COMPUTING OPTIMAL QUADRATIC LYAPUNOV FUNCTIONS FOR POLYNOMIAL NONLINEAR SYSTEMS VIA LMIS

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Abstract: The problem of estimating the domain of attraction (DA) of equilibria of polynomial systems is considered. Specifically, the computation of the quadratic Lyapunov function which maximizes the volume of the estimate is addressed. In order to solve this double non-convex optimization problem, a semi-convex approach based on Linear Matrix Inequalities (LMIs) is proposed. Moreover, for the case of odd polynomial systems, a relaxed criterion for obtaining an effective starting candidate of the optimal quadratic Lyapunov function is presented.

Keywords: Nonlinear systems, Stability, Domain of attraction, Quadratic Lyapunov functions, Linear Matrix Inequality

1. INTRODUCTION

In control systems engineering it is very important to know the domain of attraction (DA) of an equilibrium point, that is the set of initial states from which the system converges to the equilibrium point itself (Khalil, 1992). Indeed, such problem arises in both systems analysis and synthesis, in order to guarantee stable behaviours in a certain region of the state space. Unfortunately, it is well known that the DA is a very complicated set, and, in the most cases, it does not admit an exact analytic representation (Genesio *et al.*, 1985). Also, a point-to-point approximation of the set is almost impossible to obtain, due to the very heavy computational burden such procedure would require.

Therefore, the approximation of the DA via an estimate of a simpler shape has become a fundamental issue since long time (see (Genesio *et al.*, 1985)). The estimate shape is described by a Lyapunov function, generally quadratic. For a given Lyapunov function, the computation of the optimal estimate of the DA (that is, the largest estimate of the selected shape) amounts to solving a non-convex distance problem.

Within this context, a problem of primary importance is the selection of the quadratic Lyapunov function. In fact, the volume of the optimal estimate strongly depends on the Lyapunov function chosen for approximating the DA. Obviously, it would be useful to single out the function that maximizes the volume, that is the Optimal Quadratic Lyapunov Function (OQLF). Unfortunately, the computation of the OQLF amounts to solve a double non-convex optimization problem (Davison and Kurak, 1971; Michel *et al.*, 1982).

In this paper, a new technique for computing the OQLF for polynomial systems is presented. Specifically, we propose a Linear Matrix Inequality (LMI) approach based on convexification techniques recently developed for dealing with non-

convex distance problems (Chesi *et al.*, 2001*b*; Chesi, 2001; Chesi *et al.*, 2001*a*). It is shown how the OQLF can be computed, among all quadratic Lyapunov functions which ensure stability of the considered equilibrium point, solving a semi-convex optimization problem. Moreover, in order to obtain a good starting point for the non-convex step, a relaxed criterion is proposed for odd polynomial systems, based on the volume maximization of the region where the time derivative of the Lyapunov function is negative. It is shown how its solution can be computed via a one-parameter sequence of LMIs, that is the computational burden required by the computation of the DA for a fixed Lyapunov function. Simulation results show that this relaxed criterion can provide quite satisfactory candidates for the OQLF.

The paper is organized as follows. In Section 2 the problem formulation is given. Section 3 presents the semi-convex approach for computing the OQLF. Section 4 illustrates the relaxed criterion. Section 5 shows some numerical examples. Finally, Section 6 reports some brief comments.

Notation:

- 0_n: origin of $\mathbb{R}^n$;
- $\mathbb{R}_0^n$: $\mathbb{R}^n \setminus \{0_n\}$;
- I_n: identity matrix $n \times n$;
- A': transpose of matrix A;
- $A > 0$ ($A \geq 0$): symmetric positive definite (semidefinite) matrix A;
- s.t.: subject to.

