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SEPARATION PRINCIPLE FOR A CLASS OF NONLINEAR SYSTEMS

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Abstract: We address the problem of global output feedback stabilization for a class of nonlinear affine systems. For affine systems that admit an exponentially stable observer it is shown that a global separation principle holds. Our stabilization results are also presented for general nonlinear systems. An output feedback controller is designed for a flexible one-link robot arm as an illustrative example.

Keywords: Nonlinear systems, Separation, Output feedback stabilization.

1. INTRODUCTION

This paper deals with global output feedback stabilization of nonlinear systems and separation principle. The output feedback control and the generalization of the nonlinear separation principle have received a lot of attention during the past decade. The certainty equivalence approach - replacing the state in a stabilizing state feedback by the state's estimate from a converging state observer - was considered, for instance, in (Tsinias, 1991; Tsinias, 1993) as an effective methodology for global output feedback stabilization of the certain classes of nonlinear systems. More recent results on output feedback controllers concern semiglobal asymptotic stabilization of locally Lipschitz nonlinear systems by output feedback using high-gain observers (Teel and Praly, 1994; Atassi and Khalil, 1999; Shim and Teel, 2001). However it is well-known that the global separation principle does not hold for general nonlinear systems (Freeman, 1995) and the certainty equivalence does not apply in general.

In this paper by using a similar technique as in (Tsinias, 1991; Atassi and Khalil, 1999) for nonlinear affine systems that can be transformed into a form in which the nonlinearities do not depend on unmeasured states and admit an exponentially stable observer we consider the global output feedback stabilization problem and show that a global separation principle holds. The remaining of the paper is organized as follows: in section 2 we present and prove our main results on output feedback stabilization and the separation principle for affine systems. Stabilization results for general nonlinear systems are given in section 3. These results are then applied in section 4 to a flexible one-link robot arm. Conclusions end the paper in section 5.

2. AFFINE SYSTEMS

2.1 Main results

Consider an output feedback stabilization problem for nonlinear dynamical systems of the form

$$\dot{x} = A(x) + B(x)u, \quad y = h(x), \qquad (1)$$

where $x \in R^n$, $u \in R$, $y \in R$; A, B and h are sufficiently smooth.

First, let us design an observer for (1) by using ideas given in (Krener and Respondek, 1985; Krishchenko and Tkachev, 1995). System (1) without control input

$$\dot{x} = A(x), \quad y = h(x) \tag{2}$$

is locally observable at x_0 if in a neighborhood of x_0 the observability matrix

$$W(x) = \Big(\frac{\partial h(x)}{\partial x}, \frac{\partial L_A h(x)}{\partial x}, \ldots, \frac{\partial L_A^{n-1} h(x)}{\partial x}\Big)^T \tag{3}$$

has the rank n, where $L_A h(x) = \dfrac{\partial h}{\partial x} A(x)$ denotes the Lie derivative of $h(x)$ along the vector field $A(x)$, $(\cdot)^T$ stands for transpose. By using results obtained in (Krishchenko and Tkachev, 1995) the following theorem can be stated.

Theorem 2.1. A locally observable at x_0 dynamical system (2) admits around x_0 an observer canonical form

$$\dot{\chi} = D\chi + \psi(\chi_1), \quad y = H(\chi_1), \tag{4}$$

where $\chi = (\chi_1, \ldots, \chi_n)^T$, $D = (d_{ij})$ is $n \times n$ matrix with elements $d_{ij} = 1$ if $j - i = 1$ and $d_{ij} = 0$ if $j - i \neq 1$, $\psi(\chi_1) = (\psi_1(\chi_1), \ldots, \psi_n(\chi_1))^T$, and admits an observer

$$\dot{\hat{\chi}} = D\hat{\chi} + G(\hat{\chi}_1 - \chi_1) + \psi(\chi_1),$$
$$\chi_1 = H^{-1}(y),$$

where vector $G = (g_1, \ldots, g_n)^T$ determines the state estimation error dynamics and is chosen such that under $C = (1, 0, \ldots, 0)$ the matrix $D + GC$ is Hurwitz, if and only if there exists such a smooth function $p(\tau)$, $\tau \in R$, $p(\tau) > 0$, that a vector field $B_1(x)$ which is solution of

$$W(x)B_1(x) = (0, \ldots, 0, p(h(x)))^T$$

satisfies around x_0

$$[\operatorname{ad}_A^k B_1, \operatorname{ad}_A^{k+1} B_1](x) = 0, \quad k = \overline{0, n-2},$$

where $W(x)$ denotes the observability matrix (3), $[.,.]$ denotes the Lie bracket of two vector fields and $\operatorname{ad}_A^{k+1} B_1(x) = [A(x), \operatorname{ad}_A^k B_1(x)]$, $k > 0$, $\operatorname{ad}_A^0 B_1(x) = B_1(x)$.

