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ON THE CONTROL OF LINEAR SYSTEMS HAVING INTERNAL VARIATIONS, PART II–CONTROL

Moisés Bonilla Estrada [*] **Michel Malabre** [**]

[*] *CINVESTAV-IPN, Control Automático. AP 14-740. México 07000, MEXICO. mbonilla@enigma.red.cinvestav.mx.*
[**] *IRCCyN, CNRS UMR 6597, B.P. 92101, 44321 NANTES, Cedex 03, FRANCE. Michel.Malabre@irccyn.ec-nantes.fr.*

Abstract: In this paper we show how to embed the variable internal structure present in square implicit descriptions inside an (A, E, B) invariant subspace contained in the kernel of the output map. Thanks to this embedding, we make unobservable the variable internal structure, obtaining in this way a proper closed loop system with a controllable pre specified structure. *Copyright © 2002 IFAC.*

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1. INTRODUCTION

Given the implicit global description:

$$\Sigma^g : \ \mathbb{E}\,\dot{x}(t) = \mathbb{H}\,x(t) + \mathbb{B}\,u(t) \ \ ; \ \ y = Cx(t)$$
$$\mathbb{E} = \begin{bmatrix} E \\ 0 \end{bmatrix}; \ \mathbb{H} = \begin{bmatrix} A \\ D \end{bmatrix}; \ \mathbb{B} = \begin{bmatrix} B \\ 0 \end{bmatrix} \quad (1)$$

where $x(t)$, $u(t)$, and $y(t)$ are, respectively, the descriptor variable, the input, and the output. $\mathbb{E} : \mathcal{X} \to \underline{\mathcal{X}}_g$, $\mathbb{H} : \mathcal{X} \to \underline{\mathcal{X}}_g$, $\mathbb{B} : \mathcal{U} \to \underline{\mathcal{X}}_g$, and $C : \mathcal{X} \to \mathcal{Y}$ are linear operators, such that: [1]

$$\mathrm{Im}\,A + \mathcal{B} \subset \mathcal{E} \ \ ; \ \ \mathcal{E} \oplus \mathrm{Im}\,D = \underline{\mathcal{X}}_g \quad (2)$$

We are interested in finding a P.D. feedback, $u(t) = F_p x(t) + F_d \dot{x}(t)$, to solve the Problem stated in (Bonilla and Malabre 2002); namely for rendering unobservable the variation of structure, characterized by $0 = Dx(t)$, and assigning at will, for all the linear systems described by (1), the same external closed loop behaviour. The synthesis of the control law is based on the common internal structure, showed in (Bonilla and Malabre 2002), of the implicit system:

$$\Sigma^i : \quad E\dot{x}(t) = Ax(t) + Bu(t) \ ; \ y(t) = Cx(t) \quad (3)$$

[1] We write $\mathcal{B}$ and $\mathcal{E}$ to denote Im B and Im E, resp.

In Section 2 we show that the geometric condition

$$\dim(\mathrm{Ker}\,E) \le \dim\left(\mathcal{V}^* \cap E^{-1}\mathcal{B}\right) \quad (4)$$

where: [2] $\mathcal{V}^* = \sup\{\mathcal{T} \subset \mathcal{K}_C \mid A\mathcal{T} \subset E\mathcal{T} + \mathrm{Im}\,B\}$, guarantees the existence of a P.D. feedback which makes unobservable at the output the internal variation of structure. In Section 3, we show that:

$$\mathcal{R}^*_{\mathcal{X}} = \mathcal{X} \quad (5)$$

where $\mathcal{R}^*_{\mathcal{X}}$ is the reachability subspace, computed by: $\mathcal{R}^*_{\mathcal{X}} = \mathcal{V}^*_{\mathcal{X}} \cap \mathcal{S}^*_{\mathcal{X}}$, with: $\mathcal{V}^*_{\mathcal{X}} = \sup\{\mathcal{T} \subset \mathcal{X} \mid A\mathcal{T} \subset E\mathcal{T} + \mathrm{Im}\,B\}$, and $\mathcal{S}^*_{\mathcal{X}} = \inf\{\mathcal{T} \subset \mathcal{X} \mid \mathcal{T} = E^{-1}(A\mathcal{T} + \mathrm{Im}\,B)\}$, guarantees that the observable part of the closed loop system is controllable. And in Section 4 we conclude. For the notation concerning subspaces, and their associated algorithms, the reader is sent to Part I of this paper (Bonilla and Malabre 2002).

2. VARIABLE STRUCTURE REJECTION

In this Section we explore the usefulness of the (A, E, B) invariant subspace contained in $\mathcal{K}_C$,

[2] We also write $\mathcal{K}_{\mathrm{X}}$ to denote the kernel of a given X.

$\mathcal{V}^*_{\mathcal{K}_C,\Sigma^i}$, for making unobservable the variation of the internal structure in the input–output behaviour of the closed loop system. For this, in Theorem 1 we show that thanks to (4) we can embed the degree of freedom in $\mathcal{V}^*_{\mathcal{K}_C,\Sigma^i}$ (see (7)). And in Theorem 2 we show that the quotient system, $\widehat{\Sigma}^*_F(E_*, A_*, B_*, C_*)$ (see (24)), is actually strictly proper described by the state space description (27). In order to simplify the notation, in this Section we write $\mathcal{V}^*$ in place of $\mathcal{V}^*_{\mathcal{K}_C,\Sigma^i}$.

Theorem 1. If (2.a) and (4) hold, there then exists $(F^*_p\ ,\ \ F^*_d) \in \mathbf{F}(\mathcal{V}^*)$ such that:

$$\text{Ker } (E - BF^*_d) \approx \mathcal{K}_E \ ; \ \ \text{Im } (E - BF^*_d) = \mathcal{E} \quad (6)$$

$$\text{Ker } (E - BF^*_d) \subset \mathcal{V}^* \quad (7)$$

Proof of Theorem 1 This proof is done in 7 steps. We are going to write E_{F^*} and A_{F^*} to design $(E - BF^*_d)$ and $(A + BF^*_p)$, respectively.

1. Let us first decompose the space $\mathcal{X}$ and the subspaces $\mathcal{V}^*$, $E^{-1}\mathcal{B}$ and $\mathcal{K}_E$ as follows:

$$\begin{array}{c} \mathcal{X} = (\mathcal{V}^* + E^{-1}\mathcal{B}) \oplus \mathcal{X}_0 \\ \mathcal{V}^* = \mathcal{X}_{\mathcal{V}^*} \oplus (\mathcal{V}^* \cap E^{-1}\mathcal{B}) \\ E^{-1}\mathcal{B} = ((\mathcal{V}^* \cap E^{-1}\mathcal{B}) + \mathcal{K}_E) \oplus \mathcal{X}_3 \\ \mathcal{K}_E = (\mathcal{V}^* \cap \mathcal{K}_E) \oplus \mathcal{X}_{\mathcal{K}_E} \end{array} \quad (8)$$

where $\mathcal{X}_0$, $\mathcal{X}_{\mathcal{V}^*}$, $\mathcal{X}_3$, and $\mathcal{X}_{\mathcal{K}_E}$ are any complementary subspaces. Now in view of (4), there then exist $\mathcal{X}_1$ and $\mathcal{X}_2$, both subspaces of $E^{-1}\mathcal{B}$, such that (recall (8.d)):

$$\left\{ \begin{array}{c} \mathcal{V}^* \cap E^{-1}\mathcal{B} = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus (\mathcal{V}^* \cap \mathcal{K}_E) \\ \text{with:} \ \ \mathcal{X}_2 \approx \mathcal{X}_{\mathcal{K}_E} \end{array} \right. \quad (9)$$