2. PROBLEM FORMULATION

Without loss of generality, let us consider the polynomial system defined as

$$\dot{x} = Ax + \tilde{f}(x),$$
$$\tilde{f}(x) = \sum_{i=2}^{m_f} f_i(x) \qquad (1)$$

where $x \in \mathbb{R}^n$, A is a Hurwitz matrix, and $f_i(x)$ is a vector of homogeneous forms of degree i. It follows that the origin is a locally asymptotically stable equilibrium point.
Let us consider the quadratic Lyapunov function $V(P; x) = x'Px$, where $P > 0$ is such that the time derivative

$$\dot{V}(P; x) = 2x'P\left(Ax + \tilde{f}(x)\right) \qquad (2)$$

is locally negative definite. We refer to a such matrix P as *feasible* P and call $\mathcal{P}$ the set of all feasible P. In particular, $\mathcal{P}$ can be characterized as

$$\mathcal{P} = \left\{P = P' \in \mathbb{R}^{n \times n} : PA + A'P = -Q, \atop Q > 0\right\}. \qquad (3)$$

Let us define the ellipsoidal set induced by $V(P; x)$,

$$\mathcal{V}(P; c) = \left\{x \in \mathbb{R}^n : x'Px \leq c\right\}, \qquad (4)$$

and the negative time derivative region,

$$\mathcal{D}(P) = \left\{x \in \mathbb{R}^n : \dot{V}(P; x) < 0\right\} \cup \{0\}. \qquad (5)$$

Then, $\mathcal{V}(P; c)$ is an *estimate* of the domain of attraction of the origin if $\mathcal{V}(P; c) \subseteq \mathcal{D}(P)$. Moreover, the optimal estimate $\mathcal{S}(P)$ for the selected Lyapunov function is given by

$$\mathcal{S}(P) = \mathcal{V}(P; \gamma(P)),$$
$$\gamma(P) = \sup\ \{c \in \mathbb{R} : \mathcal{V}(P; c) \subseteq \mathcal{D}(P)\}. \qquad (6)$$

Let us observe that the computation of $\gamma(P)$ requires the solution of a non-convex distance problem. In fact,

$$\gamma(P) = \inf_{x \in \mathbb{R}_0^n}\ x'Px$$
$$\text{s.t.}\quad \dot{V}(P; x) = 0. \qquad (7)$$

Finally, let us define the OQLF as the quadratic Lyapunov function $V^*(P^*; x) = x'P^*x$ that maximizes the volume of the DA. Therefore,

$$P^* = \underset{P \in \mathcal{P}}{\operatorname{argmax}}\ \delta(P),$$
$$\delta(P) = \sqrt{\frac{\gamma^n(P)}{\det(P)}}, \qquad (8)$$

where $\delta(P)$ is the volume of $\mathcal{S}(P)$ up to a scale factor depending on n.
It turns out that the computation of the OQLF amounts to solve a double non-convex optimization problem. In fact, the volume function $\delta(P)$ can present local maxima in addition to the global one $\delta(P^*)$. Moreover, each evaluation of $\delta(P)$ requires the computation of $\gamma(P)$, that is the solution of the non-convex distance problem (7).

3. SEMI-CONVEX APPROACH FOR COMPUTING OQLF

In this section we show how the optimal estimate $\mathcal{S}(P)$ in (6) can be computed avoiding local minima, and, hence, how the OQLF can be found via a semi-convex approach. In particular, we exploit the convexification techniques developed in (Chesi *et al.*, 2001*b*; Chesi, 2001; Chesi *et al.*, 2001*a*), which allow us to obtain a lower bound of $\gamma(P)$ via a one-parameter sequence of LMIs (see also (Chesi *et al.*, 1997)). A simple test procedure is

available for assessing the tightness of such a lower bound. Moreover, in some cases tightness can be established a priori.