If conditions of theorem 2.1 hold there exists a local smooth nondegenerate change of coordinates $x = \Phi(\chi)$ that transforms system (2) into the form (4). It can be found by using the following (Krener and Isidori, 1983; Krishchenko and Tkachev, 1995)

$$\Phi'(\chi) = \Big(B_1(x), -\operatorname{ad}_A B_1(x), \ldots, (-1)^{n-1} \operatorname{ad}_A^{n-1} B_1(x)\Big)\Big|_{x=\Phi(\chi)},$$

where $\Phi'(\chi)$ denotes Jacobian of the mapping $\Phi : R^n \to R^n$.

Assume that vector field $B(x)$ of (1) satisfies

$$[(-1)^{i-1} \operatorname{ad}_A^{i-1} B_1, B](x) = 0, \quad i = \overline{1, n}. \tag{5}$$

Then $B(x)$ is constant in new coordinates χ. In this case system (1) in new variables χ is given by

$$\dot{\chi} = D\chi + \psi(\chi_1) + Bu, \quad B = \text{const}, \qquad y = H(\chi_1). \tag{6}$$

Next show that the system

$$\dot{\hat{\chi}} = D\hat{\chi} + G(\hat{\chi}_1 - \chi_1) + \psi(\chi_1) + Bu, \qquad \chi_1 = H^{-1}(y) \tag{7}$$

is an exponentially stable observer for (6). The state estimation error $e = \hat{\chi} - \chi$ has the following dynamics

$$\dot{e} = (D + GC)e, \tag{8}$$

where matrices D, C and G are from the statement of theorem 2.1. Therefore the error doesn't depend on control input and converges to zero exponentially fast. The latter implies that the equilibrium $e = 0$ of the estimation error equation is exponentially stable.

Remark 2.1. If only first $n - 1$ equations in (5)

$$[(-1)^{i-1} \operatorname{ad}_A^{i-1} B_1, B](x) = 0, \qquad i = \overline{1, n-1} \tag{9}$$

hold for affine system (1) then vector field $B(x)$ depends on χ_1 in new variables χ and has the form $\tilde{B}(\chi_1)$. Obviously system (7), where $B = \tilde{B}(\chi_1)$, is the exponentially stable observer for (6) with $B = \tilde{B}(\chi_1)$ and the state estimation error dynamics is of the form (8).

Assumption 2.1. System (1) can be transformed into the form (6), where $B = \text{const}$ or $B = \tilde{B}(\chi_1)$, with the change of coordinates $x = \Phi(\chi)$ generating diffeomorphism $R^n = \{\chi\}$ onto $R^n = \{x\}$, and admits exponential observer (7), where $B = \text{const}$ or $B = \tilde{B}(\chi_1)$ respectively, on the whole R^n.

Assumption 2.2. The map $\psi(\chi_1)$ is globally Lipschitz, i.e. there exists a constant $L > 0$, such that

$$|\psi(\chi_1) - \psi(\chi_1')| \leq L|\chi_1 - \chi_1'|$$

for all $\chi_1, \chi_1' \in R$.

Next let such a continously differentiable state feedback $u(\chi)$ be designed that closed-loop system (6) with $u = u(\chi)$ is globally exponentially stable at a point $\chi = \chi_*$. Since the state-vector of the system is not known, the state estimation error $e = \hat{\chi} - \chi$ one can interpret as a sensor disturbance that acts on the closed-loop system (6) through the control law $u(\chi + e) = u(\hat{\chi})$.

Consider first the case when assumption 2.1 holds with $B = \text{const}$.