From (8) and (9), $\mathcal{X}$ can be decomposed as:

$$\begin{array}{l} \mathcal{X} = \mathcal{X}_{\mathcal{V}^*} \oplus \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus (\mathcal{V}^* \cap \mathcal{K}_E) \oplus \mathcal{X}_{\mathcal{K}_E} \\ \qquad\qquad\qquad\qquad\qquad \oplus \mathcal{X}_3 \oplus \mathcal{X}_0 \\ \mathcal{V}^* = \mathcal{X}_{\mathcal{V}^*} \oplus \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus (\mathcal{V}^* \cap \mathcal{K}_E) \\ E^{-1}\mathcal{B} = \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus (\mathcal{V}^* \cap \mathcal{K}_E) \oplus \mathcal{X}_{\mathcal{K}_E} \oplus \mathcal{X}_3 \end{array} \quad (10)$$

2. Let us next note that:

$$\widehat{N} A \mathcal{V}^* \subset \widehat{N} E \mathcal{V}^* \quad (11)$$

$$\widehat{N} : \ \mathcal{X} \to \mathcal{E}/\mathcal{B} \ \text{ is the canonical projection} \quad (12)$$

Indeed, for any $(\overline{F}_p,\ \overline{F}_d) \in \mathbf{F}(\mathcal{V}^*)$: $(A + B\overline{F}_p)\mathcal{V}^* \subset (E - B\overline{F}_d)\mathcal{V}^*$, then (11) follows from (12).

3. Let us now define the control law. For this, we define the following natural projections ($\underline{\mathcal{X}}_0$ is any fixed complement of $\mathcal{B}$ in $\mathcal{E}$, i.e. $\mathcal{E} = \mathcal{B} \oplus \underline{\mathcal{X}}_0$):[3]

$$\left\{ \begin{array}{l} P : \mathcal{E} \to \mathcal{B}\ , \ \ \text{along } \underline{\mathcal{X}}_0\ , \ \ \text{such that: } PB = \mathrm{I} \\ \qquad Q_{\mathcal{V}^*} : \mathcal{X} \to \mathcal{V}^* \ \ ; \ \ Q_{\mathcal{X}_{\mathcal{V}^*}} : \mathcal{X} \to \mathcal{X}_{\mathcal{V}^*} \\ \qquad Q_2 : \mathcal{X} \to \mathcal{X}_2 \ \ ; \ \ Q_{\mathcal{X}_{\mathcal{K}_E}} : \mathcal{X} \to \mathcal{X}_{\mathcal{K}_E} \end{array} \right. \quad (13)$$

[3] The natural projections $Q_{\mathcal{X}_i}$ are projected along the complementary subspaces of $\mathcal{X}_i$, shown in (10).

the following insertion maps:

$$\left\{ \begin{array}{l} R_{\mathcal{V}^*} : \mathcal{V}^* \to \mathcal{X} \ ; \ R_{\mathcal{X}_{\mathcal{V}^*}} : \mathcal{X}_{\mathcal{V}^*} \to \mathcal{X} \\ R_2 : \ \mathcal{X}_2 \to \mathcal{X} \ \ ; \ R_{\mathcal{X}_{\mathcal{K}_E}} : \ \mathcal{X}_{\mathcal{K}_E} \to \mathcal{X} \end{array} \right. \quad (14)$$

such that $Q_{\mathcal{X}_i} R_{\mathcal{X}_i} = \mathrm{I}$, note that $Q_{\mathcal{X}_i} R_{\mathcal{X}_j} = 0$ for all $i \neq j$. Let us define the isomorphism

$$T : \ \mathcal{X}_{\mathcal{K}_E} \longleftrightarrow \mathcal{X}_2 \quad (15)$$

Let us then define the maps, $F^*_p : \ \mathcal{X} \ \to \ \mathcal{U}$ and $F^*_d : \ \mathcal{X} \ \to \ \mathcal{U}$, as follows:

$$\begin{array}{l} F^*_d = PE \left(R_2 T Q_{\mathcal{X}_{\mathcal{K}_E}} \left(R_{\mathcal{X}_{\mathcal{V}^*}} Q_{\mathcal{X}_{\mathcal{V}^*}} + R_2 Q_2 \right) \right) \\ F^*_p = -PAR_{\mathcal{V}^*} Q_{\mathcal{V}^*} \end{array} \quad (16)$$

Applying this P.D. control law, $u(t) = F^*_p x(t) + F^*_d \dot{x}(t) + v(t)$, to system (3), we get from (16), (13), (14), and (15):

$$\left\{ \begin{array}{l} PE_{F^*} \left| \mathcal{X}_{\mathcal{V}^*} \right. = PE_{F^*} R_{\mathcal{X}_{\mathcal{V}^*}} = 0 \\ PE_{F^*} \left| \mathcal{X}_1 \right. = PE \left| \mathcal{X}_1 \right. \\ PE_{F^*} \left| \mathcal{X}_2 \right. = PE_{F^*} R_2 = 0 \\ PE_{F^*} \left| (\mathcal{V}^* \cap \mathcal{K}_E) \right. = PE \left| (\mathcal{V}^* \cap \mathcal{K}_E) \right. \\ PE_{F^*} \left| \mathcal{X}_{\mathcal{K}_E} \right. = PE_{F^*} R_{\mathcal{X}_{\mathcal{K}_E}} = (PE \left| \mathcal{X}_2 \right.) T \\ PE_{F^*} \left| \mathcal{X}_3 \right. = PE \left| \mathcal{X}_3 \right. \ \ ; \ \ PE_{F^*} \left| \mathcal{X}_0 \right. = PE \left| \mathcal{X}_0 \right. \end{array} \right. \quad (17)$$

$$\left\{ \begin{array}{l} PA_{F^*} \left| \mathcal{V}^* \right. = PA_{F^*} R_{\mathcal{V}^*} = 0 \\ PA_{F^*} \left| \mathcal{X}_i \right. = PA \left| \mathcal{X}_i \right. \ , \ \ i \in \{\mathcal{K}_E, 3, 0\} \end{array} \right. \quad (18)$$

Let us note that (18) and (11) imply that $(F^*_p,\ F^*_d) \in \mathbf{F}(\mathcal{V}^*)$.