3.1 *Optimal estimate of the stability region*

Let us first observe that problem (7) is equivalent to the canonical distance problem

$$\gamma(P) = \inf_{x \in \mathbb{R}_0^n} \ x'Px$$
$$\text{s.t.} \quad w(P;x) = 0, \tag{9}$$

where $w(P;x)$ is a locally positive definite polynomial with only terms of even degree, that is $w(P;x) = \sum_{i=0}^{m} w_{2i}(P;x)$ for suitable homogeneous forms $w_{2i}(P;x)$ of degree $2i$. In fact,

- if (1) is an odd system, that is $\tilde{f}(x)$ is composed only by terms of odd degree, then the constraint function $\dot{V}(P;x)$ is composed only by terms of even degree, and, hence, $w(P;x) = -\dot{V}(P;x)$ and $m = (m_f + 1)/2$;
- otherwise, we can define the new constraint function as $w(P;x) = \dot{V}(P;x)\dot{V}(P;-x)$. It turns out that such polynomial has only terms of even degree. Hence, $m = m_f + 1$.

Our strategy consists of evaluating the constraint function $w(P;x)$ on the sets

$$\mathcal{B}(P;c) = \{x \in \mathbb{R}^n : \ x'Px = c\}. \tag{10}$$

In fact, it turns out that

$$\gamma(P) = \sup \ \{c > 0 : \ w(P;x) > 0$$
$$\forall x \in \mathcal{B}(P;c)\}. \tag{11}$$

Let us define the homogeneous form of degree $2m$

$$h(P;c;x) = \sum_{i=0}^{m} w_{2i}(P;x) \left(\frac{x'Px}{c}\right)^{m-i}. \tag{12}$$

Then, for any $c \in (0, +\infty)$ we have that

$$w(P;x) > 0 \ \forall x \in \mathcal{B}(P;c)$$
$$\Updownarrow \tag{13}$$
$$h(P;c;x) > 0 \ \forall x \in \mathbb{R}_0^n.$$

From (11) and (13) it follows that $\gamma(P)$ can be computed via a sequence of positivity tests on homogeneous forms, that is

$$\gamma(P) = \sup \ \{c > 0 : \ h(P;c;x) > 0$$
$$\forall x \in \mathbb{R}_0^n\}. \tag{14}$$

In order to perform such tests, let us introduce the Square Matricial Representation (SMR) of $h(P;c;x)$:

$$h(P;c;x) = x^{\{m\}'} H(P;c)x^{\{m\}} \ \forall x \in \mathbb{R}^n, \tag{15}$$

where the entries of vector $x^{\{m\}} \in \mathbb{R}^d$ are a base of the homogeneous forms of degree m and $H(P;c)$ is a suitable matrix. The dimension d is given by

$$d = \sigma(n,m) = \binom{n+m-1}{n-1}. \tag{16}$$

It is straightforward to verify that any homogeneous form of even degree can be represented by SMR. Let us observe that, for a fixed base $x^{\{m\}}$, the matrix $H(P;c)$ introduced in (15) is not unique. Indeed, let us introduce the following set:

$$\mathcal{L} = \Big\{ L = L' \in \mathbb{R}^{d \times d} : \ x^{\{m\}'} L x^{\{m\}} = 0$$
$$\forall x \in \mathbb{R}^n \Big\}. \tag{17}$$

It turns out that $\mathcal{L}$ is a linear space of dimension

$$d_{\mathcal{L}} = \frac{1}{2}d(d+1) - \sigma(n,2m). \tag{18}$$

Let $H(P;c)$ be any matrix satisfying (15) and define

$$H(P;c;\alpha) = H(P;c) + L(\alpha), \tag{19}$$

where $\alpha \in \mathbb{R}^{d_{\mathcal{L}}}$ is a free parameter vector and $L(\alpha) : \mathbb{R}^{d_{\mathcal{L}}} \to \mathcal{L}$ is a linear parameterization of set $\mathcal{L}$. Then, the complete SMR of $h(P;c;x)$ is given by

$$h(P;c;x) = x^{\{m\}'} H(P;c;\alpha)x^{\{m\}}$$
$$\forall \alpha \in \mathbb{R}^{d_{\mathcal{L}}} \ \forall x \in \mathbb{R}^n. \tag{20}$$

