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SHAPE CONTROL OF NOISE IN A STRUCTURAL ACOUSTIC SYSTEM

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Abstract: The considered problem is of varying the thickness of a beam along its length in order to produce an optimal design which reduces the structural noise generated into an air filled cavity by the beam vibration. The design parameter is the thickness of that beam. We consider two models which differ only in taking or not into account the back-pressure effects of the air on the vibration. We provide necessary optimality conditions which guide the optimal beam design.

Keywords: Shape control, Sensitivity analysis, Parameter optimization, Passive noise control.

1. INTRODUCTION

We are interested in reducing the noise produced by the vibration of a beam in a given two dimensional bounded domain where the beam is part of the boundary. The control parameter is the thickness of the beam along its length. We formulate two shape optimization problems related to the reduction of this structural noise. An existence result is, then, provided under a regularization assumption. Necessary optimality conditions are computed using techniques of differentiation of a min and a min-max with respect to a parameter. This allows us to avoid differentiating the state functions. The first problem will be to modify the spectrum in such a way as to avoid the audible frequency band. This problem is reduced, due to a monotonicity property of the spectrum, to the maximization and, then, the differentiation of the first eigenvalue with respect to the thickness. We note that several authors (Zolésio, 1979; Zolésio, 1981; Myslinski, 1985) have studied the differentiation properties of the first eigen value of forth order elliptic operators when the eigenvalue is not simple. Our contribution, on this issue, is to give an explicit expression of any directional derivative by proving the existence of a minimizer in the eigen vector subspace $X(0)$.

The second problem will be to minimize the acoustic radiated energy

$$j(e) = \frac{1}{2}\int_0^T \int_\Omega \left[\frac{1}{\rho c^2} p_e^2 + \rho u_e^2\right] dxdt$$

where e designates the control which is, in our case, the thickness function, p is the air pressure and u is the fluid velocity, ρ is the mass density at rest and c is the wave velocity [this is the speed of sound in the case of sound waves]. We have indexed by e those functions which depend on the thickness variations. If we assume that the sound pressure levels remain below 150 dB, the acoustic dynamics are modeled by the wave equation involving either the acoustic velocity potential ϕ or the pressure p:

$$\nabla\phi = -u \quad \text{and} \quad p = \rho\frac{\partial\phi}{\partial t}$$

See H.T. Banks, et al. (Banks *et al.*, 1996).
We will prove that the acoustic energy has, not only, directional derivatives like the first eigenvalue but also a gradient associated to the thickness variations.

2. THE SYSTEM MODEL

Consider a beam of a constant unit width, of length L ($1 << L$) and of a non-uniform thickness. The transverse displacement of such a beam is described by a one dimensional hyperbolic equation. The location of the cross section of the beam is determined by the coordinate x. Denote by $\mathbf{I}(x)$ the moment of inertia of the beam cross-section located at x with respect to the neutral axis; E designates the Young's modulus of the material. $E\mathbf{I}(x)$ is the bending stiffness (or the flexural rigidity) and $\Gamma(x)$, respectively $e(x)$, are the cross-sectional area and the thickness of the beam. The cross-sectional area and the moment of inertia are respectively given by $\Gamma(x) = e(x)$ and $\mathbf{I}(x) = \frac{1}{3}e(x)^3$. Thus, the variation of the thickness has an effect on both these functions which are present in the beam equation. This allows us to modify the vibration of the beam to satisfy our needs. Let

- $w(x,t)$ be the transverse displacement of the cross-section at the time t,
- Ω be a bounded domain of $\mathbb{R}^2$ with a smooth boundary $\Gamma = \partial\Omega$ (Ω will represent the cavity) .

The boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, $\mathrm{int}(\Gamma_0) \cap \mathrm{int}(\Gamma_1) = \emptyset$, where Γ_0 will designate the reference position of the beam.
The first model: In the context of reducing noise in relatively small cavities (inside automobiles for example), it is reasonable to neglect the back-pressure effects of the air. Therefore the coupled system modeling the transverse beam deflection and the air velocity potential is:

$$\frac{\partial^2\phi}{\partial t^2} - c^2\Delta\phi = 0 \quad \text{in } \Omega \tag{1}$$

$$\frac{\partial\phi}{\partial n} = 0 \quad \text{on } \Gamma_1 \tag{2}$$

$$\frac{\partial\phi}{\partial n} = \frac{\partial w}{\partial t} \quad \text{on} \quad \text{on } \Gamma_0 \tag{3}$$

