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HYPERBOLIC-TYPE GENERALIZED LORENZ CHAOTIC SYSTEM AND ITS CANONICAL FORM

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Abstract: This paper shows that a large class of chaotic systems, introduced in (Čelikovský and Vaněček, 1994), (Vaněček and Čelikovský, 1996) as the *generalized Lorenz system*, can be further generalized to the *hyperbolic-type generalized Lorenz system*. While the generalized Lorenz system unifies both the famous Lorenz system and new Chen's system (Ueta and Chen, 1999), (Chen and Ueta, 2000), the hyperbolic-type generalized Lorenz system introduced here is in some way complementary to it. Such a complementarity is especially clear when considering the canonical form of the generalized Lorenz system obtained in (Čelikovský and Chen, 2002), where the canonical form is characterized by the eigenvalues of the linearized part together with a key parameter $\tau \in (-1, \infty)$. The analogous canonical form of the hyperbolic-type generalized Lorenz system introduced here corresponds to the case of $\tau \in (-\infty, -1)$, while $\tau = -1$ is a single special case. This new class of chaotic systems is then analyzed, both analytically and numerically, showing its rich variety of dynamical behaviours, including bifurcation and chaos. Moreover, an algorithm for transforming the hyberbolic-type generalized Lorenz system into its canonical form, as well as its inverse scheme, are presented. *Copyright © 2002 IFAC*

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1. INTRODUCTION

Recently, a new paradigm for chaos in control systems has arisen, which consists of active design and utilization of chaotic systems. This is called the "chaos synthesis" in (Vaněček and Čelikovský, 1996) and "chaotification" or "anticontrol of chaos" in (Chen and Ueta, 1999), (Ueta and Chen, 2000).

A new type of chaotic system is introduced in (Chen and Ueta, 1999) and analyzed in detail in (Ueta and Chen, 2000). This system, referred to as the Chen system by others (Agiza *et al.*, 2000), (Wang, 1999), (Yang and Yang, 2000), (Yu and Xia, 2000), (Zhong and Tang, 2001), in a way resembles both the well-known Lorenz system and the Rössler system; and yet, it is a topologically different type of system (Chen and Ueta, 1999), (Ueta and Chen, 2000).

In (Čelikovský and Vaněček, 1994), (Vaněček and Čelikovský, 1996), on the other hand, the so-called generalized Lorenz system was already introduced and analyzed, also in detail. It turns out that both the Chen system and the Lorenz system falls within an even larger class of the generalized Lorenz systems in a broader sense. More precisely, a canonical representation of the generalized Lorenz system has recently been obtained, including the appropriate transformation algorithm, in (Čelikovský and Chen, 2002). Such

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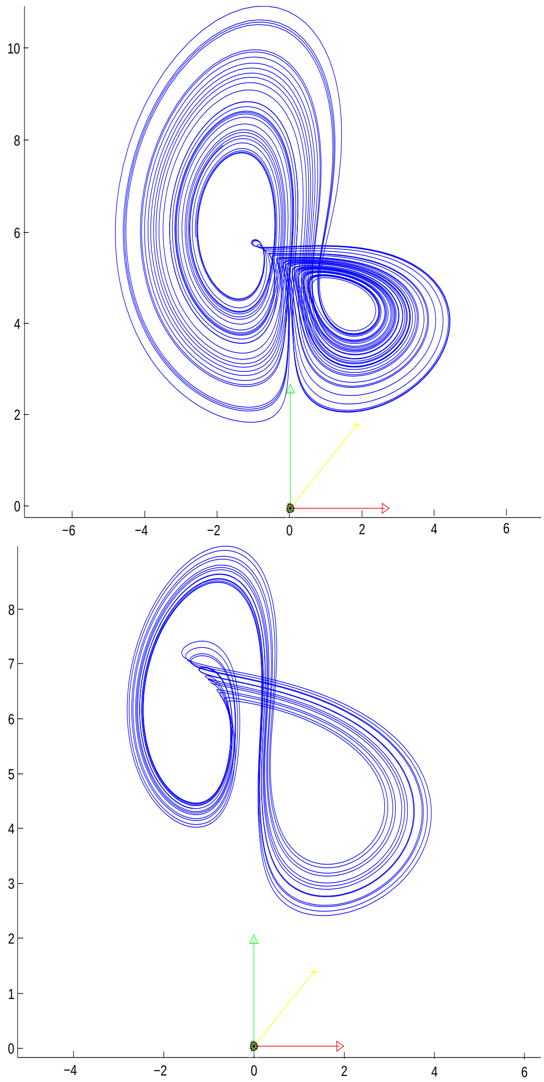