Let us introduce the quantity

$$c_\eta(P) = \sup \ \{\tilde{c} : \ \eta(P;c) > 0 \ \forall c \in (0, \tilde{c}]\}, \tag{21}$$

where

$$\eta(P;c) = \max_{t \in \mathbb{R}, \alpha \in \mathbb{R}^{d_{\mathcal{L}}}} t$$
$$\text{s.t.} \quad H(P;c;\alpha) - tI_d > 0. \tag{22}$$

It turns out that

$$c_\eta(P) \leq \gamma(P). \tag{23}$$

Therefore, $c_\eta(P)$ is a lower bound of $\gamma(P)$ and it can be computed via a one-parameter sequence of convex LMI optimizations (see (22)).

We point out that tightness of $c_\eta(P)$ is strictly related to the property of positive homogeneous forms to be represented as the sum of squares of homogeneous forms (Hardy *et al.*, 1988). Indeed, it has been proved that $c_\eta(P)$ is tight if and only if the homogeneous form $h(P;c;x)$ satisfies this property for all $c \in (0, \gamma(P))$ (Chesi, 2001). For the cases $n = 2, \forall m$ and $n = 3, m = 2$ such property is guaranteed a priori. For the other cases, extensive numerical proofs have shown that

the lower bound is almost always tight, except for ad hoc examples (Chesi *et al.*, 2001*a*). Moreover, tightness of $c_\eta(P)$ can be checked as follows:

$$c_\eta(P) = \gamma(P)$$
$$\Updownarrow \qquad (24)$$
$$\exists x \in \mathbb{R}^n : \ x^{\{m\}} \in \ker\left[H(P; c_\eta; \alpha_\eta)\right],$$

where α_η is the optimizing vector of (22). Such test can be performed solving a system of degree equal to the dimension of the found null space minus one. This means that for regular cases, in which the null space has dimension one, the test has an immediate solution. Obviously, if the lower bound should be discovered to be not tight, a standard optimization can be performed starting from the found point. This largely helps to find the global minimum avoiding local ones (Chesi *et al.*, 2001*a*).

3.2 *Computation of the OQLF*

Exploiting the convexification technique previously described, $\gamma(P)$, and hence $\delta(P)$, can be computed avoiding local optimal solutions. This lead us to formulate a semi-convex approach for computing the OQLF. In fact, let us introduce a parameterization of set $\mathcal{P}$ in (3) through the function

$$F(Q) = F(Q)' : \ F(Q)A + A'F(Q) = -Q \ (25)$$

where Q is any symmetric positive definite matrix. Moreover, since $\delta(P)$ is not affected by a positive scale factor on P, that is

$$\delta(P) = \delta(aP) \ \forall a \in \mathbb{R}, a > 0, \qquad (26)$$

and P depends linearly on Q (see (3)), the feasible set of matrices Q can be reduced by imposing a scale constraint, for example $Q_{1,1} = 1$. Therefore, let us define the set

$$\mathcal{Q} = \left\{Q \in \mathbb{R}^{n \times n} : \ Q > 0, Q_{1,1} = 1\right\} \quad (27)$$

of dimension $(n^2 + n - 2)/2$. Then, problem (8) can be equivalently rewritten as

$$P^* = F(Q^*),$$
$$Q^* = \underset{Q \in \mathcal{Q}}{\operatorname{argmax}} \ \delta\left(F(Q)\right),$$
$$\delta\left(F(Q)\right) = \sqrt{\frac{\gamma^n\left(F(Q)\right)}{\det\left(F(Q)\right)}} \qquad (28)$$

where, for each $Q \in \mathcal{Q}$, $\gamma\left(F(Q)\right)$ is computed using the technique presented in Section 3.1.
We point out that the complete SMR (20) can be systematically computed as shown in (Chesi, 2001). Moreover, the function $L(\alpha)$ has to be computed once only, lightening the computational burden of problem (28).

