\,. \qquad (13)$$

The tracking controller that achieves zero steady state error is

$$u(t) = -K_\varepsilon x(t) + H_c^{-1}(0) r\,, \qquad (14)$$

where $K_\varepsilon = \dfrac{1}{\varepsilon^2} B^{\mathrm{T}} P(\varepsilon)$ is the optimal feedback gain corresponding to the solution of (4), and $H_c(s) = C(sI - (A - BK_\varepsilon))^{-1} B$. Using (14) in closed loop with the system (1), and taking $\varepsilon \to 0$, the closed-loop transfer function $T(s)$ from the reference $R(s)$ to the output $Y(s)$ approaches

$$T(s) \approx \frac{1}{\chi_{n-m}(\frac{s}{\omega_0})} \prod_{i=1}^{m} \frac{-\frac{s}{q_i} + 1}{\frac{s}{q_i} + 1}, \quad \omega_0 = \left(\frac{\alpha^2}{\varepsilon}\right)^{\frac{1}{2(n-m)}}, \qquad (15)$$

where χ_{n-m} is a Butterworth polynomial of order $n - m$ and radius 1, ω_0 is the asymptotic radius of the Butterworth configuration of the $n - m$ closed loop poles that tend to infinity, and q_i, $i = 1, \ldots, m$, are the zeros of the open loop transfer function (13), which are assumed to be non-minimum phase. In (15), we can see the two-time scale behavior of the closed loop system previously mentioned.

3. CHEAP CONTROL PERFORMANCE LIMITATIONS OF INPUT CONSTRAINED LINEAR SYSTEMS

We next proceed to the case of performance limitations when the input is constrained. The fundamentally important roll played by input constraints (both amplitude and rate) has been eloquently explained elsewhere— see for example (Stein, 1989). Our aim here is to give additional theoretical support to the understanding of this issues. We propose as a measure of the performance limitations in the presence of constraints on the magnitude of the control the value of the L_2 norm of the tracking error when a unit step output reference is applied to a saturated limiting optimal compensator.

Specifically, we propose as a measure of performance for open-loop stable systems of the form (1) subject to the constraint $|u(t)| \leq \Delta \forall t$, the value of the cost function defined as

$$J \triangleq \lim_{\varepsilon \to 0} \int_0^\infty e^2(t) + \varepsilon^2 u(t)^2(t) dt, \qquad (16)$$

when a unit step reference, i.e., $r(t) = 1(t)$, is applied with the system initially at rest, using the control law

$$u(t) = sat_\Delta(-Kx(t) + H_c^{-1}(0) r(t)),$$

where

$$sat_\Delta(z) \triangleq \begin{cases} \Delta & \text{if } z > \Delta\,, \\ z & \text{if } |z| \leq \Delta\,, \\ -\Delta & \text{if } z < -\Delta\,. \end{cases}$$

and

$$K = \lim_{\varepsilon \to 0} K_\varepsilon. \qquad (17)$$

To evaluate the cost we assume that input saturation occurs in the first part of the evolution of the system:

Assumption 1. (A.1) The control of the closed-loop system switches between the saturation levels during a period of time $[0^+, t_{sat})$ and thereafter it never reaches the saturation levels again.

Note that this is a reasonable assumption given the high gain nature of the cheap controller.

Under assumption (A.1), the cost(16) can be separated into two components; one corresponding to the period of time where the control saturates and the other corresponding to the period of time starting when the control leaves saturation until it reaches the final steady state value $u(\infty)$, i.e.,

$$J = \lim_{\varepsilon \to 0} \int_0^{t_{sat}} e^2(t) + \varepsilon^2 u^2(t) dt + \lim_{\varepsilon \to 0} \int_{t_{sat}}^\infty e^2(t) + \varepsilon^2 u^2(t) \qquad (18)$$

We next evaluate each of the two components.

Cost during saturation

In order to evaluate the first term of the cost (18), we shall use the asymptotic properties of the cheap tracking controller, specifically the closed-loop transfer function (15), to find an input-output equivalent open-loop controller.

We see that in order to obtain (15) by an open-loop equivalent controller $Q(s)$, it must satisfy $G(s)Q(s) = T(s)$, that is, this controller should essentially approximate the inverse of the system. Hence, the open-loop controller for the system (13) equivalent to the cheap controller (14) (*i.e.,* the one that achieves the same closed-loop transfer function (15)) has the form

$$Q(s) = G_{MP}^{-1}(s) F(s)\,, \qquad F(s) = \frac{1}{\chi_{n-m}(\frac{s}{\omega_0})}\,, \qquad (19)$$

where χ_{n-m} is as in (15), and $G_{MP}(s)$ has the same poles of $G(s)$ and the reflection through the imaginary axis of the non-minimum phase zeros of $G(s)$ (recall that to highlight the methodology, we have assumed that all the zeros of $G(s)$ are non-minimum phase).