Fig. 1. The generalized Lorenz canonical form for the case of $\lambda_1 = 8$, $\lambda_2 = -16$, $\lambda_3 = -1$. From the top to the bottom: $\tau = 0.6$ and $\tau = 0$. The first attractor corresponds via linear change of coordinates to the classical Lorenz system. The second attractor presents "boundary" case between Chen's type and classical Lorenz systems.

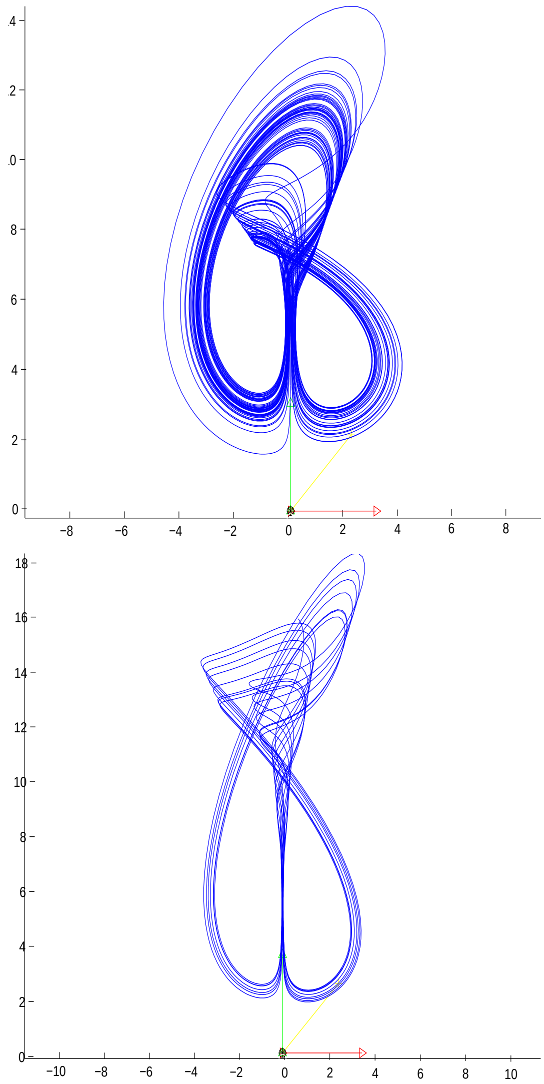

Fig. 2. The generalized Lorenz canonical form for the case of $\lambda_1 = 8$, $\lambda_2 = -16$, $\lambda_3 = -1$. From the top to the bottom: $\tau = -1$ and $\tau = -5$. The first case is the "boundary" case between generalized Lorenz system and hyperbolic-type generalized Lorenz system. The second one is equivalent to the hyperbolic-type generalized Lorenz system.

a canonical representation enables subtle analysis of the generalized Lorenz system and its chaos tuning for synthesis. Moreover, it shows that the classical Lorenz family and the Chen family of chaotic systems are intrinsically related via a continuous change of a key scalar parameter. This is especially surprising, having in mind the obviously different topological structures of the Lorenz and Chen attractors (Čelikovský and Chen, 2002). Nevertheless, several interesting connections have been discovered (Lü, *et al.*, 2002).