4. RELAXED SOLUTION VIA LMIS FOR ODD SYSTEMS

In this section a relaxed criterion for computing an initial candidate of the OQLF for odd polynomial systems is proposed. Our aim is to find, along with a moderate computational burden, an effective starting point for initializing the optimization procedure in (28).

Our criterion consists of finding the matrix P which maximizes the volume of an ellipsoid of fixed shape whose border is included in the negative time derivative region $\mathcal{D}(P)$ in (5). This criterion is based on the idea that, for obtaining a large optimal estimate of the DA, a good strategy is to enlarge $D(P)$, which clearly bounds the optimal estimate itself. Obviously, this criterion is relaxed with respect to (28), since the volume is measured via an a priori selected ellipsoidal shape, instead of the unknown one defined by the OQLF, and since we are requiring that only the border of the ellipsoid is included in $\mathcal{D}(P)$.

Let us select the ellipsoidal shape for measuring the volume of the region where the time derivative is negative as $\mathcal{V}(U; c)$, where $U \in \mathbb{R}^{n \times n}$ is a given symmetric positive definite matrix. Then, the relaxed criterion above described can be formulated as

$$\max_{P \in \mathcal{P}} \ \beta(P),$$
$$\beta(P) = \sup \ \left\{c : \ \mathcal{B}(U; c) \subseteq \mathcal{D}(P)\right\}, \qquad (29)$$

where $\mathcal{B}(U; c)$ is the border of the ellipsoid $\mathcal{V}(U; c)$, whose volume is given by

$$\sqrt{\frac{\beta(P)^n}{\det(U)}}.$$

In order to check the inclusion of $\mathcal{B}(U; c)$ in $\mathcal{D}(P)$, let us introduce the homogeneous form of degree $2m$

$$g(U; P; c; x) = \sum_{i=0}^{m} \ w_{2i}(P; x) \left(\frac{x'Ux}{c}\right)^{m-i}. \ (30)$$

Then, for any $P > 0$ and $c \in (0, +\infty)$ we have that

$$\mathcal{B}(U; c) \subseteq \mathcal{D}(P)$$
$$\Updownarrow$$
$$w(P; x) > 0 \ \forall x \in \mathcal{B}(U; c) \qquad (31)$$
$$\Updownarrow$$
$$g(U; P; c; x) > 0 \ \forall x \in \mathbb{R}_0^n.$$

Therefore, our criterion amounts to maximizing c subject to the following condition:

$$\exists P \in \mathcal{P} : \ g(U; P; c; x) > 0 \ \forall x \in \mathbb{R}_0^n. \quad (32)$$

Let us introduce the complete SMR of $g(U; P; c; x)$,

$$g(U; P; c; x) = x^{\{m\}'} G(U; P; c; \alpha) x^{\{m\}}$$
$$\forall \alpha \in \mathbb{R}^{d_\mathcal{L}} \ \forall x \in \mathbb{R}^n. \quad (33)$$

Following the strategy described in Section 3.1 for the computation of $\gamma(P)$, we relax condition (32) substituting it with

$$\exists P \in \mathcal{P}, \alpha \in \mathbb{R}^{d_\mathcal{L}} : \ G(U; P; c; \alpha) > 0. \quad (34)$$

Let us observe that condition (34) can be checked with one convex LMI optimization. Indeed, let us introduce the function

$$\mu(U; c) = \max_{Q \in \mathbb{R}^{n \times n}, t \in \mathbb{R}, \alpha \in \mathbb{R}^{d_\mathcal{L}}} t$$
$$\text{s.t.} \quad \begin{cases} G(U; F(Q); c; \alpha) - t I_d > 0 \\ Q \in \mathcal{Q} \end{cases} \quad (35)$$

where $G(U; F(Q); c; \alpha)$ depends linearly on the unknowns Q and α. Then, condition (34) is satisfied if and only if $\mu(U; c) > 0$. Hence, let us introduce the quantity