$$\gamma\, e(x)\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(E\mathbf{I}(x)\frac{\partial^2 w}{\partial x^2}\right) = g(t) \quad \text{on } \Gamma_0 \tag{4}$$

with the following initial conditions:

$$w(x,0) = w_0; \;\; \partial_t w(x,0) = w_1 \tag{5}$$

$$\phi(x,0) = \phi_0; \;\; \partial_t\phi(x,0) = \phi_1 \tag{6}$$

Assuming the beam is fixed at both ends, then the boundary conditions associated to the beam equation are:

$$w(0,t) = \frac{\partial w}{\partial x}(0,t) = 0 \tag{7}$$

$$w(L,t) = \frac{\partial w}{\partial x}(L,t) = 0 \tag{8}$$

The Second model: When the effect of the variation of the air pressure on the beam vibration must be taken into account, one has to add in the beam equation the external force $\rho\frac{\partial\phi}{\partial t}$. Therefore, equation (4) must be replaced by

$$\gamma\, e(x)\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(E\mathbf{I}(x)\frac{\partial^2 w}{\partial x^2}\right) = g(t) - \rho\frac{\partial\phi}{\partial t} \quad \text{on } \Gamma_0 \tag{9}$$

The coupled system is then described by equations (1)-(3) and (9) with the initial conditions (5)-(6) and the boundary conditions corresponding to the clamped beam (7)-(8).

For our control problem we will consider the two models and give a different approach for each one. For the first model, we will act on the first eigenvalue which will effectively move the entire spectrum due to the spectrum monotonicity property. However for the second model, we will focus on minimizing the acoustic radiated energy.

In what follows, the thickness e will be assumed to be bounded:

Assumption 2.1. Assume that $e \in L^\infty(0,L)$ with

$$0 < e_0 \le e(x) \le e_1 \quad \text{for a.e. } \; x \in [0,L].$$

This implies that

$$0 < \mathbf{I}_0 \le \mathbf{I}(x) \le \mathbf{I}_1, \quad \text{where} \;\; \mathbf{I}_0 = e_0^3/3, \mathbf{I}_1 = e_1^3/3.$$

3. THE FIRST APPROACH

3.1 Spectral Analysis

Assumption 3.1. We assume the excitation is harmonic in time with the frequency λ: $g(t) = \exp(i\lambda t)\,\tilde{g}$.

We look for solutions which are harmonic with the same frequency as the excitation:

$$\phi(t) = \exp(i\lambda t)\,\varphi; w(t) = \exp(i\lambda t)\,v.$$

Substituting g, ϕ and w in the equations (1)-(4), we obtain

$$-\lambda^2\varphi - c^2\Delta\varphi = 0 \text{ in } \Omega \quad (10)$$

$$\frac{\partial\varphi}{\partial n} = 0 \text{ on } \Gamma_1 \quad (11)$$

$$\frac{\partial\varphi}{\partial n} = i\lambda v \text{ on } \Gamma_0 \quad (12)$$

$$-\lambda^2\gamma e(x)v + \frac{d^2}{dx^2}[E\mathbf{I}(x)\frac{d^2v}{dx^2}] = \tilde{g} \text{ on } \Gamma_0 \quad (13)$$

Notice that we can restrict our study to equation (13). Therefore, by considering a particular solution of (13), the corresponding eigenvalue problem can be derived:

$$A_e v = k\, B_e v \quad (k = \lambda^2) \quad (14)$$

where

$$A_e w = \frac{d^2}{dx^2}\left[E\mathbf{I}(x)\frac{d^2w}{dx^2}\right];\ B_e \overset{def}{=} \gamma e(x)\,\mathrm{Id}_{\Gamma_0}$$

and Id_{Γ_0} is the identity operator in $L^2(\Gamma_0)$.

Properties 3.1. For any $e(.) \in L^\infty(\Gamma_0)$ satisfying assumption 2.1, the linear operators A_e and B_e have the following properties:

(i) A_e and B_e are self-adjoint.
(ii) A_e positive definite.
(iii) $(\xi/\eta)_{\Gamma_0} = 0, \forall\xi \in H_0^2(\Gamma_0)$ implies $\eta = 0$.
(iv) There exists a constant α_0 such that

$$(B_e v/v)_{\Gamma_0} \le \alpha_0 \langle A_e v, v\rangle_{\Gamma_0}$$

where $\langle .\,,.\rangle_{\Gamma_0}$ and $(./.)_{\Gamma_0}$ denote respectively the duality product between $H^{-2}(\Gamma_0)$ and $H_0^2(\Gamma_0)$ and the $L^2(\Gamma_0)$ inner product.