Using the equivalent open-loop controller, the problem of the saturated limiting optimal compensator can be posed using the scheme shown in figure 1, *i.e.,* this scheme reproduces the control $u = sat_{\Delta}(-Kx(t) + H_c^{-1}(0)r(t))$. In this scheme, $Q(s)$

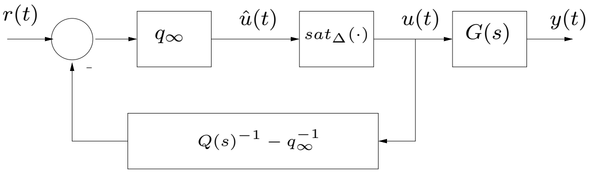

Fig. 1. Anti-windup open-loop control.

is the bi-proper transfer function (19), and q_∞ is its high frequency gain. It is easy to show that when the system is not saturated, the loop in figure 1 reduces to $Q(s)$.

From figure 1, we see that after applying a unit step signal in $r(t)$, the control $u(t)$ typically saturates since the gain q_∞ is usually large for small values of ε. We thus assume, without loss of generality, that $u(0^+) = \Delta$. The control will then switch between Δ and $-\Delta$ until $t = t_{sat}$ when it leaves saturation to continue with a linear evolution. The crucial step in this analysis is then to evaluate the switching times during the saturated regime. For clarity of exposition, we illustrate the ideas by taking the case of first and second order systems with, at most, one right half plane zero. For these cases, we have only one switch in the saturation regime. In this case, during the saturation period $[0^+, t_{sat})$, $\hat{u}(t)$ (*i.e.,* the signal at the input of the saturation function, see figure 1) is given by

$$\hat{u}(t) = q_\infty (1 - \mathcal{L}^{-1}\{(Q(s)^{-1} - q_\infty^{-1})\frac{\Delta}{s}\}), \qquad (20)$$

where $\mathcal{L}^{-1}\{\cdot\}$ denotes the inverse Laplace transform operator. In addition, the control signal leaves saturation when the following condition is satisfied

$$\hat{u}(t_{sat}) = \Delta\,. \qquad (21)$$

Using (20) and (21) we can determine the time instant t_{sat} at which the control leaves saturation. Also from figure 1, we see that the tracking error during the saturation interval can be calculated as

$$e(t) = \mathcal{L}^{-1}\{G(s)\frac{\Delta}{s}\} - 1\,.$$

With the expressions for $e(t)$ and t_{sat}, all the ingredients to evaluate the first term of (18) are available. After performing the limits and the integration, we can obtain an analytical expression for the first term of the cost.

Cost after saturation

Once the control signal leaves saturation, the problem reduces to the unconstrained cheap control compensator. To find the associated cost we will use the properties of the slow evolution of the cheap control state in the singular hyperplane.

The first step is to recognize that when the system leaves saturation, the state is on the singular hyperplane. This is easy to show by contradiction: Suppose that the system leaves saturation and never saturates again, and also that the state is not on the singular hyperplane. Then as the control is not saturated the system behaves like the unconstrained problem, and since the state is not on the singular hyperplane there will be a singularity in the control that will make the control saturate. Therefore, once the control leaves saturation, the state must be on the singular hyperplane.

Consequently, once the control leaves saturation at $t = t_{sat}$, the state of the system follows the same trajectories that the unconstrained state would have followed if it had started from the initial condition $[\tilde{\eta}(t_{sat})\tilde{z}(t_{sat})]'$. Hence, the cost after saturation is

$$\int_{t_{sat}}^{\infty} e^2(t)dt = \tilde{\eta}(t_{sat})' P_0 \tilde{\eta}(t_{sat}), \qquad (22)$$

where $\tilde{\eta}(t_{sat})$ is determined from

$$\begin{bmatrix}\tilde{\eta}(t_{sat})\\ \tilde{z}(t_{sat})\end{bmatrix} = e^{-A_n t_{sat}} \begin{bmatrix}\tilde{\eta}(0)\\ \tilde{z}(0)\end{bmatrix} + \int_0^{t_{sat}} e^{-A_n(t-\tau)} B_n \Delta d\tau\,, \qquad (23)$$

where

$$A_n \triangleq \begin{bmatrix} A_0 & [B_0\,0\,0] \\ A_1 & A_2 \end{bmatrix} \quad B_n \triangleq \begin{bmatrix} 0 \\ B_1 \end{bmatrix}.$$

Equation (22) allows us to complete the calculation of the cost by evaluating the second term of of the right hand side of (18).

For systems with only one non-minimum phase zero, the evaluation of the cost once the system leaves saturation can be considerably simplified since it is not necessary to solve equation (23). The procedure is explained in the following.

