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THRESHOLD POLICIES IN THE CONTROL OF PREDATOR-PREY MODELS

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Abstract: Threshold policies are defined and analysed for different types of one species and predator-prey type models. It is shown that such policies can be designed by suitable choice of so called virtual equilibrium points. The simplest threshold policies are discontinuous, which leads to some drawbacks. It is also shown how to design continuous threshold policies that retain most of the advantages of their discontinuous versions but do not have the major drawback of chatter in the control. Threshold policies are also seen to be robust to uncertainty of model parameters, and initial conditions, as well as to delays between stock assessment and policy enactment. The models which are controlled are the Noy-Meir herbivore-vegetation model, subject to linear consumption curves, the two species Rosenzweig–MacArthur model, and a three-dimensional chemostat model.

Keywords: Virtual equilibrium point, variable structure, Rosenzweig–MacArthur model, robustness, chemostat model.

1. INTRODUCTION

A pure threshold policy is of the on-off type. In simple terms, whenever the predator (prey) population is above a certain level, harvesting of the predator (respectively, prey) is allowed. If not, then harvesting is prohibited. Such threshold policies or controls do, in fact, occur both in mathematical models of ecosystems as well as in real-life ecosystems. This paper points out that a suitably chosen threshold policy can modify the dynamics of a predator-prey system in such a way that a new robustly stable equilibrium is introduced. The design of this policy is based on the concept of virtual equilibria: these are to be introduced in such a manner that the controlled system has the desired behavior. The discontinuity of the pure threshold policies leads to undesirable chattering or high-frequency on-off behavior of the control in order to maintain the system at the desired equilibrium which is undesirable and unimplementable in real ecosystems. This paper also shows that a straightforward piecewise-linear continuous version of the discontinuous policy retains most of the good features of the latter,

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in addition to alleviating the chattering problem. The analysis, although a little more elaborate in this case, can still be carried out in terms of real and virtual equilibria.

The following models are studied in this paper. In the single species case, Noy-Meir's stock removal herbivore-vegetation model is first studied as a simple case in which the effects of a threshold policy as well as its continuous version are clearly seen.

In terms of previous work in this area, the paper (Costa *et al.*, 2000) proposed a type of threshold policy known as a weighted escapement policy (WEP) in which a threshold is built from a weighted (or linear) combination of prey and predator densities. This discontinuous policy was used to stabilize a Lotka–Volterra model under simultaneous harvesting of both predator and prey. Thus the present paper can be viewed as carrying the results of (Costa *et al.*, 2000) further, proposing continuous control and showing the usefulness as well as ubiquity of threshold type policies in mathematical ecology.

Studies of switching in population dynamics, more specifically switching of predator behavior in the context of optimal foraging, leading to chattering behavior were made in (Křivan, 1996; Křivan and Sikder, 1999).

The book (Emel'yanov *et al.*, 1998) proposes a general methodology, referred to as induced internal feedback, for the control of uncertain nonlinear dynamic systems. In particular, continuous versions of threshold-type control are proposed for the Lotka-Volterra model.

Finally, it should be mentioned that there is a large literature on variable structure sliding mode control that has appeared and continues to appear in control journals, although to the best of the authors' knowledge, an analysis in terms of virtual equilibria and the specific application to predator-prey models does not occur in this literature. Accounts of variable structure control can be found in the books (Utkin, 1992; Edwards and Spurgeon, 1998).

2. SINGLE SPECIES STOCK REMOVAL MODELS

Threshold policies can be implemented in fisheries (Quinn and Deriso, 2000) as well as in stock removal in herbivore-vegetation models (Noy-Meir, 1975). In his work, Noy-Meir presents a stock removal strategy and comments on its possible effects on herbivore productivity. The vegetation-herbivore interaction is modeled as follows.

$$\frac{dV}{dt} = G(V) - Hc(V) \quad V(0) = V_0,$$

where V is the vegetation density, $G(V)$ is the vegetation growth rate, H is the herbivore density (considered constant) and $c(V)$ is the vegetation consumption rate.

The vegetation dynamics under stock removal and a threshold policy can be defined as follows:

$$\frac{dV}{dt} = G(V) - \phi(V)\,Hc(V), \qquad (1)$$

where

$$\begin{cases} \phi(V) = 1 \text{ if } V > V_{th} \\ \phi(V) = 0 \text{ if } V \le V_{th}. \end{cases}$$

Choosing the threshold value V_{th} amounts to defining a threshold policy. The herbivore density is normalized, i.e., $H = 1$.