4. From (10) we have that the map $PE \left| E^{-1}\mathcal{B} \right. :$ $E^{-1}\ \mathcal{B} \ \to \ \mathcal{B}$ is characterized by the following matrix:

$$M = \left[\, X_1\ \ X_2\ \ 0\ \ 0\ \ X_3 \,\right] \quad (19)$$

$$X_1 = PE \left| \mathcal{X}_1 \right. ; \ \ X_2 = PE \left| \mathcal{X}_2 \right. ; \ \ X_3 = PE \left| \mathcal{X}_3 \right. \quad (20)$$

The map $\left[\, X_1\ \ X_2\ \ X_3 \,\right]$ is an isomorphism.
Indeed, since $\mathcal{B} \subset \mathcal{E}$, we get: Im $M = PEE^{-1}\mathcal{B} = P\,(\mathcal{E} \cap \mathcal{B}) = P\mathcal{B} = \mathcal{B}$, which implies that M is epic, namely, so does $\left[\, X_1\ \ X_2\ \ X_3 \,\right]$. On the other hand (recall (2.a) and (10)): $\dim \mathcal{K}_M \ = \ \dim E^{-1}\mathcal{B} \ - \dim \mathcal{B} = \dim \mathcal{K}_E + \dim\,(\mathcal{B} \cap \mathcal{E}) - \dim \mathcal{B} = \dim \mathcal{K}_E = \dim E^{-1}\mathcal{B} - (\dim \mathcal{X}_1 + \dim \mathcal{X}_2 + \dim \mathcal{X}_3)$, and thus, the map $\left[\, X_1\ \ X_2\ \ X_3 \,\right]$ is monic.

5. Noting that for any feedback $F_d : \mathcal{X} \to \mathcal{U}$:

$$E^{-1}\mathcal{B} = (E - BF_d)^{-1}\mathcal{B} \quad (21)$$

We have that the operator $PE_{F^*} \left| E^{-1}\mathcal{B} \right. : E^{-1}\mathcal{B} \to \mathcal{B}$ is characterized by the following matrix (recall (10), (17), and (20)):

$$M_F = \left[\, X_1\ \ 0\ \ 0\ \ X_2\ \ X_3 \,\right] \quad (22)$$

Now, in view that (recall (13) and (21)):

$$\left\{ \begin{array}{l} \mathcal{K}_M = E^{-1}\mathcal{B} \cap E^{-1}\mathcal{K}_P = \mathcal{K}_E \\ \mathcal{K}_{M_F} = E^{-1}_{F^*}\mathcal{B} \cap E^{-1}_{F^*}\mathcal{K}_P = \mathcal{K}_{E_{F^*}} \end{array} \right. \quad (23)$$

from (19) and (22), we get: $\mathcal{K}_{E_{F^*}} \approx \mathcal{K}_E$.
6. From (2.a) and (6.a) we get (6.b).
7. In view of (22), (23.b), and (10), we get:[4] Ker $E_{F^*} \subset \mathcal{V}^*$. □

Theorem 2. Let $(F_p^*, F_d^*) \in \mathbf{F}(\mathcal{V}^*)$, as in Theorem 1. There then exist unique maps E_*, A_*, B_*, and C_* satisfying:

$$\begin{array}{c} \Pi A_{F^*} = A_* \Phi \ ; \ \Pi E_{F^*} = E_* \Phi \\ \Pi B = B_* \ ; \ C = C_* \Phi \end{array} \qquad (24)$$

where: $E_{F^*} = (E - BF_d^*)$, $A_{F^*} = (A + BF_p^*)$ and

$$\begin{array}{lll} \Pi : \mathcal{E} \to \mathcal{E}/E_{F^*}\mathcal{V}^* & , & \text{the canonical projection} \\ \Phi : \mathcal{X} \to \mathcal{X}/\mathcal{V}^* & , & \text{the canonical projection} \end{array} \qquad (25)$$

Moreover, the induced map E_* is an isomorphism. Furthermore, when the system (1) is fed back with the pair (F_p^*, F_d^*), namely

$$u(t) = F_p^* x(t) + F_d^* \dot{x}(t) + v(t) \qquad (26)$$

it is externally equivalent to the quotient system:

$$\dot{\hat{x}} = E_*^{(-1)} A_* \hat{x} + E_*^{(-1)} B_* v \ ; \ y = C_* \hat{x} \qquad (27)$$

where: $\hat{x} = \Phi x$.

Proof of Theorem 2 Proved in 4 steps:
1. (24) is satisfied. Since $(F_p^*, F_d^*) \in \mathbf{F}(\mathcal{V}^*)$, then $A_{F^*}\mathcal{V}^* \subset E_{F^*}\mathcal{V}^*$, which implies: $A_{F^*}\mathcal{K}_\Phi \subset \mathcal{K}_\Pi$ and $E_{F^*}\mathcal{K}_\Phi = \mathcal{K}_\Pi$.
2. E_* is an epic map. From (6.b), (24) and (25) we get: Im $E_* = E_* \mathcal{X}/\mathcal{V}^* = E_* \Phi \mathcal{X} = \Pi E_{F^*} \mathcal{X} = \Pi E \mathcal{X} = \mathcal{E}/E_{F^*}\mathcal{V}^*$.
3. E_* is a monic map. From previous item, (7) and (6.a), we get: $\dim \mathcal{K}_{E_*} = \dim \mathcal{X}/\mathcal{V}^* - \dim \text{Im}\, E_* = \dim \mathcal{X}/\mathcal{V}^* - \dim \mathcal{E}/E_{F^*}\mathcal{V}^* = \dim \mathcal{K}_E - \dim(\mathcal{V}^* \cap \mathcal{K}_{E_{F^*}}) = \dim \mathcal{K}_E - \dim \mathcal{K}_{E_{F^*}} = 0$.
4. Quotient System. applying the control law (26) to system (3) we get (recall (24)):

$$E_* \dot{\hat{x}} = A_* \hat{x} + B_* v \ ; \ y = C_* \hat{x}$$

Let us define the monic maps, $\Upsilon : \mathcal{X}/\mathcal{V}^* \to \mathcal{X}$ and $L_3 : \mathcal{E}/E_{F^*}\mathcal{V}^* \to \mathcal{E}$, such that $\mathcal{X} = \mathcal{V}^* \oplus \text{Im}\ \Upsilon$ and $\mathcal{E} = E_{F^*}\mathcal{V}^* \oplus \text{Im}\ L_3$. In these bases the matrix representation of the closed loop system, (1) fed back by (26), is:

$$\begin{bmatrix} Y_1 & Y_2 \\ 0 & E_* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\tilde{x}} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \\ 0 & A_* \\ D_{i1} & D_{i2} \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \hat{x} \end{bmatrix} + \begin{bmatrix} 0 \\ B_* \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & C_* \end{bmatrix} \begin{bmatrix} \tilde{x} & \hat{x} \end{bmatrix}^T$$

which implies (27). This pencil is solvable since E_* is an isomorphism and the maps $Y_1 = E_{F^*}\ \mathcal{V}^* | \ E_{F^*} | \mathcal{V}^*$ and $D_i : \mathcal{X} \to \underline{\mathcal{X}}_D$ are epic. □

[4] Recall that the map $\begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}$ is an isomorphism.

3. CLOSED LOOP REACHABILITY

In this Section we show, in Theorem 3, that the reachability condition (5) implies the reachability of the closed loop system (27).