Except for certain singular cases, the canonical form of the generalized Lorenz system is defined by three eigenvalues of its linear part and a key scalar parameter $\tau \in (-1, \infty)$ (Čelikovský and Chen, 2002). This parameter τ is the only one responsible for the subtle chaos and bifurcation tuning, while the three eigenvalues have to fullfil some (very robust) inequality-type conditions (Vaněček and Čelikovský, 1996), (Čelikovský and Chen, 2002). Therefore, a natural question to ask is what would happen for the canonical form with $\tau \in (-\infty, -1)$? Later, it will be shown that $\tau = -1$ is a single special case. It turns out that this case corresponds to a simple variation of the generalized Lorenz system (GLS), tentatively called the *hyperbolic-type generalized Lorenz system* hereafter. The linear parts of both classes of systems have the same structure while the nonlinear parts are slightly different. The term *hyperbolic* is employed based on the fact that the skew-symmetric matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

defines the nonlinear part of the GLS while the nonlinear part of the hyperbolic-type GLS is defined by the symmetric matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

This paper offers a rather detailed investigation of this new hyperbolic-type generalized Lorenz system.

The rest of the paper is organized as follows. In the next section the basic definition of the generalized Lorenz system is first reviewed, followed by the introduction of its hyperbolic-type version. Section 3 then presents the canonical form of the hyperbolic-type generalized Lorenz system and derives the inverse transformations for both types of generalized Lorenz canonical forms. Section 4 further provides some qualitative analysis of the hyperbolic-type generalized Lorenz canonical form, with some simulations illustrated in Section 5. Conclusions are drawn in the final section, with some future research outlooks.

2. THE HYPERBOLIC-TYPE GENERALIZED LORENZ SYSTEM

To begin with, recall the following concept introduced in (Vaněček and Čelikovský, 1996):

Definition 2.1. The nonlinear system of ordinary differential equations in $\mathbb{R}^3$ of the following form is called the generalized Lorenz system:

$$\dot{x} = \begin{bmatrix} A & 0 \\ 0 & \lambda_3 \end{bmatrix} x + x_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} x, \qquad (1)$$

where $x = [x_1 \;\; x_2 \;\; x_3]^\top$, $\lambda_3 \in \mathbb{R}$, and A is a (2×2) real matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad (2)$$

with eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$-\lambda_2 > \lambda_1 > -\lambda_3 > 0. \qquad (3)$$

Moreover, the generalized Lorenz system is said to be nontrivial if it has at least one solution that goes neither to zero nor to infinity nor to a limit cycle.

Motivation for studying this generalized Lorenz system has been thoroughly discussed in (Vaněček and Čelikovský, 1996), (Čelikovský and Chen, 2002). In particular, it is now well understood the inequality condition (3) on the system eigenvalues, in view of Shilnikov's criterion. Since the eigenvalues requirement (3) is the only one, the generalized Lorenz system represents a quite general class of autonomous systems in $\mathbb{R}^3$. The interesting question thereafter is under what parameterization the generalized Lorenz system can be systematically classified in order to simplify its chaos synthesis. The following result has been obtained in (Čelikovský and Chen, 2002):

Theorem 2.2. For the nontrivial generalized Lorenz system $(1)-(3)$, there exists a nonsingular linear change of coordinates, $z = Tx$, which takes (1) into the following *generalized Lorenz canonical form*:

$$\dot{z} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} z + cz \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & \tau & 0 \end{bmatrix} z, \qquad (4)$$

where $z = [z_1, z_2, z_3]^\top$, $c = [1, -1, 0]$ and parameter $\tau \in (-1, \infty)$.