Theorem 2.2. Let assumption 2.2 be hold for system (6) with $B = \text{const}$. Assume that there exists a continously differentiable state feedback law $u(\chi)$ that makes the equilibrium position $\chi = \chi_*, u = u_*$ of (6) globally exponentially stable. Then system (6) in closed-loop form with the feedback law $u(\chi + e) = u(\hat{\chi})$ is globally asymptotically stable at $\chi = \chi_*$.

In case assumption 2.1 holds with $B = \tilde{B}(\chi_1)$ one can state the following version of theorem 2.2.

Theorem 2.3. Let assumption 2.2 be hold for system (6) with $B = \tilde{B}(\chi_1)$. Assume that vector field $\tilde{B}(\chi_1)$ is globally Lipschitz and furthermore there exists a continously differentiable globally bounded state feedback law $u(\chi)$ that makes the equilibrium position $\chi = \chi_*, u = u_*$ of (6) with $B = \tilde{B}(\chi_1)$ globally exponentially stable. Then system (6) with $B = \tilde{B}(\chi_1)$ in closed-loop form with the feedback law $u(\chi + e) = u(\hat{\chi})$ is globally asymptotically stable at $\chi = \chi_*$.

The following results are direct corollaries of theorem 2.2, theorem 2.3, assumption 2.1 and present the global separation principle for the considered class of systems.

Theorem 2.4. Suppose that for system (1) 1) assumption 2.1 holds; 2) (5) are satisfied on R^n and the conditions of theorem 2.2 are fulfilled (or (9) are satisfied on R^n and the conditions of theorem 2.3 are fulfilled). Then system (1) in closed-loop form with the estimated state feedback $u_0(\hat{x}) = u(\hat{\chi}), \hat{\chi} = \Phi^{-1}(\hat{x})$, where $u(\chi)$ comes from the statement of theorem 2.2 (or theorem 2.3), is globally asymptotically stable at point $x = x_* = \Phi(\chi_*)$.

Theorem 2.5. Suppose that for system (1) 1) assumptions 2.1 and 2.2 hold; 2) the mappings Φ and Φ^{-1} are globally Lipschitz; 3) (5) are satisfied on R^n (or (9) are satisfied on R^n and vector field $B(\chi_1)$ is globally Lipschitz); 4) there exists a continously differentiable (and globally bounded in case $B = \tilde{B}(\chi_1)$) state feedback law $u(x)$ that makes system (1) globally exponentially stable. Then system (1) in closed-loop form with the estimated state feedback $u_0(\hat{x}) = u(\hat{x}) = u(\Phi(\hat{\chi}))$, $\hat{x} = \Phi(\hat{\chi})$ is globally asymptotically stable.

One can see that theorem 2.5 follows from theorem 2.4. The fact that Φ and Φ^{-1} are globally Lipschitz implies that closed-loop system (6) with $u = u(\Phi(\chi))$ is globally exponentially stable.

2.2 Proof of theorem 2.2

Consider the following system

$$\begin{aligned} \dot{\hat{\chi}} &= D\hat{\chi} + GCe + \psi(\hat{\chi}_1 - e_1) + Bu(\hat{\chi}), \\ \dot{e} &= (D + GC)e. \end{aligned} \tag{10}$$

As it was stressed in previous section the equilibrium $e = 0$ of error equation (8) is globally exponentially stable and so according to (Krasovskiy, 1959) there exists a quadratic Lyapunov function $W(e)$ that for all $e \in R^n$ satisfies

$$\begin{aligned} &l_1|e|^2 \le W(e) \le l_2|e|^2, \quad \left|\frac{\partial W(e)}{\partial e}\right| \le l_3|e|, \\ &\dot{W}(e)\Big|_{(8)} \le -l_4|e|^2, \end{aligned} \tag{11}$$

where l_i, $i = \overline{1, 4}$ are some positive constants, $|\cdot|$ denotes the euclidean norm on R^n.