Theorem 3. Given any pair (F_p^*, F_d^*), as in Theorem 2, and the closed loop system, $\Sigma_{F^*}^i : E_{F^*}\dot{x} = A_{F^*}x + Bv$, and if in addition $\mathcal{R}_{\mathcal{X},\Sigma^i}^* = \mathcal{X}$, then:

$$\mathcal{R}_{\mathcal{X},\Sigma_{F^*}^i}^* = \mathcal{X} \qquad (28)$$

$$\sum_{j=0}^{n-1} \left(E_*^{(-1)} A_* \right)^j \text{Im} \ \left(E_*^{(-1)} B \right) = \mathcal{X}/\mathcal{V}_{\mathcal{K}_C,\Sigma^i}^* \qquad (29)$$

We need the following Lemma:

Lemma 4. Given any pair of feedbacks $F_d : \mathcal{X} \to \mathcal{U}$ and $F_p : \mathcal{X} \to \mathcal{U}$, and the closed loop system $\Sigma_F^i : E_F \dot{x} = A_F x + Bv$, with $E_F = E - BF_d$ and $A_F = A + BF_p$, such that $\mathcal{K}_{E_F} \approx \mathcal{K}_E$ and if Im $A + \mathcal{B} \subset \mathcal{E}$, then:

$$\text{Im}\ A_F + \mathcal{B} \subset \text{Im}\ E_F \qquad (30)$$

$$\mathcal{V}_{\mathcal{X},\Sigma_F^i}^* = \mathcal{X} \ ; \ \mathcal{R}_{\mathcal{X},\Sigma_F^i}^* = \mathcal{S}_{\mathcal{X},\Sigma_F^i}^* \qquad (31)$$

Moreover, the algorithm to compute $\mathcal{S}_{\mathcal{X},\Sigma_F^i}^*$ satisfies the following two equations:

$$\begin{aligned} \mathcal{S}_{\mathcal{X},\Sigma_F^i}^{k+1} &= (E^{-1}A)^{k+1}\mathcal{K}_{E_F} \\ &\quad + (E^{-1}A)^k E^{-1}\mathcal{B} + \mathcal{S}_{\mathcal{X},\Sigma^i}^k \end{aligned} \qquad (32)$$

$$\begin{aligned} \mathcal{S}_{\mathcal{X},\Sigma_F^i}^{k+1} &= (E_F^{-1}A_F)^{k+1}\mathcal{K}_{E_F} \\ &\quad + \sum_{j=0}^{k} (E_F^{-1}A_F)^j E_F^{-1}\mathcal{B} \end{aligned} \qquad (33)$$

Furthermore, if $\mathcal{K}_{E_F} \subset \mathcal{V}_{\mathcal{K}_C,\Sigma^i}^*$ and $(F_p,\ F_d) \in \mathbf{F}(\mathcal{V}_{\mathcal{K}_C,\Sigma^i}^*)$, then:

$$(E_F^{-1}A_F)^{k+1}\mathcal{K}_{E_F} \subset \mathcal{V}_{\mathcal{K}_C,\Sigma^i}^* \qquad (34)$$

Proof of Lemma 4
1. Since $\mathcal{K}_{E_F} \approx \mathcal{K}_E$ implies Im $E_F = \mathcal{E}$, we get together with Im $A + \mathcal{B} \subset \mathcal{E}$ that: Im $A_F + \mathcal{B} \subset$ Im $A + \mathcal{B} \subset \mathcal{E} =$ Im E_F. From[5] [7] and (30), we get: $\mathcal{V}_{\mathcal{X},\Sigma_F^i}^0 = \mathcal{X}$ and $\mathcal{V}_{\mathcal{X},\Sigma_F^i}^1 = A_F^{-1}$ (Im E_F + $\mathcal{B}$) $= \mathcal{X}$, which implies (31.a). Equation (31.b) directly follows from (31.a) and [10].
2. Let us note that: $E_F^{-1}\ (A_F\mathcal{S} + \mathcal{B}) = E^{-1}(A\mathcal{S} + \mathcal{B})$ (this is a standard result, see for instance Wonham 1985). Then, from [8], (2.a), and noting that $\mathcal{B} + \text{Im}\ A \subset \mathcal{E}$ implies:

$$E^{-1}(A\mathcal{S} + B\mathcal{T}) = E^{-1}A\mathcal{S} + E^{-1}B\mathcal{T} \qquad (35)$$

[5] We write [×] for the equation's numbers which are in (Bonilla and Malabre 2002), Part I of this paper.

we have that the remaining steps, $i \geq 0$, of the algorithm to compute $\mathcal{S}^*_{\mathcal{X},\Sigma^i_F}$ are also equal to:

$$\mathcal{S}^{i+1}_{\mathcal{X},\Sigma^i_F} = E^{-1}A\mathcal{S}^i_{\mathcal{X},\Sigma^i_F} + E^{-1}\mathcal{B} \quad (36)$$

Let us now note that (2.a), [8], (36) and (35) imply that: $\mathcal{S}^0_{\mathcal{X},\Sigma^i_F} = \mathcal{K}_{E_F}$, $\mathcal{S}^1_{\mathcal{X},\Sigma^i_F} = E^{-1}A\mathcal{K}_{E_F} + E^{-1}\mathcal{B} = E^{-1}A\mathcal{K}_{E_F} + \mathcal{K}_E + E^{-1}\mathcal{B}$, $\mathcal{S}^2_{\mathcal{X},\Sigma^i_F} = \left(E^{-1}A\right)^2\mathcal{K}_{E_F} + (E^{-1}A)\mathcal{K}_E + (E^{-1}A)E^{-1}\ \mathcal{B} + E^{-1}\ \mathcal{B}, \ldots, \mathcal{S}^{k+1}_{\mathcal{X},\Sigma^i_F} = \left(E^{-1}A\right)^{k+1}\mathcal{K}_{E_F} + (E^{-1}A)^k\ \mathcal{K}_E + (E^{-1}A)^k\ E^{-1}\ \mathcal{B} + \sum_{j=0}^{k-1}(E^{-1}A)^j\ E^{-1}\ \mathcal{B} = \left(E^{-1}A\right)^{k+1}\ \mathcal{K}_{E_F} + (E^{-1}A)^k\ E^{-1}\ \mathcal{B} + \mathcal{S}^k_{\mathcal{X},\Sigma^i}$. Equation (33) directly follows from [8], (30) and (35).

3. Let us first note that: $(F_p,\ F_d) \in \mathbf{F}(\mathcal{V}^*_{\mathcal{K}_C,\Sigma^i})$, implies that: $A_F\ \mathcal{V}^*_{\mathcal{K}_C,\Sigma^i} \subset E_F\ \mathcal{V}^*_{\mathcal{K}_C,\Sigma^i}$. Let us next note that: $\mathcal{K}_{E_F} \subset \mathcal{V}^*_{\mathcal{K}_C,\Sigma^i}$, implies that: $A_F\ \mathcal{K}_{E_F} \subset A_F\ \mathcal{V}^*_{\mathcal{K}_C,\Sigma^i} \subset E_F\mathcal{V}^*_{\mathcal{K}_C,\Sigma^i}$, namely: $E_F^{-1}\ A_F\ \mathcal{K}_{E_F} \subset E_F^{-1}E_F\mathcal{V}^*_{\mathcal{K}_C,\Sigma^i} = \mathcal{V}^*_{\mathcal{K}_C,\Sigma^i} + \mathcal{K}_{E_F} = \mathcal{V}^*_{\mathcal{K}_C,\Sigma^i}$. And thus: $(E_F^{-1}\ A_F)^{k+1}\ \mathcal{K}_{E_F} \subset \mathcal{V}^*_{\mathcal{K}_C,\Sigma^i}$.