3. THRESHOLD POLICIES WITH LINEAR VEGETATION CONSUMPTION RATE

We first analyze the system subject to a variable structure threshold policy under the assumption that the vegetation consumption rate is linear, i.e., $c(V) = c_{max}V$.

Let the vegetation growth rate be given by the following logistic function:

$$G(V) = gV\left(1 - \frac{V}{V_{\max}}\right),$$
$$c(V) = c_{\max}V,$$

where $g > 0$ is a constant denoting the intrinsic vegetation growth rate. As mentioned above, a threshold policy applied to the system (1) generates two systems: a free system (i.e., system without grazing) when $\phi(V) = 0$, and a grazed system when $\phi(V) = 1$. The equilibrium points are calculated for each value of ϕ. When the equilibrium point corresponding to $\phi = 1$ (i.e., $V > V_{th}$) occurs in the region $V < V_{th}$, then this point is called *virtual* (Costa *et al.*, 2000) and will never be attained by the system. A similar statement holds for the equilibrium point calculated for $\phi = 0$.

For $\phi = 0$, the equilibrium points are calculated by setting

$$gV\left(1 - \frac{V}{V_{max}}\right) = 0,$$

thus the equilibrium points of the free system are

$$V_{sp1} = 0, \quad V_{sp2} = V_{max}.$$

For $\phi = 1$, the equilibrium points are calculated by setting

$$gV\left(1 - \frac{V}{V_{max}}\right) - c_{max}V = 0,$$

so that the equilibrium points of the grazed system are

$$V_{sp1} = 0, \quad V_{sp2} = V_{\max}\left(1 - \frac{c_{max}}{g}\right).$$

The graph of the logistic curve $G(V)$ is a convex parabola intercepting the V-axis at the origin, where it has slope g, and at the point $V_{\max}$. The consumption curve is a straight line through the origin with slope c. Clearly if $c > g$, then the consumption curve and the logistic curve intersect only at the origin, which becomes the (undesirable) equilibrium of the system (no vegetation survives the action of the herbivores). Thus, in the absence of grazing control, it is necessary that c be less than g, in order that the free system possess a nonzero equilibrium, which, from the graph will lie between 0 and $V_{\max}$, more precisely at $(1 - \frac{c}{g})V_{\max}$. The question that arises is the following: is it possible to do better by the introduction of a threshold policy? In other words, can a threshold policy induce the system to stabilize at a higher vegetation level V? If so, a greater yield can be expected from the herbivore, since it is known that a larger V leads to a larger consumption (the consumption curve has been assumed linear with positive slope), and, as a consequence larger production by the herbivore.

Figures 1 and 2 show that this can indeed be the case. In the first figure, the slope of the consumption curve is above the maximum level g, implying that the uncontrolled grazed system equilibrium, V_{gr}, corresponds to zero vegetation. Applying a threshold policy with, for example, $V_{th} = V_{\max}/2$, the system subject to on-off grazing stabilizes at vegetation level $V_{\max}/2$, which is clearly superior to the equilibrium V_{gr}. Figure 2 shows that, even if the slope of the consumption curve is below the maximum level $c < c_{\max} = g$, and the uncontrolled system reaches equilibrium at $V_{gr} = (1 - \frac{c}{g})V_{\max} < V_{\max}$ (since $c < g$), it is still possible to choose the threshold level V_{th}, such that $V_{gr} < V_{th} < V_{\max}$, resulting in an increase in the stabilized vegetation level.

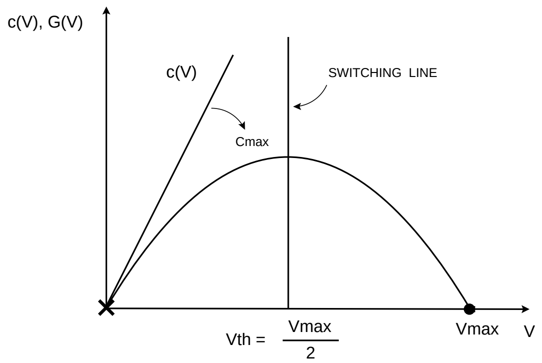

Fig. 1. Equilibria with consumption curve $c(V)$ linear with large slope ($c > g$). Free system equilibrium point – •. Grazed system equilibrium point – ×. Parameter values: $g = 1$, $c = 1.2$, $V_{th} = 0.5$, $V_{\max} = 1$.