Proposition 3.1. The eigenvalues associated to (14) are real, positive. Moreover, if $k_1, k_2, \cdots, k_n, \cdots$ are the distinct values of the eigenvalues, and if we denote by $\mathcal{E}_i$ the eigenvector space associated to k_i, we have the following characterization:

$$k_{m+1} = \min_{v \in (\oplus_{i=1}^m \mathcal{E}_i)^\perp} \frac{\langle A_e v, v\rangle_{\Gamma_0}}{(B_e v/v)_{\Gamma_0}},\ m \in \mathbb{N} \quad (15)$$

Proof. For a proof of this result one can refer to chapter 3 in either (Weinberger, 1974) or (Chen and Zhou, 1993).

3.2 Shape sensitivity analysis for the first eigenvalue

To compute the derivative of the first eigenvalue with respect to the thickness, we will use a result of differentiation of a min with respect to a parameter, see (Delfour and Zolésio, 2001). Assume $e \in \mathrm{int}\{f \in L^\infty(0,L) \text{ s.t. } e_0 \le f(x) \le e_1 \text{ a.e. }\}$ and consider a direction of perturbation, h, of the thickness function e. Denoting

$$A(s) = A_{e+sh},\ B(s) = B_{e+sh}$$

the functional will be, for a suitable s_0,

$$F : [0, s_0] \times H_0^2(0,L) \longrightarrow \mathbb{R}$$

$$(s, v) \longrightarrow \frac{\langle A(s)v, v\rangle_{\Gamma_0}}{(B(s)v/v)_{\Gamma_0}}$$

For each $s \in [0, s_0]$, introduce

$$f(s) \overset{def}{=} \inf\{F(s,x) : x \in X\} = k_1(s)$$
$$X(s) \overset{def}{=} \{x \in X : F(s,x) = f(s)\}.$$

Proposition 3.2. For any direction $h \in L^\infty(0,L)$, the mapping $s \to k_1(s) \overset{def}{=} k_1(e+sh)$ has a semi-derivative at $s = 0$. More precisely, there exists $\overline{v} \in X(0)$ such that $(B_e\overline{v}/\overline{v})_{\Gamma_0} = 1$ and

$$\mathrm{d}k_1(e;h) = \int_0^L (Ee^2(\frac{d^2\overline{v}}{dx^2})^2 - k_1(e)\gamma\overline{v}^2)\, h\, dx$$

Proof. The direction h being given, there exists $s_0 > 0$ small enough so that $(e + sh)$ satisfies assumption 2.1 for any $s \in [0, s_0]$. $X(s) \neq \emptyset$ for any $s \in [0, s_0]$ since properties 3.1 are still satisfied.
For any $v \in H_0^2(0,L) \setminus \{0\}$, the mapping

$$s \longrightarrow F(s,v) = \frac{\langle A(s)v, v\rangle_{\Gamma_0}}{(B(s)v/v)_{\Gamma_0}}$$

is differentiable in $[0, s_0]$. Indeed, using the dominated convergence theorem (Rudin, 1966), we easily prove the differentiability of $s \to \langle A(s)v/v\rangle$ and $s \to (B(s)v, v)$ and we obtain

$$\partial_s\langle A(s)v, v\rangle = \int_0^L E\,(e+sh)^2 h \left(\frac{d^2v}{dx^2}\right)^2 dx;$$

$$\partial_s(B(s)v, v) = \int_0^L \gamma\, h\, v^2\, dx.$$

Then, let $\mathcal{T}_X$ be the weak topology associated with $H_0^2(0,L)$ and consider a sequence $\{s_n\} \subset]0, s_0]$ which converges to 0.
To any s_n, we associate the eigenvalue

$$k_1(s_n) = \inf_{v \in H_0^2(0,L)} F(s_n, v)$$

and an eigenvector $v_1(s_n)$ satisfying:

$$A(s_n)v_1(s_n) = k_1(s_n)B(s_n)v_1(s_n) \quad (16)$$

The vector $v_1(s_n)$ can be chosen such that

$$(B(s_n)v_1(s_n)/v_1(s_n))_{\Gamma_0} = 1$$

Then using equation (16) we prove that for some constant c, we have

$$\|v_1(s_n)\|^2_{\Gamma_0} \le c \inf_{v \in H_0^2(0,L)} \{\|v\|^2_{\Gamma_0} \text{ s.t. } |v|_{L^2} = 1\}$$