Proof of Theorem 3 Since $\mathcal{R}^*_{\mathcal{X},\Sigma^i} = \mathcal{X}$ and $\mathcal{R}^*_{\mathcal{X},\Sigma^i} = \mathcal{V}^*_{\mathcal{X},\Sigma^i} \cap \mathcal{S}^*_{\mathcal{X},\Sigma^i}$, then: $\mathcal{S}^*_{\mathcal{X},\Sigma^i} = \mathcal{X}$, and thus, equation (28) directly follows from (31.b) and (32). Equation (29) is proved in 4 steps:

1. Let us first show that:

$$\Phi E^{-1}_{F^*}\mathcal{S} \subset E^{-1}_*\Pi\mathcal{S} \quad (37)$$

Indeed, let $x \in \Phi E^{-1}_{F^*}\mathcal{S}$, there then exist $s \in \mathcal{S}$ and $z \in E^{-1}_{F^*}\mathcal{S}$ such that: $x = \Phi z$ and $E_{F^*}z = s$, which imply (recall (24.b)): $E_*x = E_*\Phi z = \Pi E_{F^*}z = \Pi s$, namely: $x \in E_*^{-1}\Pi s$.

2. Let us next note that (34) and (25.b) imply:

$$\Phi\left(E^{-1}_{F^*}A_{F^*}\right)^{n+1}\mathcal{K}_{E_{F^*}} \subset \Phi\mathcal{V}^*_{\mathcal{K}_C,\Sigma^i} = \{0\} \quad (38)$$

3. Let us now show that (37) and (24) imply:

$$\begin{array}{l}\Phi(E^{-1}_{F^*}A_{F^*})(E^{-1}_{F^*}A_{F^*})\cdots(E^{-1}_{F^*}A_{F^*})E^{-1}_{F^*}\mathcal{B} \\ \subset E^{-1}_*\Pi A_{F^*}(E^{-1}_{F^*}A_{F^*})\cdots(E^{-1}_{F^*}A_{F^*})E^{-1}_{F^*}\mathcal{B} \\ \subset (E^{-1}_*A_*)(E^{-1}_*A_*)\cdots(E^{-1}_*A_*)E^{-1}_*\mathrm{Im}\ B_*\end{array} \quad (39)$$

4. Finally, from (28), (31.b), (33), (25.b), (38) and (39), we have that: $\mathcal{X} = \mathcal{R}^*_{\mathcal{X},\Sigma^i_{F^*}} = (E^{-1}_{F^*}A_{F^*})^{n+1}\mathcal{K}_{E_{F^*}} + \sum_{j=0}^n(E^{-1}_{F^*}A_{F^*})^jE^{-1}_{F^*}\mathcal{B}$, and that: $\Pi\mathcal{X} = \mathcal{X}/\mathcal{V}^*_{\mathcal{X},\Sigma^i} = \sum_{j=0}^n\Phi(E^{-1}_{F^*}A_{F^*})^jE^{-1}_{F^*}\mathcal{B} \subset \sum_{j=0}^n(E^{-1}_*A_*)^jE^{-1}_*\mathrm{Im}\ B_* \subset \mathcal{X}/\mathcal{V}^*_{\mathcal{X},\Sigma^i}$. Namely: $\sum_{j=0}^n(E^{-1}_*A_*)^jE^{-1}_*\mathrm{Im}\ B_* = \mathcal{X}/\mathcal{V}^*_{\mathcal{X},\Sigma^i}$, which together with the Cayley-Halmiton Theorem, gives (29) (recall that we have also proved in Theorem 2 that the induced map E_* is an isomorphism). □

Let us come back to the illustrative example [2]:

$$\begin{array}{rl}\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}\dot{x}(t) =& \begin{bmatrix}0&1&-1\\1&0&-1\end{bmatrix}x(t) + \begin{bmatrix}0\\1\end{bmatrix}u(t)\\ 0 =& \begin{bmatrix}\alpha&\beta&1\end{bmatrix}x(t)\\ y(t) =& \begin{bmatrix}0&0&1\end{bmatrix}x(t)\end{array}$$

Let us note that: $\mathcal{V}^* = \mathcal{K}_C = \{e_1, e_2\}$, $\mathcal{K}_E = \{e_3\}$, and $E^{-1}\mathcal{B} = \{e_2, e_3\}$, then (see (8), (9) and (10): $\mathcal{X}_{\mathcal{V}^*} = \{e_1\}$, $\mathcal{X}_1 = \{0\}$, $\mathcal{X}_2 = \{e_2\}$, $\mathcal{V}^*\cap\mathcal{K}_E = \{0\}$, $\mathcal{X}_{\mathcal{K}_E} = \{e_3\}$, $\mathcal{X}_3 = \{0\}$, $\mathcal{X}_0 = \{0\}$. From (13), (14) and (15), we have: $P = \begin{bmatrix}0&1\end{bmatrix}$, $Q_{\mathcal{V}^*} = \begin{bmatrix}1&0&0\\0&1&0\end{bmatrix} = R^T_{\mathcal{V}^*}$, $Q_{\mathcal{X}_{\mathcal{V}^*}} = \begin{bmatrix}1&0&0\end{bmatrix} = R^T_{\mathcal{X}_{\mathcal{V}^*}}$, $Q_{\mathcal{X}_2} = \begin{bmatrix}0&1&0\end{bmatrix} = R^T_{\mathcal{X}_2}$, $Q_{\mathcal{X}_{\mathcal{K}_E}} = \begin{bmatrix}0&0&1\end{bmatrix} = R^T_{\mathcal{X}_{\mathcal{K}_E}}$, and $T^{\mathcal{X}_2}_{\mathcal{X}_{\mathcal{K}_E}} = \begin{bmatrix}-1\end{bmatrix}$; which together with (16) imply: $F^*_{d1} = \begin{bmatrix}0&-1&-1\end{bmatrix}$ and $F^*_{p1} = \begin{bmatrix}-1&0&0\end{bmatrix}$. In view of Theorem 3, we can assign the dynamics of the strictly proper quotient system (27), so we choose: $F^*_{p2} = \begin{bmatrix}0&0&(1-1/\tau_0)\end{bmatrix}$. And thus, $u = F^*_{d1}\ \dot{x} + (F^*_{p1} + F^*_{p2})x + R/\tau_0$. The closed loop system is:

$$\begin{array}{rl}\begin{bmatrix}1&0&0\\0&0&\mathbf{1}\end{bmatrix}\dot{x}(t) =& \begin{bmatrix}0&1&-1\\0&0&-\mathbf{1}/\tau_{\mathbf{0}}\end{bmatrix}x + \begin{bmatrix}0\\\mathbf{1}/\tau_{\mathbf{0}}\end{bmatrix}\mathrm{R}\\ y(t) =& \begin{bmatrix}0&0&\mathbf{1}\end{bmatrix}x\end{array}$$

And thus, the degree of freedom has been made unobservable, in other words, the *variation of structure* is no longer present at the output. The closed loop system behaves as: $\tau_0\dot{y}(t) + y(t) = \mathrm{R}(t)$, whatever be the active constraint: $0 = \begin{bmatrix}\alpha&\beta&1\end{bmatrix}x(t)$. For details on the practical implementation of the proposed P.D. control law by a proper controller, see (Bonilla *et al.* 1993). And for details on the synthesis of a descriptor variable reconstructor see (Bonilla *et al.* 2000).

4. CONCLUDING REMARKS

We have proposed here a synthesis procedure for giving the closed loop system an unique and pre specified external behaviour.