The discontinuous (on-off) threshold policy leads to stabilization at the threshold level ($V = V_{th}$),

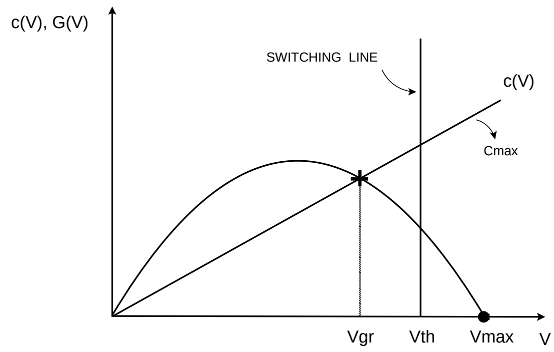

Fig. 2. Equilibria with consumption curve $c(V)$ linear with medium slope ($c < g$). Free system equilibrium point – •. Grazed system equilibrium point – ×. Parameter values: $g = 1$, $c = 0.3$, $V_{th} = 0.85$, $V_{\max} = 1$.

with very rapid alternation of grazing and grazing supression. The vegetation density also 'stabilizes' in a high frequency oscillation around the threshold value V_{th} – this is known as a *sliding mode* in the literature on variable structure systems (Utkin, 1992). This makes the application of such a policy impractical and motivates the next development, which is the design of a continuous policy that has similar features.

Design of a continuous threshold policy

Given the model studied in the previous section subject to a threshold policy $\phi(V)$,

$$\frac{dV}{dt} = gV\left(1 - \frac{V}{V_{max}}\right) - \phi(V)\,Hc(V), \qquad (2)$$

this section is concerned with the design of a new policy $\phi(\cdot)$ that is a continuous function of V.

An obvious way to modify the discontinuous threshold policy and turn it into a continuous one is to incline the vertical segment at V_{th}.

The expression for ϕ is then given by $\phi(V) = \varepsilon_1$, if $V > V_{th} + \sigma$; $\varepsilon_1\left(\frac{V - V_{th} + \sigma}{2\sigma}\right)$, if $V_{th} - \sigma \leq V \leq V_{th} + \sigma$; 0, if $V < V_{th} - \sigma$.

Note that, in order to analyze this policy it will be necessary to consider three regions. The idea is to choose ε_1 such that the system has its real (and desired) equilibrium in the linear region of the control, i.e., $V_{th} - \sigma \leq V \leq V_{th} + \sigma$ and that the equilibria in the remaining two regions ($V > V_{th} + \sigma$ and $V < V_{th} - \sigma$) are virtual. This analysis follows.

• $V_{th} - \sigma \leq V \leq V_{th} + \sigma$: In this region the system is described by the equation

$$\dot{V} = gV\left(1 - \frac{V}{V_{max}}\right) - \phi(V)\,c_{max}V,$$

which implies that the equilibria are given by:

$$V_{sp1} = 0,\ V_{sp2} = \frac{g + (V_{th} - \sigma)\dfrac{\varepsilon_1 c_{\max}}{2\sigma}}{\dfrac{g}{V_{max}} + \dfrac{\varepsilon_1 c_{\max}}{2\sigma}}.$$

Requiring that $V_{sp2} = V_{th}$, implies that ε_1 must have the following value:

$$\varepsilon_1 = \frac{2g}{c_{\max}}\left(1 - \frac{V_{th}}{V_{max}}\right).$$

A sample simulation result is given below in figure 3.

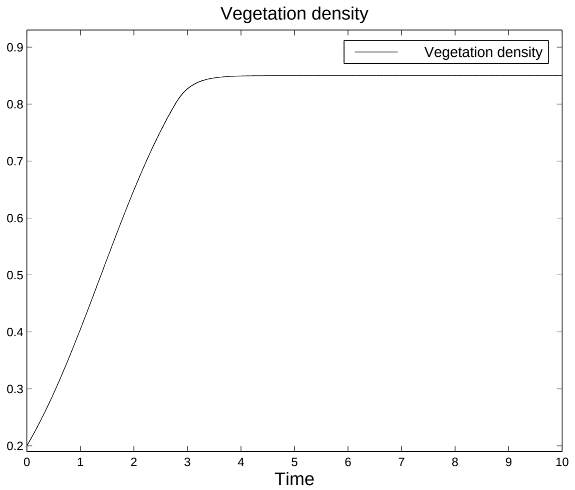

Fig. 3. Vegetation density $G(V)$, as a function of time, under continuous threshold policy with $c = 0.3$, $V_{gr} = 0.7$ and $V_{th} = 0.85$.