All this implies that there exists a subsequence, still denoted $\{s_n\}$, $\overline{k}_1$ and $v^0 \in H_0^2(0,L)$ such that when n goes to infinity $k_1(s_n) \to \overline{k}_1$ and

$$v_1(s_n) \to v^0 \text{ weakly in } H_0^2(0,L)$$

In fact v^0 belongs to $X(0)$ which means that it is an eigenvector associated to $k_1(0)\,(= k_1(e))$. The weak formulation of (16) converges to a similar expression where s_n, $k_1(s_n)$ and $v_1(s_n)$ are replaced respectively by 0, $\bar{k}_1$ and v^0. This shows that $\bar{k}_1$ is the first eigenvalue for $s = 0$ and that $v^0 \in X(0)$. Obviously we have $(B(0)v^0, v^0)_{\Gamma_0} = 1$. The continuity of the mapping $(s, v) \longrightarrow \partial_s F(s, v)$ holds when $H_0^2(\Gamma_0)$ is endowed with its strong topology. Finally and according to (Delfour and Zolésio, 2001) we deduce the desired result.

3.3 Existence and necessary optimality condition

It is known, see (Baranger and Temam, 1975), that there is no existence result corresponding to the maximization of the first eigenvalue only under the boundedness assumption 2.1. A regularization term must be added. So the cost functional will be:

$$\max_{e \in \mathcal{U}_{ad}} \left\{ k_1(e) - \frac{\varepsilon}{2} \|e\|^2_{H^1(\Gamma_0)} \right\} \tag{17}$$

where $\varepsilon > 0$ is arbitrarily small and $\mathcal{U}_{ad} = \{e \in H^1(\Gamma_0); 0 < e_0 \leq e(x) \leq e_1 \text{ in } \Gamma_0\}$.

Proposition 3.3. The maximization problem (17) has at least one solution.

The proof is based on classical boundedness arguments.

Proposition 3.4. Let $\bar{e}(.)$ be a solution of (17). For any $h \in H^1(\Gamma_0)$, we have

$$\mathrm{d}k_1(\bar{e}; h) - \frac{\varepsilon}{2}((\bar{e}/h))_{H^1(\Gamma_0)} \leq 0 \tag{18}$$

where $((./.))_{H^1(\Gamma_0)}$ denotes the inner product in $H^1(\Gamma_0)$.

This result is an immediate consequence of proposition 3.2.

4. SECOND APPROACH

In this section, we take into account the air back-pressure effects on the beam vibration.
Assuming the beam is clamped at both ends, a weak formulation of our system is:

$$\frac{\rho}{c^2}\langle \frac{\partial^2 \phi}{\partial t^2}, \ \xi \rangle_{(\bar{H}^1)', \bar{H}^1} + \int_\Omega \rho \nabla\phi \ \nabla\xi \, dxdy$$
$$+\gamma\langle e(x)\frac{\partial^2 w}{\partial t^2}, \ \eta\rangle_{\Gamma_0} + \int_{\Gamma_0} E\mathbf{I}(x)\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \eta}{\partial x^2}\, dx$$
$$+\rho \int_{\Gamma_0} \frac{\partial \phi}{\partial t}\eta - \frac{\partial w}{\partial t}\xi \, dx = \int_{\Gamma_0} g\,\eta\, dx$$

This formulation suggests introducing the spaces

$$H = \bar{L}^2(\Omega) \times L^2(\Gamma_0) \ \text{ and } \ V = \bar{H}^1(\Omega) \times H_0^2(\Gamma_0)$$

where $\bar{L}^2(\Omega) = \{\xi \in L^2(\Omega), \int_\Omega \xi \, dxdy = 0\}$ and $\bar{H}^1(\Omega) = H^1(\Omega) \cap \bar{L}^2(\Omega)$. The inner product associated to each space: $\Phi = (\phi, w), \Psi = (\xi, \eta)$

$$\langle \Phi, \Psi \rangle_H = (\phi/\xi) + (w/\eta)_{\Gamma_0}$$
$$\langle \Phi, \Psi \rangle_V = (\nabla\phi/\nabla\xi) + (\frac{\partial^2 w}{\partial x^2}/\frac{\partial^2 \eta}{\partial x^2})_{\Gamma_0}$$

where (/) is the inner product in $L^2(\Omega)$; $|\,|$, $|\,|_{\Gamma_0}$ will denote respectively the norm in $L^2(\Omega)$ and $L^2(\Gamma_0)$.