In this paper, we consider the following counterpart of the above generalized Lorenz system:

Definition 2.3. The nonlinear system of ordinary differential equations in $\mathbb{R}^3$ of the following form is called the hyperbolic-type generalized Lorenz system:

$$\dot{x} = \begin{bmatrix} A & 0 \\ 0 & \lambda_3 \end{bmatrix} x + x_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x, \qquad (5)$$

where $x = [x_1 \;\; x_2 \;\; x_3]^\top$, $\lambda_3 \in \mathbb{R}$, and A is a (2×2) real matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad (6)$$

with eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$-\lambda_2 > \lambda_1 > -\lambda_3 > 0. \qquad (7)$$

Moreover, the hyperbolic-type generalized Lorenz system is said to be nontrivial if it has at least one solution that goes neither to zero nor to infinity nor to a limit cycle.

Analogous comments as for the case of the generalized Lorenz system also apply here. In particular, the corresponding analogue of Theorem 2.2 is important and interesting, which will be derived below.

3. MAIN THEORETICAL RESULTS

The canonical form of the hyperbolic-type generalized Lorenz system is described as follows:

Theorem 3.1. For the nontrivial hyperbolic-type generalized Lorenz system $(5)-(7)$, there exists a nonsingular linear change of coordinates, $z = Tx$, which takes (1) into the following *generalized Lorenz canonical form*:

$$\dot{z} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} z + cz \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & \tau & 0 \end{bmatrix} z , \quad (8)$$

where $z = [z_1, z_2, z_3]^\top$, $c = [1, -1, 0]$ and the parameter $\tau \in (-\infty, -1)$.

Proof It is analogous to that of Theorem 2.2 presented in (Čelikovský and Chen, 2002), and so its full version is ommited for brevity. Its idea may be geometrically sketched as follows. As $\lambda_1 > 0$, $\lambda_2 < 0$, the matrix A has two linearly independent real eigenvectors. Therefore, there exists a suitable change of coordinates making A diagonal (see (Čelikovský and Chen, 2002) for the explicite description of a series of transformations). Let us denote these new coordinates as $\tilde{z}$. It is straightforward to check that the nonlinear part is transformed by the above coordinate change to

$$(a, b, 0)\tilde{z} \begin{bmatrix} 0 & 0 & -cb \\ 0 & 0 & ca \\ d & e & 0 \end{bmatrix} \tilde{z}$$

where $a, b, c, d, e \in \mathbb{R}$. Finally, if all of $a, b, c, d, e \in \mathbb{R}$ are nonzero, using several diagonal transformations one can take the nonlinear part into the final form (8), cf (Čelikovský and Chen, 2002). Notice, that further diagonal changes do not affect the linear part, as it already has been diagonalized. If some of a, b, c, d, e is zero, the similar approach leads either to (8) with $\tau = 0$ or to one of several "singular" cases that all exibit trivial behaviour (i.e. all their solutions are either unbounded, or asymptotically tends to equilibrium points and/or limit cycles).

In (Čelikovský and Chen, 2002), the detailed and explicit realization of the above geometrical sketch is given. Thanks to its constructive algebraic character, as a by-product, a computer MATLAB-based code is available to transform arbitrary (hyperbolic-type) GLS into its canonical form. ■

The following result applies to both the hyperbolic-type and the non-hyperbolic-type generalized Lorenz systems. It shows that each of these canonical forms, for every $\tau \neq -1$, is equivalent either to the generalized Lorenz system or to the hyperbolic-type generalized Lorenz system.

Theorem 3.2. System (4), with any $\tau \neq -1$, is state equivalent to the following system:

$$\dot{x} = \begin{bmatrix} A & 0 \\ 0 & \lambda_3 \end{bmatrix} x + x_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -sign(\tau+1) \\ 0 & 1 & 0 \end{bmatrix} x , \quad (9)$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} ,$$

$$\begin{aligned} a_{11} &= \left[\lambda_1 + (\lambda_2 - \lambda_1)(\tau+1)^{-1}\right] , \\ a_{12} &= -\left[(\lambda_2 - \lambda_1)(\tau+1)^{-1}\right] , \\ a_{21} &= \left[\lambda_1 - \lambda_2 + (\lambda_2 - \lambda_1)(\tau+1)^{-1}\right] , \\ a_{22} &= \left[\lambda_2 - (\lambda_2 - \lambda_1)(\tau+1)^{-1}\right] . \end{aligned} \quad (10)$$