Since the feedback law $u(\chi)$ globally exponentially stabilizes system (6) at $\chi = \chi_*$, $u = u_*$ and the right-hand side of the closed-loop system (6) without output is continously differentiable, according to (Krasovskiy, 1959) there exists a Lyapunov function $V_1(\chi - \chi_*)$ such that for all $\chi \in R^n$ the following inequalities hold

$$c_1|\chi - \chi_*|^2 \le V_1(\chi - \chi_*) \le c_2|\chi - \chi_*|^2, \tag{12}$$

$$\left|\frac{\partial V_1(\chi - \chi_*)}{\partial \chi}\right| \le c_3|\chi - \chi_*|, \tag{13}$$

$$\frac{\partial V_1(\chi - \chi_*)}{\partial \chi}(D\chi + \psi(\chi_1) + Bu(\chi)) \le -c_4|\chi - \chi_*|^2, \tag{14}$$

where c_1, c_2, c_3, c_4 are some positive constants. As a Lyapunov function candidate for system (10) let us take $V(\hat{\chi} - \chi_*, e) = kV_1(\hat{\chi} - \chi_*) + W(e)$, where k is some positive constant to be given later. Obviously $V(\chi - \chi_*, e) > 0$, $(\chi - \chi_*, e) \ne 0$. The derivative $\dot{V}$ of V along the trajectories of (10) can be written as follows

$$\begin{aligned} \dot{V}(\hat{\chi} - \chi_*, e)\Big|_{(10)} &= k\frac{\partial V_1(\hat{\chi} - \chi_*)}{\partial \hat{\chi}}(D\hat{\chi} + \psi(\hat{\chi}_1) \\ &+ Bu(\hat{\chi})) - k\frac{\partial V_1(\hat{\chi} - \chi_*)}{\partial \hat{\chi}}(\psi(\hat{\chi}_1) - \psi(\chi_1)) \\ &+ k\frac{\partial V_1(\hat{\chi} - \chi_*)}{\partial \hat{\chi}}GCe + \dot{W}(e)|_{(10)}. \end{aligned}$$

By (11)-(14) one gets

$$\begin{aligned} \dot{V}(\hat{\chi} - \chi_*, e)\Big|_{(10)} &\le -kc_4|\hat{\chi} - \chi_*|^2 \\ &+ k\left|\frac{\partial V_1(\hat{\chi} - \chi_*)}{\partial \hat{\chi}}\right||\psi(\hat{\chi}_1) - \psi(\chi_1)| \\ &+ k\left|\frac{\partial V_1(\hat{\chi} - \chi_*)}{\partial \hat{\chi}}\right||G||C||e| - l_4|e|^2 \\ &\le -kc_4|\hat{\chi} - \chi_*|^2 + kc_3L|\hat{\chi} - \chi_*||e| + kc_3|G||C| \\ &\times |e||\hat{\chi} - \chi_*| - l_4|e|^2. \end{aligned}$$

$$\dot V(\hat\chi-\chi_*,e)\Big|_{(10)} \le -kc_4|\hat\chi-\chi_*|^2 + (kc_3L+kc_3|G||C|)|e||\hat\chi-\chi_*| - l_4|e|^2.$$

So, if one chooses k such that

$$k < 4c_4l_4/(c_3L+c_3|G||C|)^2$$

then $\dot V(\chi-\chi_*,e)\Big|_{(10)}$ is negative definite on $R^n\times R^n$ and the following inequality holds

$$\dot V(\chi-\chi_*,e)\Big|_{(10)} \le -\lambda|(\chi-\chi_*,e)^T|^{\,2}$$

for some positive constant λ. By using (11) and (12)-(14) one gets

$$V(\chi-\chi_*,e) = kV_1(\chi-\chi_*) + W(e) \le kc_2|\chi-\chi_*|^2 + l_2|e|^2 \le c_2'|(\chi-\chi_*,e)^T|^{\,2},$$
$$V(\chi-\chi_*,e) \ge kc_1|\chi-\chi_*|^2 + l_1|e|^2 \ge c_1'|(\chi-\chi_*,e)^T|^{\,2},$$

where $c_1'=\min\{kc_1,l_1\}$, $c_2'=\max\{kc_2,l_2\}$. Then according to (Krasovskiy, 1959) the equilibrium $\hat\chi=\chi_*,\ e=0$ of (10) is globally exponentially stable. The linear change of coordinates

$$\hat\chi=\hat\chi,\ \ e=\hat\chi-\chi \tag{15}$$

transforms system (10) into the form

$$\begin{aligned}\dot\chi &= D\chi+\psi(\chi_1)+Bu(\hat\chi),\\ \dot{\hat\chi} &= D\hat\chi+GC(\hat\chi-\chi)+\psi(\chi_1)+Bu(\hat\chi).\end{aligned} \tag{16}$$