Pertinent remarks here are that the policy is continuous and so the problem of high-frequency on-off switching of the policy is removed. The vegetation rate still stabilizes at the designed threshold level V_{th}.

3.1 Robustness of threshold policies

This section shows that the threshold policy discussed in section 3 is robust to uncertainties in measurement. In the grazing model, such an uncertainty can occur either in the measurement of the vegetation V, and is denoted ΔV, or as a small delay Δt in the switching from one value of the control ϕ to the next. In order to model these uncertainties, the switching control is modified to: $\phi\left(V\left(t_\Delta\right)\right) = \varepsilon_1$, if $V\left(t_\Delta\right) > V_\Delta + \sigma$; $\varepsilon_1\left(\frac{V(t_\Delta) - V_{th} + \sigma}{2\sigma}\right)$, if $V_\Delta - \sigma \leq V\left(t_\Delta\right) \leq V_\Delta + \sigma$; 0, if $V\left(t_\Delta\right) < V_\Delta - \sigma$, where $t_\Delta = t - \Delta t$; $V_\Delta = V_{th} \mp \Delta V$; $\Delta V = 0.6 * \frac{(V_{max} - V_{gr})}{2}$; $V_{th} = \frac{V_{max} + V_{gr}}{2}$; $\varepsilon_1 = \frac{2g}{c_{\max}}\left(1 - \frac{V_{th}}{V_{max}}\right)$.

Figure 4 shows a simulation result from which it may be concluded that threshold policy is robust to errors in the measurement of the vegetation ΔV. Moreover, these errors may be fairly large ($\simeq 10.5\%$), as long as it is guaranteed that the switching threshold is in such a region as to guarantee appropriate virtual and real equilibria. The effect of the uncertainty ΔV is to introduce a corresponding error (offset) in the equilibrium value of V which goes from 0.85 ($= V_{th}$) in the unperturbed case, to 0.76 ($= V_{th} - \Delta V$) in the case where measurement error (ΔV) and delay (Δt) are both present. Furthermore, small delays in the application of switching are also tolerable. As is to be expected, small oscillations are introduced, but the system stabilizes in a neighborhood of the desired equilibrium. Calculations of estimates in measurement errors and proofs of robustness are omitted here for the lack of space, but can be found in (Meza *et al.*, 2001).

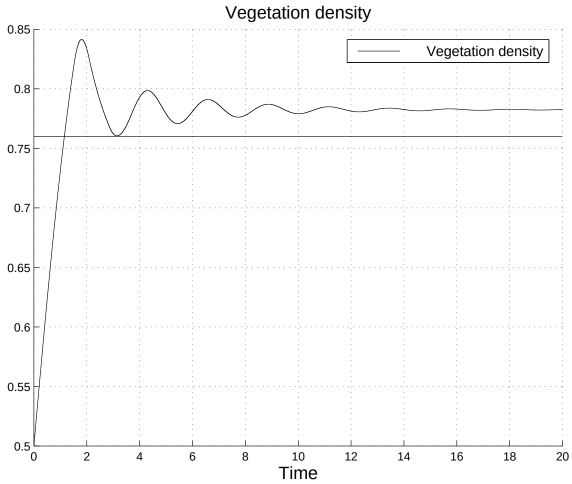

Fig. 4. Vegetation density $G(V)$, as a function of time, under continuous threshold policy with uncertainty and delay, where $g = 1$, $c = 0.3$, $V_{th} = 0.85$, $V_{gr} = 0.7$, $V_{max} = 1$, $\Delta t = 25\,h$ or $\Delta t = 0.625$time unit, $\Delta V = 0.09$, h is the integration span.