If the system has only one non-minimum phase zero the singular hyperplane becomes a line in $\mathbf{R}^n$. Indeed from (10) and (3) we see that

$$\begin{aligned}
\dot{\tilde{\eta}} &= (A_0 - B_0 B_0^{\mathrm{T}} P_0)\tilde{\eta}\,,\\
\tilde{z}_1 &= -B_0^{\mathrm{T}} P_0 \tilde{\eta}\,,\\
\tilde{z}_2 &= -B_0^{\mathrm{T}} P_0 (A_0 - B_0 B_0^{\mathrm{T}} P_0)\tilde{\eta}\,,\\
\tilde{z}_3 &= -B_0^{\mathrm{T}} P_0 (A_0 - B_0 B_0^{\mathrm{T}} P_0)^2\tilde{\eta}\,,\\
&\vdots\\
\tilde{z}_{n-1} &= -B_0^{\mathrm{T}} P_0 (A_0 - B_0 B_0^{\mathrm{T}} P_0)^{n-2}\tilde{\eta}\,.
\end{aligned}$$

Once the system leaves saturation, it behaves like the unconstrained cheap control problem, and since the singular hypeplane is a line, the output $e = \tilde{z}_1$ after saturation has the same evolution as the unconstrained output shifted in time. As a consequence, we can evaluate the cost after saturation of the constrained cheap control problem using a partial cost of the unconstrained cheap control problem as

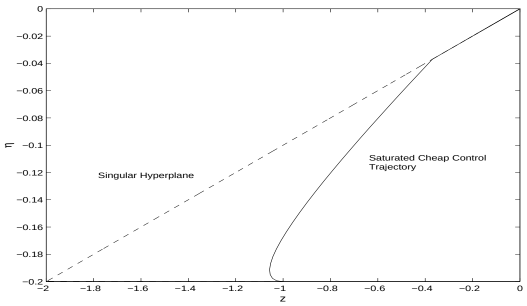

Fig. 2. Phase portrait for unconstrained and saturated cheap tracking control.

$$\int_{t_{sat}}^{\infty} e^2 dt \approx \int_{t^*}^{\infty}\left(1-\mathcal{L}^{-1}\left[\left(\frac{-\frac{s}{q}+1}{\frac{s}{q}+1}\right)\frac{1}{s}\right]dt\right)^2 = \frac{2}{q}e^{-2qt^*}. \tag{24}$$

The approximation in (24) comes from neglecting the high frequency poles in the Butterworth arrange of (15). The value t^* is determined from the condition

$$\mathcal{L}^{-1}\left[G(s)\frac{\Delta}{s}\right]\Bigg|_{t_{sat}} = \mathcal{L}^{-1}\left[\left(\frac{-\frac{s}{q}+1}{\frac{s}{q}+1}\right)\frac{1}{s}\right]\Bigg|_{t^*}.$$

Comparing (24) with (12) for the case of a single non-minimum phase zero, we see that the partial cost (24) is smaller than the total unconstrained cost (12). However, whereas the transition to the singular hyperplane is costless in the unconstrained case, it has a nonzero cost in the constrained case since the state cannot "jump" to the singular hyperplane but has a slow evolution while the control is saturated. The combination of the two partial costs yields a cost larger than the unconstrained cost (12), as we will see in the examples of section 4.

To illustrate the above ideas, figure 2 shows a simulation result for the second order example explained in detail in section 4. In this figure, we see the state trajectory using scheme of figure 1, and also using unconstrained cheap control. We next present two examples.

4. EXAMPLES

Example 1: Consider the following system

$$G(s) = \frac{1}{\tau s+1}.$$

Then, the equivalent open loop cheap controller is given by

$$Q(s) = \frac{\tau s+1}{\beta s+1},$$

where $\beta = \sqrt{\epsilon}$. It also follows that,

$$q_\infty = \frac{\tau}{\beta} \tag{25}$$

$$Q(s)^{-1} - q_\infty^{-1} = \frac{(\tau-\beta)}{\tau(\tau s+1)} \tag{26}$$

Using (25) and (26) in (20) we have

$$\hat{u}(t) = \frac{\tau}{\beta}\left[1 - \frac{(\tau-\beta)}{\tau}(1-e^{-\frac{t}{\tau}})\Delta\right]$$