Our approach is based on the use of geometric tools (see Wonham 1985). Namely, our control law makes unobservable the degree of freedom which was initialy present in the external behaviour of the system (see Theorems 1 and 2).
Moreover, under some controllability condition (which includes the effect of the degree of freedom), the strictly proper closed loop system obtained in that way not only has an unique behaviour, but also is completely controllable by its control input (see Theorem 3).
Our solution is based on the use of $(P,\ D)$ feedbacks which are *friends* of the supremal $(A,\ E,\ B)$ invariant subspace included in $\mathcal{K}_C$.

5. REFERENCES

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APPENDIX FOR PART I PROOFS OF (BONILLA AND MALABRE 2002)

Remember that [×] refers to equation's number (×) in (Bonilla and Malabre 2002), Part I of this paper.

Proof of Lemma 5 Proved in the 3 steps:

1. *The internal properness is equivalent to:*

$$\mathcal{V}^*_{\mathcal{K}_D,\Sigma_0^i} \oplus \mathcal{K}_E = \mathcal{X} \; ; \; \mathcal{S}^*_{\mathcal{K}_D,\Sigma_0^i} = \mathcal{K}_E \qquad \text{(A.1)}$$

Let us first note that regularity implies:

$$\mathcal{X} = \mathcal{A}^*_{1,\Psi^g} \oplus \mathcal{A}^*_{2,\Psi^g} \qquad \text{(A.2)}$$

Let us now note that $\mathcal{A}^*_{1,\Psi^g} = \mathcal{K}_E$. Indeed the absence of infinite zeros of order greater than one is equivalent to card{infinite elementary divisors of order ≥ 2} $= 0$, which in turns is equivalent to (see Malabre 1989):

$$\dim\left(\frac{\mathcal{A}^*_{2,\Psi^g} + \mathcal{A}^2_{1,\Psi^g}}{\mathcal{A}^*_{2,\Psi^g} + \mathcal{A}^1_{1,\Psi^g}}\right) = 0 \qquad \text{(A.3)}$$

Furthermore, in view that the geometric algorithm [13] is nondecreasing, i.e. $\mathcal{A}^{\mu}_{1,\Psi^g} \subset \mathcal{A}^{\mu+1}_{1,\Psi^g}$, (A.2) implies $\mathcal{A}^*_{2,\Psi^g} + \mathcal{A}^{\mu}_{1,\Psi^g} = \mathcal{A}^*_{2,\Psi^g} \oplus \mathcal{A}^{\mu}_{1,\Psi^g}$, which together with (A.3) imply:

$$\mathcal{A}^1_{1,\Psi^g} = \mathcal{A}^2_{1,\Psi^g} = \mathcal{A}^*_{1,\Psi^g} \qquad \text{(A.4)}$$

Then, from [12] and (A.4) we get (recall [16]): $\mathcal{A}^*_{1,\Psi^g} = \mathcal{A}^1_{1,\Psi^g} = \mathbb{E}^{-1}\,\mathbb{H}\,\mathcal{A}^0_{1,\Psi^g} = \text{Ker}\;\mathbb{E} = \mathcal{K}_E$. *From [12] and [16], we get by simple induction:* $\mathcal{A}^{\mu+1}_{1,\Psi^g} = \mathcal{S}^{\mu}_{\mathcal{K}_D,\Sigma_0^i}$. *Finally from [13] and [7], we get by simple induction* $\mathcal{A}^{\mu}_{2,\Psi^g} = \mathcal{V}^{\mu}_{\mathcal{K}_D,\Sigma_0^i}$.

2. *We next show that:* $\mathcal{K}_D \cap \mathcal{K}_E \subset \mathcal{K}_A$. Indeed, the limit of [8] satisfies $\mathcal{S}^*_{\mathcal{K}_D,\Sigma_0^i} = E^{-1}A(\mathcal{K}_D \cap \mathcal{S}^*_{\mathcal{K}_D,\Sigma_0^i})$, that is because of (A.1.b): $E^{-1}A(\mathcal{K}_D \cap \mathcal{K}_E) = \mathcal{K}_E$, which implies: $\mathcal{E} \cap A(\mathcal{K}_D \cap \mathcal{K}_E) = \{0\}$, and, due to [17.a]: $A(\mathcal{K}_D \cap \mathcal{K}_E) = \{0\}$.

3. We are now in position to prove that properness implies [18]:

Let us first note that (A.1.a) implies:

$$E\mathcal{V}^*_{\mathcal{K}_D,\Sigma_0^i} = \mathcal{E} \; ; \; \mathcal{V}^*_{\mathcal{K}_D,\Sigma_0^i} \cap \mathcal{K}_E = \{0\} \qquad \text{(A.5)}$$

Let us next show that:

$$A\mathcal{V}^*_{\mathcal{K}_D,\Sigma_0^i} = A\mathcal{K}_D \; ; \; A\mathcal{K}_D + A\mathcal{K}_E = \text{Im}\;A \qquad \text{(A.6)}$$

Indeed, the limit of [7] satisfies $\mathcal{V}^*_{\mathcal{K}_D,\Sigma_0^i} = \mathcal{K}_D \cap A^{-1}E\mathcal{V}^*_{\mathcal{K}_D,\Sigma_0^i}$, which together with (A.5) implies:

$$\begin{array}{c} \mathcal{V}^*_{\mathcal{K}_D,\Sigma_0^i} = \mathcal{K}_D \cap A^{-1}\mathcal{E} \\ \mathcal{K}_E \cap \mathcal{K}_D \cap A^{-1}\mathcal{E} = \{0\} \end{array} \qquad \text{(A.7)}$$

From (A.7.a) and [17.a] we have: $A\mathcal{V}^*_{\mathcal{K}_D,\Sigma_0^i} = A\mathcal{K}_D \cap \text{Im}\;A \cap \mathcal{E} = A\mathcal{K}_D \cap \text{Im}\;A = A\mathcal{K}_D$. Applying the operator A to equality (A.1.a) and taking into account (A.6.a), we get (A.6.b).
Now, in view of the definition of $\mathcal{V}^*_{\mathcal{K}_D,\Sigma_0^i}$ *and taking into account (A.6.a), we get:*

$$A\mathcal{K}_D = A\mathcal{V}^*_{\mathcal{K}_D,\Sigma_0^i} \subset E\mathcal{V}^*_{[\mathcal{K}_D,(A,E,0)]} \subset E\mathcal{K}_D$$

Hence, $\mathcal{K}_D$ is an $(A,E,0)$ invariant subspace contained in $\mathcal{K}_D$ and from the supremality of $\mathcal{V}^*_{\mathcal{K}_D,\Sigma_0^i}$, we get: $\mathcal{K}_D \subset \mathcal{V}^*_{\mathcal{K}_D,\Sigma_0^i}$, namely $\mathcal{V}^*_{\mathcal{K}_D,\Sigma_0^i} = \mathcal{K}_D$. □