The corresponding change of coordinates is

$$\begin{aligned} x_1 &= \sqrt{|\tau+1|}\,[z_1 - z_2] , \\ x_2 &= \sqrt{|\tau+1|}\,[z_1 + \tau z_2] , \\ x_3 &= |\tau+1| z_3 . \end{aligned} \quad (11)$$

Proof We have

$$\begin{gathered} \dot{x}_1 = \sqrt{|\tau+1|}\,[\dot{z}_1 - \dot{z}_2] = \sqrt{|\tau+1|}\,[\lambda_1 z_1 - \lambda_2 z_2] = \\ \sqrt{|\tau+1|}\,[\lambda_1(z_1 - z_2) - (\lambda_2 - \lambda_1)z_2] = \\ \left[\lambda_1 + (\lambda_2 - \lambda_1)(\tau+1)^{-1}\right] \sqrt{|\tau+1|}\,[z_1 - z_2] \\ - \left[(\lambda_2 - \lambda_1)(\tau+1)^{-1}\right] \sqrt{|\tau+1|}\,[z_1 + \tau z_2] = \\ a_{11}x_1 + a_{12}x_2 . \end{gathered}$$

Further

$$\begin{gathered} \dot{x}_2 = \sqrt{|\tau+1|}\,[\dot{z}_1 + \tau \dot{z}_2] = \\ \left[\lambda_1 - \lambda_2 + (\lambda_2 - \lambda_1)(\tau+1)^{-1}\right] \sqrt{|\tau+1|}\,[z_1 - z_2] \\ + \left[\lambda_2 - (\lambda_2 - \lambda_1)(\tau+1)^{-1}\right] \sqrt{|\tau+1|}\,[z_1 + \tau z_2] \\ -\sqrt{|\tau+1|}\,[z_1 - z_2]\,(\tau+1)z_3 = \\ a_{21}x_1 + a_{22}x_2 - sign(\tau+1)x_1x_3 . \end{gathered}$$

Finally

$$\begin{gathered} \dot{x}_3 = |\tau+1|\dot{z}_3 = \\ |\tau+1|\,[\lambda_3 z_3 + (z_1 - z_2)(z_1 + \tau z_2)] = \lambda_3 x_3 + x_1 x_2 . \end{gathered}$$

■

Remark 1. The case $\tau = -1$ is not equivalent neither to the generalized Lorenz system nor to the hyperbolic-type generalized Lorenz system. That is understable, because it constitutes a boundary between these two cases, which have qualitatively different structures in their nonlinear parts, so that they cannot be continuously changed from one to another. The case of $\tau = -1$ may also have chaotic behaviour, which has the similar character as for $\tau \neq -1$, as will be seen in the next section. To summarize, the *canonical Lorenz form* is a more preferable description as it provides a good unification for many systems that seemingly appear to be very different.

Remark 2. Notice that the trace of A in (10) is equal to $\lambda_1 + \lambda_2$ while its determinant to $\lambda_1\lambda_2$, as expected due to its equivalence to $\mathrm{diag}\{\lambda_1, \lambda_2\}$. Nevertheless, the entries of the matrix A in (10) fulfills also the additonal equalities $a_{12} + a_{21} = \lambda_1 - \lambda_2$ and $a_{11} + a_{12} = \lambda_1$ that are obviously not always valid for matrices having eigenvalues λ_1, λ_2. Therefore, the matrix A in (10) is still in a special form. That explains the relative simplicity of the transformation (11) with respect to the laborious proof of Theorem 3.1. In other words, taking an arbitrarily (hyperbolic-type) GLS, transforming it into the generalized Lorenz canonical form along the proof of Theorem 3.1 and then applying transformation (11), one would not necessarily obtain the GLS he started with. In this respect, the previously used term “inverse” is a slight abuse of notation.