4. CONTINUOUS THRESHOLD POLICY FOR THE ROSENZWEIG–MACARTHUR MODEL

The Rosenzweig–MacArthur model, subject to the threshold policy on the predators is given by the equations below

$$\begin{aligned}\dot{x} &= x\left(r\left(1 - \frac{x}{K}\right) - \frac{y}{x + A}\right),\\ \dot{y} &= y\left(\frac{sA\left(x - J\right)}{\left(J + A\right)\left(x + A\right)} - \phi\left(y\right)\varepsilon_2\right),\end{aligned} \qquad (3)$$

where

$$\phi\left(y\right) = \begin{cases} 1 & \text{if} \quad y > y_{th} + \sigma \\ \left(\dfrac{y - y_{th} + \sigma}{2\sigma}\right) & \text{if} \quad y_{th} - \sigma \leq y \leq y_{th} + \sigma \\ 0 & \text{if} \quad y < y_{th} - \sigma. \end{cases}$$

The value of ε_2 for which the system stabilizes at the threshold value y_{th} must be calculated. System (3) submitted to this control consists of three structures: (i) no harvesting with $\phi = 0$; (ii) constant harvesting effort for predator $\phi = 1$; and (iii) linear harvesting effort for predator with $\phi = \frac{y - y_{th} + \sigma}{2\sigma}$. In (i) and (ii) the equilibrium points should be virtual, and in (iii) the equilibrium point should be real, i.e., the unique real equilibrium point belongs to this region. Analyzing the system in the region $y_{th} - \sigma \leq y \leq y_{th} + \sigma$, leads to:

$$x\left(r\left(1-\frac{x}{K}\right)-\frac{y}{x+A}\right)=0,$$
$$y\left(\frac{sA\,(x-J)}{(J+A)\,(x+A)}-\frac{\varepsilon_2}{2\sigma}\,(y-y_{th}+\sigma)\right)=0.$$

Since it is desired to stabilize the system at the threshold value y_{th}, this means that the stable equilibrium points are (x_i, y_{th}) for $i = 1, 2$, and ε_2 is given by the expression

$$\varepsilon_2 = \frac{2sA\,(x_i - J)}{(J+A)\,(x_i+A)},$$

while the x_i's are given by

$$x_{1,2} = \frac{K - A \pm \sqrt{(A-K)^2 - 4\left(\frac{K}{r}\,y_{th} - KA\right)}}{2}.$$

From the expression for ε_2, we must have $x_i > J$. A sample simulation of predator and prey densities is shown in Figure 5. In the simulations the following values were used: $x_1 = 44.0394$ and $x_2 = 5.9606$. From $x_1 > J$ the value of ε_2 is obtained as 0.2966 and the parameter values shown in the figure captions were used. Note that this system without control exhibits a stable limit cycle (Gurney and Nisbet, 1998), so that the threshold policy has successfully introduced a robustly stable equilibrium into the controlled system.

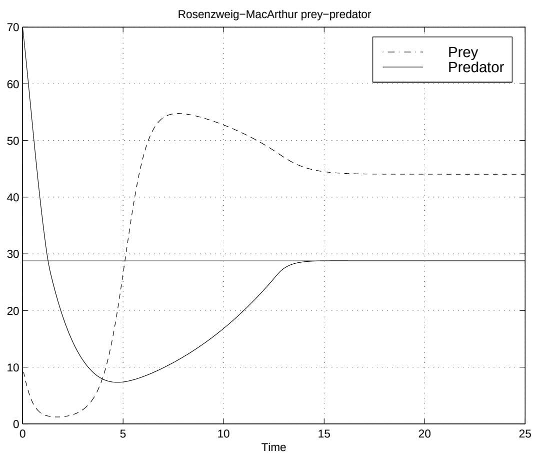

Fig. 5. Time evolution of predator and prey densities $x(t)$ and $y(t)$ under continuous threshold control with $r = 2$, $K = 60$, $s = 1$, $y_{th} = 28.75$, $A = 10$, $\varepsilon_2 = 0.2966$, $J = 20$ and $\sigma = 2.5$.

5. CONTINUOUS THRESHOLD POLICY AS A CONTROL STRATEGY IN A SIMPLE CHEMOSTAT

The system is given by the following equations

$$\begin{aligned}
\dot{x} &= (x^0 - x)D - \frac{m_1 xy}{a_1 + x} - \frac{m_2 xz}{a_2 + x}, & x(0) &= x_0,\\
\dot{y} &= y\left(\frac{m_1 x}{a_1 + x} - \bar{D}_1\right), & y(0) &= y_0 \quad (4)\\
\dot{z} &= z\left(\frac{m_2 x}{a_2 + x} - \phi\,(z)\,\bar{D}_2\right), & z(0) &= z_0,
\end{aligned}$$

where $\phi\,(z)$ is showed in equation (5).