$$c_\mu = \sup \ \{c : \ \mu(U; c) > 0\} \quad (36)$$

and let $\hat{Q}$ be the optimizing matrix Q of problem (35) for $c = c_\mu$. We define the solution of our relaxed criterion as

$$\hat{P} = F(\hat{Q}). \quad (37)$$

Let us observe that $\hat{P}$ can be computed via a one-parameter sequence of convex LMI optimizations, that is about the same computational burden required by each evaluation of the volume function $\delta(P)$. More specifically, $\hat{P}$ requires the optimizations (35) in $d_\mathcal{L} + n(n + 1)/2$ parameters, and $\delta(P)$ requires the optimizations (22) in $d_\mathcal{L} + 1$ parameters.

5. EXAMPLES

In this section we present some examples of the proposed technique. More specifically, the matrices $\hat{P}$ and P^* and the corresponding optimal DA estimates are computed. Matrix $\hat{P}$ is computed as in (35)-(37) with $U = F(I_n)$. Matrix $\mathcal{P}^*$ is calculated solving (28) from the initial solution $Q = -\hat{P}A - A'\hat{P}$ with a simplex algorithm that evaluates the function $\delta(F(Q))$ using the convexification approach described in Section 3.1. Moreover, we have iterated the computation of $\hat{P}$ setting U, at each step, equal to the matrix $\hat{P}$ obtained at the previous step. The matrix so obtained after i iterations has been denoted by $\hat{P}^{(i)}$.

The following systems have been considered:

$$(S1) \begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -2x_1 - 3x_2 + x_1^2 x_2 \end{cases}$$

$$(S2) \begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -4x_1 - 5x_2 - x_1^3 + x_2^3 \end{cases}$$

$$(S3) \begin{cases} \dot{x}_1 = 4x_2 + x_1^3, \\ \dot{x}_2 = -12x_1 - 16x_2 - 4x_2^3 \end{cases}$$

$$(S4) \begin{cases} \dot{x}_1 = -x_1 - 2x_2 + x_1^2 x_2, \\ \dot{x}_2 = x_1 - x_2 - x_2^3 \end{cases}$$

$$(S5) \begin{cases} \dot{x}_1 = -2x_1 + x_2 + x_1^3 + x_2^5, \\ \dot{x}_2 = -x_1 - x_2 + x_1^2 x_2^3 \end{cases}$$

$$(S6) \begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = -4x_1 - 3x_2 - 2x_3 + 0.1x_1^3 \\ \quad\quad + x_1^2 x_2 + x_1^2 x_3 \end{cases}$$

$$(S7) \begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \dot{x}_3 = -4x_1 - 3x_2 - 2x_3 + x_1^2 x_2 + x_1^2 x_3 \end{cases}$$

For all these systems the lower bound $c_\eta(P)$ is tight a priori, being $n = 2$ in systems S1–S5 and $n = 3, m = 2$ in S6–S7 (see Section 3.1), and, hence, $\gamma(P) = c_\eta(P)$.

Table 1 shows the results obtained for these systems. The quantity n^* denotes the number of

system	$\delta(\hat{P})$	$\delta(P^*)$	$\delta(\hat{P}^{(i)})$
S1	8.35	10.21	10.21
		$n^* = 239$	$i = 16$
S2	1.63	2.99.	2.35
		$n^* = 218$	$i = 7$
S3	7.31	8.69	7.97
		$n^* = 184$	$i = 7$
S4	3.29	7.00	6.83
		$n^* = 215$	$i = 4$
S5	1.24	1.97	1.88
		$n^* = 125$	$i = 4$
S6	6.02	7.84	7.68
		$n^* = 378$	$i = 10$
S7	6.07	7.65	7.68
		$n^* = 416$	$i = 11$

Table 1. Volumes provided by $\hat{P}$, P^* and $\hat{P}^{(i)}$ for systems S1–S7.

evaluations of the volume function $\delta(P)$ required for computing P^* in (28).