4. QUALITATIVE ANALYSIS OF THE GENERALIZED LORENZ CANONICAL FORM

The generalized Lorenz canonical form (4,8) has a natural symmetry under the coordinates transform $(z_1, z_2, z_3) \to (-z_1, -z_2, z_3)$, which persists for all values of the system parameters. It can be easily verified that the z_3-axis ($z_1 = z_2 = 0$) of the system is invariant since all system orbits starting from the z_3-axis satisfy $\dot{z}_1 = \dot{z}_2 = 0$ and $\dot{z}_3 = \lambda_3 z_3$, where $\lambda_3 < 0$.

The variation of the volume $V(t)$ of a small element $\delta V(t) = \delta z_1 \delta z_2 \delta z_3$ in the phase space is determined by the divergence of the orbit flow:

$$\nabla V = \frac{\partial \dot{z}_1}{\partial z_1} + \frac{\partial \dot{z}_2}{\partial z_2} + \frac{\partial \dot{z}_3}{\partial z_3}.$$

It follows from system (4) and the condition (3) that

$$\nabla V = \lambda_1 + \lambda_2 + \lambda_3 < 0\,, \tag{12}$$

so the system is dissipative, with an exponential contraction rate: $e^{\lambda_1+\lambda_2+\lambda_3}$ for the volume element $\delta V(t) = \delta z_1 \delta z_2 \delta z_3$. That is, a volume element V_0 is contracted by the flow as $t \to \infty$. Thus, all orbits ultimately are confined to a specific subset having zero volume. This is the well-known necessary condition for the asymptotic motion to settle onto an attractor, which has also been confirmed by computer simulations (see the next section).

Besides the trivial equilibrium point $E^0 = (0,0,0)'$, the system has two mutually symmetric nontrivial equilibrium points

$$E^{\pm} = \begin{pmatrix} \pm\dfrac{\lambda_2}{\lambda_1 - \lambda_2}\sqrt{\dfrac{\lambda_1\lambda_2\lambda_3}{-(\lambda_2 + \tau\lambda_1)}} \\ \pm\dfrac{\lambda_1}{\lambda_1 - \lambda_2}\sqrt{\dfrac{\lambda_1\lambda_2\lambda_3}{-(\lambda_2 + \tau\lambda_1)}} \\ \dfrac{\lambda_1\lambda_2}{\lambda_2 - \lambda_1} \end{pmatrix} \tag{13}$$

where, obviously, under condition (3), $E^{\pm}$ are complex conjugate points if $\tau > -\lambda_2/\lambda_1$ and they are both at infinity if $\tau = -\lambda_2/\lambda_1$. Therefore, only the range $\tau < -\lambda_2/\lambda_1$ is interesting.

5. NUMERICAL SIMULATION RESULTS

We now illustrate our view point, given above, that the overall Lorenz canonical form provides a good unification for many systems that seemingly appear to be very different.

The series of simulations on Figures 1 and 2 for eigenvalues $\lambda_1 = 8, \lambda_2 = -16, \lambda_3 = -1$ starts with the value of the parameter $\tau = 0.6$, which corresponds to the classical Lorenz system. The next simulations therefore illustrate how the classical Lorenz family changes gradually into Chen’s family and then to the hyperbolic-type configuration.

6. CONCLUSIONS AND FUTURE RESEARCH OUTLOOK

Apart from its theoretical contribution as a kind of unique and unified classification for a very large class of chaotic sytems, the hyperbolic-type generalized Lorenz system and its canonical form have been studied, both analytically and numerically, which demonstrate to be a new and usefull tool for chaos synthesis.

Future research along this line would be some potential applications of the new classification and the new chaos synthesis methods obtained in our whole series of studies of the subject, (Čelikovský, and Vaněček, 1994), (Vaněček and Čelikovský, 1996), (Čelikovský and Chen, 2002), (Chen and Ueta, 1999), (Ueta and Chen, 2000), (Kunin and Chen, 2000), (Lü *et al.*, 2001, 2002).

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