Theorem 4.1. Assume $g \in L^2(0, T; L^2(\Gamma_0))$, $\Phi_0 = (\phi_0, w_0) \in V$ and $\Phi_1 = (\phi_1, w_1) \in H$. There exists a unique $\Phi = (\phi, w) \in L^2(0, T; V)$ solution of (1)-(3) and (9) such that $\dfrac{d\Phi}{dt} \in L^2(0, T; H)$.

For a proof one can refer to (Lions, 1971).

4.1 Acoustic energy minimization

We are interested in minimizing the radiated acoustic energy in the cavity Ω with respect to the thickness. Expressing the air pressure and velocity in terms of the potential velocity field, the acoustic energy, for a given thickness $e()$, becomes

$$J(\phi_e) \stackrel{def}{=} \frac{\rho}{2}\int_0^T \int_\Omega \frac{1}{c^2}(\frac{\partial \phi_e}{\partial t})^2 + |\nabla\phi_e|^2 \, dxdt$$

We have the following existence result.

Proposition 4.1. For any arbitrarily small $\varepsilon > 0$, there exists $\bar{e}$ solution of

$$\min_{e \in \mathcal{U}_{ad}} \{J(\phi_e) + \frac{\varepsilon}{2}\|e\|^2_{H^1(\Gamma_0)}\} \tag{19}$$

4.2 Shape sensitivity analysis

To obtain information on the way to modify the thickness in order to minimize the acoustic energy, we need to compute the derivative of the cost functional with respect to the thickness. For that purpose we introduce the Lagrangian $\mathcal{L}$ associated to our cost functional. If we denote respectively by W, H_0 and H_T the spaces

$\{(\phi, w) \in L^2([0,T]; V) \,/\, (\dfrac{d\phi}{dt}, \dfrac{dw}{dt}) \in L^2([0,T], H)\}$,

$\{(\phi, w) \in W \,/\, (\phi, w)(0) = \Phi_0, (\partial_t\phi, \partial_t w)(0) = \Phi_1\}$

and

$\{(\xi, \eta) \in W \,/\, \xi(T) = \partial_t\xi(T) = \eta(T) = \partial_t\eta(T) = 0\}$.

Then

$$\mathcal{L}(e; (\phi, w); (\xi, \eta)) = J(\phi) + L(e; (\phi, w); (\xi, \eta)) \tag{20}$$

where $L : H_0 \times H_T \longrightarrow \mathbb{R}, L(e; (\phi, w); (\xi, \eta))$

$$= \int_0^T \frac{\rho}{c^2} \langle \frac{\partial^2 \phi}{\partial t^2}, \xi \rangle_{(\bar{H}^1)', \bar{H}^1} + \gamma \langle e(x) \frac{\partial^2 w}{\partial t^2}, \eta \rangle_{\Gamma_0} \, dt$$

$$+ \int_0^T \rho (\nabla \phi / \nabla \xi) + E \left(\mathbf{I}(x) \frac{\partial^2 w}{\partial x^2} / \frac{\partial^2 \eta}{\partial x^2} \right)_{\Gamma_0} \, dt$$

$$+ \int_0^T \rho \left(\frac{\partial \phi}{\partial t} / \eta \right)_{\Gamma_0} - \rho \left(\frac{\partial w}{\partial t} / \xi \right)_{\Gamma_0} - (g/\eta)_{\Gamma_0} \, dt$$

According to the previous notations, we have

Lemma 4.1. The acoustic energy associated to a given thickness function $e(.)$ has the following expression

$$J(\phi_e) = \inf_{(\phi,w) \in H_0} \sup_{(\xi,\eta) \in H_T} \mathcal{L}(e; (\phi, w); (\xi, \eta)) \quad (21)$$

Assume $e \in \text{int}\{f \in L^\infty(0, L) \text{ s.t. } e_0 \leq f(x) \leq e_1 \text{ a.e.}\}$ and let h be a direction of perturbation and $s_0 > 0$ be small enough such that $e_0 \leq e + sh \leq e_1$, $\forall s \in [0, s_0]$. Define the mapping $j(e + sh) \overset{def}{=}$

$$\inf_{(\phi,w) \in H_0} \sup_{(\xi,\eta) \in H_T} \mathcal{L}(e + sh; (\phi, w); (\xi, \eta)) \quad (22)$$