As we can see, the relaxed criterion provides a good trade-off of accuracy and computational burden. In fact, few iterations yield a quite good approximation $\hat{P}^{(i)}$ of P^* in many cases (really, in system S1 the relaxed criterion provides the OQLF in 16 iterations only). Such approximation $\hat{P}^{(i)}$ can be used like initialization of problem (28) for computing P^*, since it is expected that a better starting solution avoids local maxima. Indeed,

let us consider system S7. The found P^* (416 evaluations of function $\delta(P)$) is a local maxima, as shown by $\hat{P}^{(11)}$ (11 computations of c_μ and 11 evaluations of $\delta(P)$), which provides a larger volume. Initializing (28) with $\hat{P}^{(11)}$ we find the global maximum at 7.78.

As a final remark, we would like to underline the importance of using the convex method described in Section 3.1 for evaluating function $\delta(F(Q))$ in (28). In fact, problem (7) can present local minima, especially if the Lyapunov function is close to the OQLF, as shown in Fig. 1 where the optimal estimates of the DA provided by $\hat{P}$ and P^* are illustrated. It is obvious that, if a local minimum is found in (7) instead of the global one, the computation of P^* totally fails.

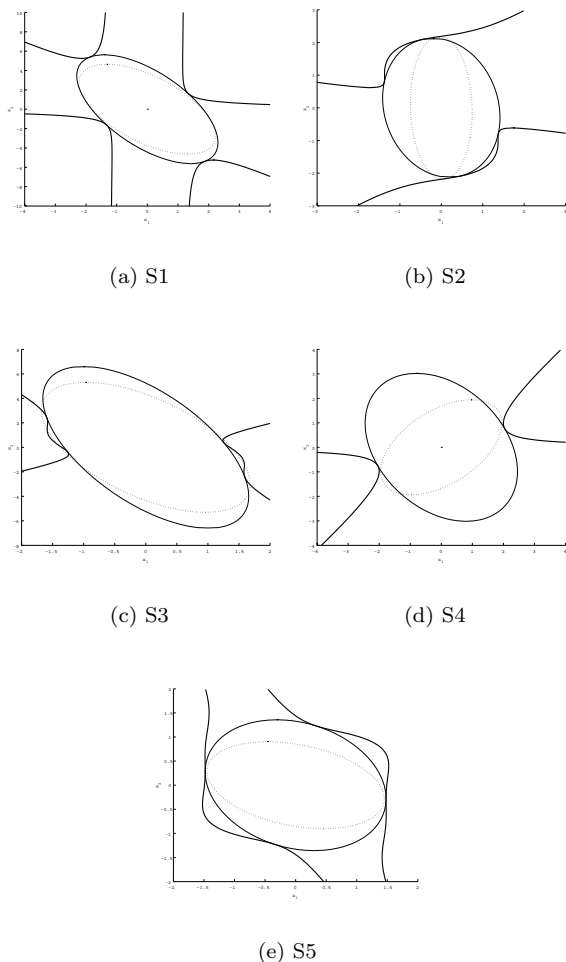

(a) S1   (b) S2

(c) S3   (d) S4

(e) S5

Fig. 1. Systems S1–S5. Optimal estimates of the domain of attraction given by P^* (solid line) and $\hat{P}$ (dotted line), and constraint set $\dot{V}(P^*; x) = 0$ (thick solid line).

6. CONCLUSION

A semi-convex approach for computing the quadratic Lyapunov function which maximizes the volume of the domain of attraction estimate for polynomial systems, has been presented. For a fixed Lyapunov function, the proposed technique allows one to compute, via a sequence of Linear Matrix Inequalities (LMIs), the optimal estimate avoiding local minima. This is a necessary step of any procedure for computing optimal quadratic Lyapunov functions (OQLFs). Moreover, a relaxed criterion has been presented for odd polynomial systems, which provides candidates for the OQLF via a one-parameter sequence of convex LMI optimizations. Simulation results have shown that these candidates are very effective starting points for the computation of the OQLF.

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