Since the exponential stability property remains invariant under linear change of coordinates (15) it follows that the equilibrium $\chi=\chi_*,\ \hat\chi=\chi_*$ of (16) is also globally exponentially stable. By using this fact one gets for (6) in closed-loop form with $u(\chi+e)$ the following

$$\begin{aligned}|\chi(t)-\chi_*| &= |\chi(t)-\hat\chi(t)+\hat\chi(t)-\chi_*|\\ &\le |\hat\chi(t)-\chi_*|+|\hat\chi(t)-\chi(t)|\\ &\le |(\hat\chi(t)-\chi_*,e(t))^T|+|e(t)| \le \beta_1e^{-\alpha_1t}\\ &\times|(\hat\chi(0),e(0))^T-(\chi_*,0)^T|+\beta_2e^{-\alpha_2t}|e(0)|,\end{aligned}$$

where $\alpha_1>0,\ \alpha_2>0$ and $\beta_1>0,\ \beta_2>0$ are corresponding positive constants. Hence $|\chi(t)-\chi_*|\le\beta e^{-\alpha t}$, where $\alpha=\min\{\alpha_1,\alpha_2\}$, $\beta=\max\{\beta_2|e(0)|,\ \beta_1|(\hat\chi(0),e(0))^T-(\chi_*,0)^T|\}$. Therefore system (6) in closed-loop form with $u(\chi+e)=u(\hat\chi)$ is globally asymptotically stable.

2.3 Proof of theorem 2.3

One can show that the derivative $\dot V$ of $V(\chi-\chi_*,e)=kV_1(\chi-\chi_*)+W(e)$ along the trajectories of system (10) with $B=\tilde B(\chi_1)$ can be written as follows

$$\begin{aligned}\dot V(\hat\chi-\chi_*,e)\Big|_{(10)} &= k\frac{\partial V_1(\hat\chi-\chi_*)}{\partial\hat\chi}(D\hat\chi+\psi(\hat\chi_1)\\ &+\tilde B(\hat\chi_1)u(\hat\chi)) - k\frac{\partial V_1(\hat\chi-\chi_*)}{\partial\hat\chi}(\psi(\hat\chi_1)-\psi(\chi_1))\\ &-k\frac{\partial V_1(\hat\chi-\chi_*)}{\partial\hat\chi}(\tilde B(\hat\chi_1)-\tilde B(\chi_1))u(\hat\chi)\\ &+k\frac{\partial V_1(\hat\chi-\chi_*)}{\partial\hat\chi}GCe+\dot W(e)|_{(10)}.\end{aligned}$$

So one can see, that if there exists some positive constant M such that $|u(\hat\chi)|<M$ for all $\hat\chi\in R^n$, then theorem 2.3 can be proved analogous to theorem 2.2.

3. GENERAL CASE

The results stated above can be generalized in case one considers the nonlinear dynamical systems given by

$$\dot x = f(x,u),\quad y=h(x), \tag{17}$$

where $x\in R^n$, $u\in R^m$, $y\in R^p$, f and h are sufficiently smooth functions.

Assumption 3.1. The mapping $f: R^n\times R^m\to R^n$ is globally Lipschitz with respect to x uniformly on u with a constant L_f.

Assumption 3.2. System (17) admits on R^n a global observer of the form $\dot{\hat x}=g(\hat x,h(x),u)$, such that the state estimation error equation has the form $\dot e=F(e,u,t)$, $e=\hat x-x$, $F(0,u,t)=0$ for all $u\in R^m$, $t\ge0$ and its equilibrium $e=0$ is globally exponentially stable with a quadratic Lyapunov function $W(e)=e^TPe$, $P=P^T>0$ satisfying the inequalities analogous to (11), where the mapping $F: R^n\times R^m\times R\to R^n$ is continously differentiable and globally Lipschitz with respect to x uniformly on u and t with a Lipschitz constant L_F.

Notice that assumption 3.2 is feasible for observer considered in the previous section and, for instance, nonlinear observers introduced in (Thau, 1973) and (Arcak and Kokotović, 1999).