Therefore, as formulated, we have a saddle point to differentiate with respect to a parameter. To do so, the mapping

$$s \longrightarrow \mathcal{L}(s; ., .) \overset{def}{=} \mathcal{L}(e + sh; ., .)$$

must satisfy hypothesis of a result due originally to R. Correa and A. Seeger (Correa and Seeger, 1985). Let the real number $s_0 > 0$, the sets X and Y and the functional $F : [0, s_0] \times X \times Y \to \mathbb{R}$ be given. For each $s \in [0, s_0]$ define

$$f(s) = \inf_{x \in X} \sup_{y \in Y} F(s; x, y)$$

and the sets

$$X(s) = \{x^s \in X : \sup_{y \in Y} F(s; x^s, y) = f(s)\}$$

$$Y(s, x) = \{y^s \in Y : F(s; x, y^s) = \sup_{y \in Y} F(s; x, y)\}$$

Similarly, define

$$l(s) = \sup_{y \in Y} \inf_{x \in X} F(s; x, y)$$

and the sets $Y(s)$ and $X(s, y)$. Finally let

$$S(s) = \{(x, y) \in X \times Y : f(s) = F(s; x, y) = l(s)\}$$

The cited result can be applied, in our case, with

$F \equiv \mathcal{L}$, $X = H_0$, $Y = H_T$, and $f(s) = j(e + sh)$

Lemma 4.2. For any $s \in [0, s_0]$, $S(s) \neq \emptyset$.

For a saddle-point existence hypothesis, see I. Ekeland and R. Temam (Ekeland and Temam, 1976).

Remark 4.1. If $((\bar{\phi}, \bar{w}), (\bar{\xi}, \bar{\eta})) \in H_0 \times H_T$ is a saddle point of $\mathcal{L}$ if and only if $\forall (\xi, \eta) \in H_T$ and $\forall (\phi, w) \in W_0$.

$$L(s; (\bar{\phi}, \bar{w}), (\xi, \eta)) = 0 \quad (23)$$

$$L(s; (\phi, w), (\bar{\xi}, \bar{\eta})) + \int_0^T (g/\bar{\eta})_{\Gamma_0} = \quad (24)$$

$$-\frac{\rho}{c^2} \int_0^T (\frac{\partial \bar{\phi}}{\partial t} / \frac{\partial \phi}{\partial t}) \, dt - \rho \int_0^T (\nabla \bar{\phi} / \nabla \phi) \, dt \quad (25)$$

where $W_0 = \{(\phi, w) \in W / \phi(0) = \partial_t \phi(0) = w(0) = \partial_t w(0) = 0\}$.

The first identity is the state system (1)-(3) and (9). The second one is the associated adjoint system which can be rewritten as follows

$$\frac{1}{c^2} \frac{\partial^2 \bar{\xi}}{\partial t^2} - \Delta \bar{\xi} = 2 \Delta \bar{\phi} \text{ in } \Omega \quad (26)$$

$$\frac{\partial \bar{\xi}}{\partial n} = \frac{\partial \bar{\eta}}{\partial t} - \frac{\partial \bar{\phi}}{\partial n} \text{ on } \Gamma_0 \quad (27)$$

$$\frac{\partial \bar{\xi}}{\partial n} = 0 \text{ on } \Gamma_1 \quad (28)$$

$$\gamma e(x) \frac{\partial^2 \bar{\eta}}{\partial t^2} + \frac{\partial^2}{\partial x^2} \left(E \mathbf{I}(x) \frac{\partial^2 \bar{\eta}}{\partial x^2} \right) \quad (29)$$

$$+\rho \frac{\partial \bar{\xi}}{\partial t} = 0 \text{ on } \Gamma_0$$

with the following boundary conditions satisfied by $\bar{\eta}$

$$\bar{\eta}(0, t) = \frac{\partial \bar{\eta}}{\partial x}(0, t) = 0 \quad (30)$$

$$\bar{\eta}(L, t) = \frac{\partial \bar{\eta}}{\partial x}(L, t) = 0 \quad (31)$$