Proof of Proposition 8 Proved in 4 steps:
1. Let us first prove that, if Im $A + \mathcal{B} \subset \mathcal{E}$ then:

$$\mathcal{V}^*_{\mathcal{X},\Sigma^g} = \mathcal{K}_D \qquad \text{(A.8)}$$

Indeed, from algorithm [7] we have for the first step (recall [17.a]): $\mathcal{V}^1_{\mathcal{X},\Sigma^g} = \begin{bmatrix} A \\ D \end{bmatrix}^{-1} \left(\begin{bmatrix} E \\ 0 \end{bmatrix} \mathcal{V}^0_{\mathcal{X},\Sigma^g} + \begin{bmatrix} B \\ 0 \end{bmatrix} \mathcal{U} \right) = \begin{bmatrix} A \\ D \end{bmatrix}^{-1} \left(\begin{bmatrix} E \\ 0 \end{bmatrix} \mathcal{X} + \begin{bmatrix} B \\ 0 \end{bmatrix} \mathcal{U} \right) = \begin{bmatrix} A \\ D \end{bmatrix}^{-1} \text{Im} \begin{bmatrix} E \\ 0 \end{bmatrix} = \mathcal{K}_D$. Now, since (recall [18]): $E\mathcal{V}^1_{\mathcal{X},\Sigma^g} = E\mathcal{K}_D = \mathcal{E} = E\mathcal{X} = E\mathcal{V}^0_{\mathcal{X},\Sigma^g}$, we get: $\mathcal{V}^2_{\mathcal{X},\Sigma^g} = \mathcal{V}^1_{\mathcal{X},\Sigma^g} = \mathcal{K}_D = \mathcal{V}^*_{\mathcal{X},\Sigma^g}$.
2. From Proposition 7, [18], [17.a] and (A.8), we get that *the non algebraic redundant part of the global system,* Σ^g, *is restricted to* $\mathcal{K}_D$ *in the domain and to* $E\mathcal{K}_D$ *in the codomain.*
3. Let us now prove that, the pair of induced maps $\widehat{\mathbb{E}} : \mathcal{X}/\mathcal{V}^*_{\mathcal{X},\Sigma^g} \to \underline{\mathcal{X}}_g/(\mathbb{E}\,\mathcal{V}^*_{\mathcal{X},\Sigma^g} + \text{Im}\;\mathbb{B})$ and $\widehat{\mathbb{H}} : \mathcal{X}/\mathcal{V}^*_{\mathcal{X},\Sigma^g} \to \underline{\mathcal{X}}_g/(\mathbb{E}\,\mathcal{V}^*_{\mathcal{X},\Sigma^g} + \text{Im}\;\mathbb{B})$ satisfy: $(\widehat{\mathbb{E}}, \widehat{\mathbb{H}}) = (0, \text{I})$.
Indeed, noting first that [19] implies that $\mathcal{X}/\mathcal{V}^*_{\mathcal{X},\Sigma^g} \approx \mathcal{K}_E$ and $\underline{\mathcal{X}}_g/(\mathbb{E}\,\mathcal{V}^*_{\mathcal{X},\Sigma^g} + \text{Im}\;\mathbb{B}) \approx D\mathcal{K}_E$, and defining next the following pair of maps:

$Q : \mathcal{X} \to \mathcal{K}_E$, natural projection along $\mathcal{K}_D$
$\underline{Q} : \underline{\mathcal{X}}_g \to D\mathcal{K}_E$, natural projection along $E\mathcal{K}_D$

we have that (recall [19] and [16]):

$$\mathbb{E}\,\text{Ker}\; Q = \mathbb{E}\,\mathcal{K}_D = E\mathcal{K}_D \subseteq \text{Ker}\; \underline{Q}$$
$$\mathbb{H}\,\text{Ker}\; Q = \mathbb{H}\,\mathcal{K}_D = A\mathcal{K}_D \subset \mathcal{E} = E\mathcal{K}_D \subseteq \text{Ker}\; \underline{Q}$$

There then exists a pair of unique maps $(\widehat{\mathbb{E}}, \widehat{\mathbb{H}})$ such that (see Fig. 1 in (Bonilla and Malabre 2002)):

$$\widehat{\mathbb{E}}\, Q = \underline{Q}\,\mathbb{E} \quad \text{and} \quad \widehat{\mathbb{H}}\, Q = \underline{Q}\,\mathbb{H}$$

which implies that (recall [19] and [16]):

$$\text{Im}\; \widehat{\mathbb{E}} = \widehat{\mathbb{E}}\,\text{Im}\; Q = \underline{Q}\text{Im}\; \mathbb{E} = \underline{Q}E\mathcal{K}_D = \{0\}$$
$$\text{Im}\; \widehat{\mathbb{H}} = \widehat{\mathbb{H}}\,\text{Im}\; Q = \underline{Q}\text{Im}\; \mathbb{H} = D\mathcal{K}_E = \text{Im}\; \underline{Q}$$

namely, $(\widehat{\mathbb{E}}, \widehat{\mathbb{H}}) = (0, \text{I})$.

4. Let us finally prove that, the global system [16], $\Sigma^g : (\mathbb{E}, \mathbb{H}, \mathbb{B}, C)$, is externally equivalent to the reduced state space description [24], Σ^s : $(\text{I}, A_0, B_0, \overline{C})$.

Let us first note that (A.8), [17.a] and Proposition 7, imply that $\Sigma^g : (\mathbb{E}, \mathbb{H}, \mathbb{B}, C)$ is externally equivalent to the restricted system $\overline{\Sigma}$: $(\overline{E}, \overline{A}, \overline{B}, \overline{C})$. Let us next note that $\overline{E}$ is an invertible map. Indeed, from [23.a], [16] and [21] we have: $\mathcal{K}_{\overline{E}} = \text{Ker}\; (\underline{V}\overline{E}) = \text{Ker}\; (\mathbb{E}\, V) = V^{-1}\mathcal{K}_{\mathbb{E}} = V^{-1}\mathcal{K}_E = \{0\}$, $\overline{E}$ is thus monic. On the other hand, from [18] directly follows that $\mathcal{K}_D \approx \mathcal{E}$, and then, $\overline{E}$ is a square monic map. Let us finally note that definitions [20] and [21.b] imply $\underline{P}\;\underline{V} = \text{I}$, which together with [23] and [16] we get [26]. □

Proof of Lemma 10 *Let us first note that* [17.a] implies that $\mathcal{B}^*_{1,\Psi^i} = \mathcal{E}$ (just apply algorithm [14] with $\Psi^i = [\lambda E - A]$) and then $\mathcal{B} \subset \mathcal{B}^*_{1,\Psi^i} + \mathcal{B}^*_{2,\Psi^i}$, which together with Proposition 9 prove the first part. *Let us next note that* (2.b) implies that $\text{Im}\; \mathbb{B} \subset \text{Im}\,[\lambda\,\mathbb{E} - \mathbb{H}\,]$ which together with Proposition 9 prove the second part. *Let us finally note that* [17.a] and [29] imply that $\text{Im}\,(A + BF_p) + \text{Im}\; B \subset \text{Im}\; A + \text{Im}\; B \subset \text{Im}\; E = \text{Im}\,(E - BF_d)$ and then $\mathcal{B}^*_{1,\Psi^i_F} = \text{Im}\,(E - BF_d)$ (just apply algorithm [14] with $\Psi^i_F = [\lambda(E - BF_d) - (A + BF_p)]$), then $\mathcal{B} \subset \mathcal{B}^*_{1,\Psi^i_F} + \mathcal{B}^*_{2,\Psi^i_F}$, which together with Proposition 9 prove the third part. □

Proof of Theorem 11 For the proof of this theorem we write E_F and A_F to denote $(E - BF_d)$ and $(A + BF_p)$, respectively, and Ψ^i_F is the pencil $[\lambda E_F - A_F]$. This Theorem is proved in 4 steps:

1. Let us first note that [17.a] and [29] imply:

$$\mathcal{V}^1 = \text{Ker}\; C \quad ; \quad \text{Im}\; A_F \subset \text{Im}\; E_F \qquad \text{(A.9)}$$

Indeed, taking into account [17.a] in [7] we get (A.9.a). And from [17.a] and [29] we get: $\text{Im}\; A_F \subset \text{Im}\; A + \mathcal{B} \subset \mathcal{E} = \text{Im}\; E_F$.