In this strategy we need to know the value of ε_1 such that we can reach the desired equilibrium point, in this case we want to reach z_{min}. System (4) submitted to this control consists of three structures: (i) dilution rate with $\phi = \alpha$; (ii) dilution rate with $\phi = \varepsilon_1$; and (iii) linear dilution rate for microorganism with $\phi = \beta$. In (i) and (ii) the equilibrium points should be virtual, and in (iii) the equilibrium point should be real, i.e., the unique real equilibrium point belongs to this region. The threshold policy is given as follows

$$\phi\,(z) = \begin{cases} \varepsilon_1 & \text{if} \quad z > z_{min} + \sigma \\ \beta & \text{if} \quad z_{min} - \sigma \leq z \leq z_{min} + \sigma \\ \alpha & \text{if} \quad z < z_{min} - \sigma, \end{cases} \qquad (5)$$

where $\beta = \alpha + (\varepsilon_1 - \alpha)\,((z - z_{min} + \sigma)/2\sigma)$.

The system must be analyzed in the linear region $z_{min} - \sigma \leq z \leq z_{min} + \sigma$ in order to calculate an appropriate value of ε_1. After some algebra the equilibrium is calculated as $x_{eq} = (\bar{D}_1 a_1)/(m_1 - \bar{D}_1)$; $z_{eq} = z_{min}$ which implies that $\varepsilon_1 = \frac{2}{\bar{D}_2}\left(\frac{m_2 x_{eq}}{a_2 + x_{eq}} - \frac{\alpha}{2}\bar{D}_2\right)$.

Figure 6 shows the time plots of $x(t)$, $y(t)$ and $z(t)$, respectively.

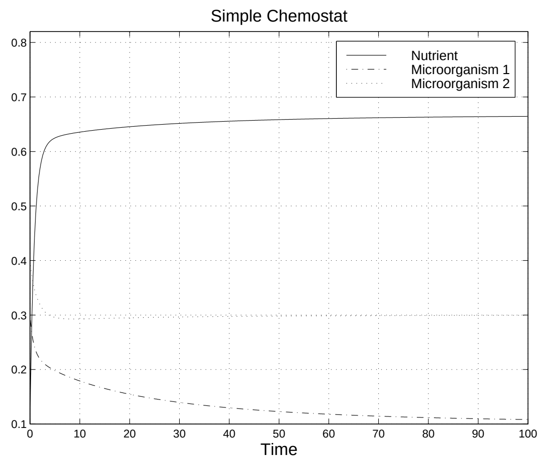

Fig. 6. Time evolution of the nutrient $x(t)$ and the microorganisms $y(t)$, $z(t)$ under continuous threshold policy with $\sigma = 0.015, \varepsilon_1 = 0.5909$.

This policy engenders coexistence of the microorganisms as can be seen in Figure 6, in contrast to the use of a proportional control, which results in extinction of microorganism 1 (Meza *et al.*, 2001).

6. CONCLUDING REMARKS

Simple on-off or threshold type policies, which are discontinuous, as well as their continuous versions, have been shown to be effective in the control of one (Noy–Meir) and two (Rosenzweig–MacArthur) species predator-prey type models commonly used in mathematical population biology. In addition a three-dimensional chemostat model is also controlled successfully.

The design of the continuous threshold policies is based on an analysis of the equilibria of the system: the control is chosen so as to make all the equilibria virtual, with the sole exception of the desired one. The novelty in this paper is the analysis via real and virtual equilibria, which the authors find intuitive for design purposes. Simulations in section 3.1 show that the strategies proposed here are robust to uncertainty of model parameters, initial conditions and delays between stock assessment and policy enactment. The latter may be described by a slight translation of the switching line, which does not alter the dynamics significantly, provided that the equilibrium points are all virtual. Moreover, some counter-intuitive results can be achieved by threshold policies. For example, in the herbivore-vegetation model under a threshold policy, maximum herbivore consumption (and consequently, production, assuming it is directly proportional to consumption) is guaranteed for high levels of herbivore densities, which would drive vegetation to extinction in the absence of this policy. Further details, such as plots of control inputs, comparison of discontinuous and continuous threshold policies, and proofs excluded here for lack of space can be found in (Meza *et al.*, 2001).

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