Lemma 4.3. For any $(\phi, w) \in H_0$ and $(\xi, \eta) \in H_T$, the mapping $s \longrightarrow \mathcal{L}(s; (\phi, w), (\xi, \eta))$ is differentiable on $[0, s_0]$ and we have

$$\partial_s \mathcal{L}(s; (\phi, w), (\xi, \eta)) = \int_0^T \gamma \langle h(x) \frac{\partial^2 w}{\partial t^2}, \eta \rangle_{\Gamma_0} \, dt$$

$$+ \int_0^T E \left((e + sh)^2(x) h(x) \frac{\partial^2 w}{\partial x^2} / \frac{\partial^2 \eta}{\partial x^2} \right)_{\Gamma_0} \, dt$$

Moreover the mapping

$$(s, (\phi, w), (\xi, \eta)) \longrightarrow \partial_s \mathcal{L}(s; (\phi, w), (\xi, \eta))$$

is continuous in the strong topology of W.

Lemma 4.4. For any sequence $\{s_n\}$, $s_n \to 0$, there exists $((\phi^0, w^0), (\xi^0, \eta^0)) \in H_0 \times H_T$ and a subsequence (still denoted $\{s_n\}$) such that

$$\left.\begin{array}{l} (\phi_n, w_n) \longrightarrow (\phi^0, w^0) \\ (\xi_n, \eta_n) \longrightarrow (\xi^0, \eta^0) \end{array}\right\} \text{strongly in } W$$

and $((\phi^0, w^0), (\xi^0, \eta^0))$ is a saddle point for $s = 0$.

Proof. For each s_n, we associate $((\phi_n, w_n), (\xi_n, \eta_n)) \in H_0 \times H_T$ solution of

$$L(s; (\phi_n, w_n), (\xi, \eta)) = 0 \quad \forall (\xi, \eta) \in H_T$$

$$L(s; (\phi, w), (\xi_n, \eta_n)) + \int_0^T (g/\eta_n)_{\Gamma_0} =$$

$$\int_0^T -\frac{\rho}{c^2}(\frac{\partial \phi_n}{\partial t}/\frac{\partial \phi}{\partial t}) - \rho(\nabla \phi_n / \nabla \phi) dt \, \forall (\phi, w) \in W_0$$

Since $(e + s_n h)$ converges strongly, in $L^\infty(\Gamma_0)$, to e, we can prove using the same arguments as in the proof of the existence result 4.1 that there exists a subsequence, still denoted (ϕ_n, w_n), which converges strongly, in H_0, to (ϕ^0, w^0). The fact that (ϕ_n, w_n) is bounded in H_0 implies the boundedness of (ξ_n, η_n) in H_T. Hence, the weak convergence, in H_T, to (ξ^0, η^0) is obtained. Also the strong convergence can be derived. Using these convergence properties, it is easy to prove that $((\phi^0, w^0), (\xi^0, \eta^0))$ is a saddle point for $s = 0$.

The lemmas 4.2 to 4.4 allow us to justify the following differentiation result. We denote by G_e, the $L^1(\Gamma_0)$ function defined by

$$G_e(x) = \int_0^T \left[-\gamma \frac{\partial \bar{w}}{\partial t} \frac{\partial \bar{\eta}}{\partial t} + E \, e^2(x) \frac{\partial^2 \bar{w}}{\partial x^2} \frac{\partial^2 \bar{\eta}}{\partial x^2} \right] dt$$

where $((\bar{\phi}, \bar{w}), (\bar{\xi}, \bar{\eta})) \in H_0 \times H_T$ solution of (23)-(25).

Proposition 4.2. The mapping $s \longrightarrow j(e + sh)$ has a derivative at $s = 0$ denoted $dj(e; h)$:

$$\mathrm{d}j(e; h) = \int_{\Gamma_0} G_{\bar{e}} \, h(x) \, dx dt \tag{32}$$

Now, we are able to formulate a necessary optimality condition for the minimization problem (19).

Proposition 4.3. Assume $\bar{e}$ to be a solution of (19). Then

$$\int_{\Gamma_0} G_{\bar{e}}(x) \, h(x) \, dx + \varepsilon((\bar{e}, h)) = 0, \; \forall h \in H^1(\Gamma_0) \tag{33}$$

Conclusion: The expression for the directional derivatives with respect to the design parameter, namely the thickness of the beam, of the first eigenvalue of problem (14) and of the radiated acoustic energy (21) shows that, from the design point of view, the two approaches considered seem to have equivalent structures.

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