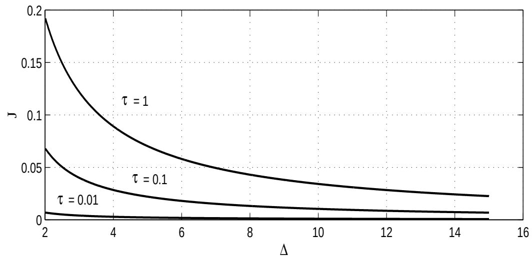

Fig. 3. Cost vs. Δ for example 1

This last expression is valid until t_{sat} which, using(21), and taking the limit as $\beta \to 0$ is found to be

$$t_{sat} \to -\tau \ln\left[\frac{\Delta-1}{\Delta}\right] \tag{27}$$

On the other hand, until t_{sat} the tracking error is

$$e(t) = \Delta(1-e^{-\frac{t}{\tau}}) - 1 \tag{28}$$

Finally, since the system is minimum phase, and using (28) and (27), we obtain the following expression for the cost (18):

$$J = -\tau(1-\Delta)^2 \ln\left[\frac{\Delta-1}{\Delta}\right] + \frac{2\Delta(\Delta-1)}{\tau}\left(\left[\frac{\Delta-1}{\Delta}\right]^{\tau^2}-1\right) - \frac{\Delta^2}{2\tau}\left(\left[\frac{\Delta-1}{\Delta}\right]^{2\tau^2}-1\right) \tag{29}$$

Figure 3 shows a graphical representation of the value of the cost (29) as a function of Δ for different values of τ . It should be noted that the set point value of the control signal should be feasible, i.e., $\Delta > 1$ for this case.

Note that since this system is minimum phase, the limiting cheap cost for $\Delta \to \infty$ is zero. Also, as expected, the faster the plant, then the lower the cost, since slow plants require more control effort that contribute to saturation in this case. However, the relation (29) is not trivial.

Example 2: Consider the following system

$$G(s) = \frac{2(1-\sigma s)}{(s+1)(s+2)}$$

In this case, the equivalent open loop cheap controller is given by

$$Q(s) = \frac{(s+1)(s+2)}{2(1+\sigma s)(1+\beta s)}.$$

It then follows that:

$$q_\infty = \frac{1}{2\sigma\beta} \tag{30}$$

$$Q(s)^{-1} - q_\infty^{-1} = \frac{2\left[(\sigma+\beta-3\sigma\beta)s + (1-2\sigma\beta)\right]}{(s+1)(s+2)} \tag{31}$$

Using (30) and (31) in (20) we have that as $\beta \to 0$

$$t_{sat} \to -\ln\left[\frac{1-\sigma-\sqrt{(\sigma-1)^2+(1+2\sigma)\frac{(\Delta-1)}{\Delta}}}{1-2\sigma}\right] \tag{32}$$

The tracking error on the interval $[0^+ t_{sat})$ is

$$e(t) = \Delta\left[1-2(1+\sigma)e^{-t}+(1-2\sigma)e^{-2t}\right]-1 \tag{33}$$

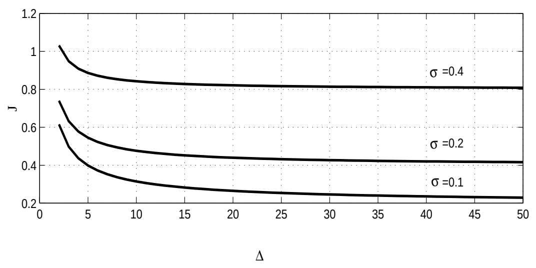

Fig. 4. Cost vs. Δ for example 2

The value t^* is given by

$$t^* = -\sigma \ln \left[\frac{1-\Delta}{2} + \frac{\Delta(1+\sigma)}{2} e^{-t_{sat}} - \frac{\Delta(1+2\sigma)}{2} e^{-2t_{sat}} \right] \tag{34}$$

Using (32), (33), (34) and (24) we can evaluate the cost numerically. The results are shown in figure 4 for three different positions of the non minimum phase zero and plot the cost versus Δ. Note that as $\Delta \to \infty$ the limiting cost approaches 2σ, which is consistent with the results of unconstrained cheap control. The results shown in figure 4 give insight into the effect of the input constrained achievable performance. It is interesting, for instance, to note that a constraint $\Delta = 5$ (which is five times the steady state input necessary in this case) changes the performance limit associated with a non minimum phase zero at 10 ($\sigma = 0.1$) to be equivalent to the performance limit achieved *without constraints* for a non minimum phase zero at 5. This illustrates the fact that, depending on conditions, the effect of input constraint can swap linear effects due to right half plane poles or zeros. This is in accord with intuition.

5. CONCLUSIONS

Under the assumption that the system switches between the saturation levels only for a certain period of time and never thereafter, the cost can be evaluated in terms of two components. The first component corresponds to the saturation period. We present a way to evaluate this component and non trivial analytical expressions are obtained as a function of the constraint level. The second term corresponds to the period of time from the instant the control leaves saturation until the control reaches its steady state. This cost is zero for minimum phase systems and different from zero for non minimum phase systems. For the non minimum phase case, we give a way to calculate this cost, which is very simple if the system has only one non minimum phase zero.

The proposed method seems to be of value in gaining a deeper understanding of the relationship between the dynamics of a system and the constraints essential at the stage of determining the authority of the actuator for some applications.

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