2. Let us next show that (A.9) imply that:

$$E_F\mathcal{A}^*_{1,\Psi^i_F} = A_F\mathcal{A}^*_{1,\Psi^i_F} \qquad \text{(A.10)}$$

Indeed, from (A.9.b) and [12] we get: $E_F\mathcal{A}^*_{1,\Psi^i_F} = E_F E_F^{-1} A_F\mathcal{A}^*_{1,\Psi^i_F} = \text{Im}\; E_F \cap A_F\mathcal{A}^*_{1,\Psi^i_F} = A_F\mathcal{A}^*_{1,\Psi^i_F}$.

3. Let us then prove that the solution y of the closed loop system Σ^i_F is unique if and only if

$$\mathcal{A}^*_{1,\Psi^i_F} \subset \mathcal{V}^* \qquad \text{(A.11)}$$

Let us first note that (A.9.b) and [13] imply that $\mathcal{A}^*_{2,\Psi^i_F} = \mathcal{X}$, then, from Proposition 9, the solution y of Σ^i_F is unique iff $\mathcal{A}^*_{1,\Psi^i_F} \subset \mathcal{K}_C$.

Let us now show that $\mathcal{A}^*_{1,\Psi^i_F} \subset \mathcal{K}_C$ implies that $\mathcal{A}^*_{1,\Psi^i_F} \subset \mathcal{V}^*$. Indeed, in view of (A.9.a) let us assume that $\mathcal{A}^*_{1,\Psi^i_F} \subset \mathcal{V}^\mu$, then from (A.10) we get $A_F\mathcal{A}^*_{1,\Psi^i_F} = E_F\mathcal{A}^*_{1,\Psi^i_F} \subset E_F\mathcal{V}^\mu \subset E_F\mathcal{V}^\mu + \mathcal{B}$, which implies $\mathcal{A}^*_{1,\Psi^i_F} \subset \mathcal{A}^*_{1,\Psi^i_F} + \mathcal{K}_{A_F} = A_F^{-1}A_F\mathcal{A}^*_{1,\Psi^i_F} \subset A_F^{-1}(E_F\mathcal{V}^\mu + \mathcal{B}) = A^{-1}(E\mathcal{V}^\mu + \mathcal{B})$ and then $\mathcal{A}^*_{1,\Psi^i_F} = \mathcal{A}^*_{1,\Psi^i_F} \cap \mathcal{K}_C \subset \mathcal{K}_C \cap A^{-1}(E\mathcal{V}^\mu + \mathcal{B}) = \mathcal{V}^{\mu+1}$.

4. Let us finally conclude that if the solution y of the closed loop system Σ^i_F is unique then [30] and [31] are satisfied. From (A.11) and [12] we get [30]. From [29] and [30] we get: $\mathcal{K}_E \approx \mathcal{K}_{E_f} \subset \mathcal{V}^* \cap E_F^{-1}\mathcal{B} = \mathcal{V}^* \cap E^{-1}\mathcal{B}$. □

Proof of Theorem 12 In view of Theorem 11 (and its proof) we only need to prove that [30] and [31] imply $\mathcal{A}^*_{1,\Psi^i_F} \subset \mathcal{V}^*$ with $(F_p^*\;,\; F_d^*) \in \mathbf{F}(\mathcal{V}^*_{\mathcal{K}_C,\Sigma^i})$. Indeed from algorithm [12] and [30] we get $\mathcal{A}^1_{1,\Psi^i_F} = \mathcal{K}_{E_F} \subset \mathcal{V}^*$. Let us now suppose that $\mathcal{A}^\mu_{1,\Psi^i_F} \subset \mathcal{V}^*$, we then have from [12] and [30]: $\mathcal{A}^{\mu+1}_{1,\Psi^i_F} = E_F^{-1}A_F\mathcal{A}^\mu_{1,\Psi^i_F} \subset E_F^{-1}A_F\mathcal{V}^* \subset E_F^{-1}E_F\mathcal{V}^* = \mathcal{V}^* + \mathcal{K}_{E_F} \subset \mathcal{V}^*$. □

Proof of Lemma 14 *Let us first prove* [35.a]. From [7], [25], [26], and Fig. 1, we get (recall that $\mathcal{E} = E\mathcal{K}_D$ and that $\mathcal{B} \subset \mathcal{E}$): $\mathcal{V}^0_{\mathcal{K}_D,\Sigma^s} = \mathcal{K}_D$ and $\mathcal{V}^1_{\mathcal{K}_D,\Sigma^s} = \mathcal{K}_D \cap \left(\overline{E}^{\;(-1)}\,\overline{A}\right)^{-1}\left(\mathcal{K}_D + \overline{E}^{\;(-1)}\,\mathcal{B}\right) = \mathcal{K}_D \cap \overline{A}^{\;-1}\left(\overline{E}\mathcal{K}_D + \mathcal{B}\right) = \mathcal{K}_D \cap \overline{A}^{\;-1}\left(\underline{P}\,\mathbb{E}\,V\mathcal{K}_D + \mathcal{B}\right) = \mathcal{K}_D \cap \overline{A}^{\;-1}(\mathcal{E} + \mathcal{B}) = \mathcal{K}_D \cap \overline{A}^{\;-1}\mathcal{E} = \mathcal{K}_D$, which implies: $\mathcal{V}^*_{\mathcal{K}_D,\Sigma^s} = \mathcal{K}_D$. *Let us next prove* [35.b]. From algorithm [7] we have for the first step (recall that $\text{Im}\; A \subset \mathcal{E}$): $\mathcal{X} \supset \mathcal{V}^1_{\mathcal{X},\Sigma^i} = A^{-1}\left(E\mathcal{V}^0_{\mathcal{X},\Sigma^i} + \mathcal{B}\right) = A^{-1}(\mathcal{E} + \mathcal{B}) \supset A^{-1}\mathcal{E} = \mathcal{X}$. Hence, $\mathcal{V}^*_{\mathcal{X},\Sigma^i} = \mathcal{V}^1_{\mathcal{X},\Sigma^i} = \mathcal{V}^0_{\mathcal{X},\Sigma^i} = \mathcal{X}$. *Let us finally prove* [35.c]. The assumption $\text{Im}\; A \subset \mathcal{E}$ and the algorithm [7] imply: $\mathcal{V}^0_{\mathcal{X},\Sigma^i_0} = \mathcal{X}$ and $\mathcal{V}^1_{\mathcal{X},\Sigma^i_0} = A^{-1}\mathcal{E} = A^{-1}(\text{Im}\; A \cap \mathcal{E}) = A^{-1}\text{Im}\; A = A^{-1}A\mathcal{X} = \mathcal{X} + \mathcal{K}_A = \mathcal{X}$.