Theorem 3.1. Suppose that assumptions 3.1 and 3.2 hold and furthermore there exists a continously differentiable state feedback $u(x)$ that makes the equilibrium position $x=x_*, u=u_*$ of (17) globally exponentially stable. Then the same equilibrium of system (17) in closed-loop form with the feedback law $u(\hat x)=u(x+e)$ is globally asymptotically stable.

3.1 Proof of theorem 3.1

Consider the system

$$\begin{aligned}\dot x &= f(x,u(x+e)),\\ \dot e &= F(e,u(x+e),t).\end{aligned} \tag{18}$$

Note that assumption 3.2 implies that the equilibrium $e=0$ of the estimation error equation is globally exponentially stable with a quadratic Lyapunov function $W(e)=e^TPe$, $P=P^T>0$ satisfying the inequalities analogous to (11).

Since the feedback law $u(x)$ globally exponentially stabilizes system (17) and the right-hand side of the closed-loop system (17) without output is continously differentiable, according to (Krasovskiy, 1959) there exists a Lyapunov function $V_1(x-x_*)$ such that for all $x \in R^n$ the following inequalities hold

$$c_1|x-x_*|^2 \le V_1(x-x_*) \le c_2|x-x_*|^2, \quad (19)$$

$$\Big|\frac{\partial V_1(x-x_*)}{\partial x}\Big| \le c_3|x-x_*|, \quad (20)$$

$$\frac{\partial V_1(x-x_*)}{\partial x} f(x,u(x)) \le -c_4|x-x_*|^2, \quad (21)$$

where $c_1,\ c_2,\ c_3,\ c_4$ are some positive constants.

The linear change of coordinates

$$x = z - e, \ \ e = e$$

transforms system (18) into the form

$$\begin{aligned} \dot{z} &= f(z-e,u(z)) + F(e,u(z),t), \\ \dot{e} &= F(e,u(z),t). \end{aligned} \quad (22)$$

Consider the following Lyapunov function candidate for system (22)

$$V(z-z_*,e) = kV_1(z-z_*) + W(e),$$

where $z_* = x_* =$ const and k is some positive constant to be determined later. Obviously $V(z-z_*,e) > 0$, $(z-z_*,e) \ne 0$. The derivative $\dot{V}$ of V along the trajectories of (22) satisfies

$$\begin{aligned} \dot{V}(z-z_*,e)\Big|_{(22)} &= k\frac{\partial V_1(z-z_*)}{\partial z}\dot{z} \\ +\dot{W}(e)|_{(22)} &= k\frac{\partial V_1(z-z_*)}{\partial z} f(z-e,u(z)) \\ &+k\frac{\partial V_1(z-z_*)}{\partial z}F(e,u(z),t) + \dot{W}(e)|_{(22)} \\ &= k\frac{\partial V_1(z-z_*)}{\partial z} f(z,u(z)) + k\frac{\partial V_1(z-z_*)}{\partial z} \\ &\times(f(z-e,u(z)) - f(z,u(z)) + k\frac{\partial V_1(z-z_*)}{\partial z} \\ &\times F(e,u(z),t) + \dot{W}(e)|_{(22)}. \end{aligned}$$

From (19)-(21) and the inequalities analogous to (11) it follows that

$$\begin{aligned} \dot{V}(z-z_*,e)\Big|_{(22)} &\le -kc_4|z-z_*|^2 \\ &+(kc_3L_f + kc_3L_F)|e||z-z_*| - l_4|e|^2. \end{aligned}$$

So, if one chooses k such that

$$k < 4c_4l_4/(c_3L_f + c_3L_F)^2,$$

then $\dot{V}(z-z_*,e)\Big|_{(22)}$ is negative definite on $R^n \times R^n$ and similar to the proof of theorem 2.2 one can show that the equilibrium $x = x_*$, $e = 0$ of (18) is globally exponentially stable and system (17) in closed-loop form with $u(x+e)$ is globally asymptotically stable at $x = x_*$.

4. ILLUSTRATIVE EXAMPLE

Consider the model for a flexible one-link robot arm (Marino and Tomei, 1995). The dynamics are given by

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -M_1 \sin x_1 - k_1(x_1 - x_3), \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= -b_1x_4 + k_2(x_1 - x_3) + 1/Ju, \\ y &= h(x) = x_1, \end{aligned} \quad (23)$$

where x_1 is the angle of the arm, x_2 is the angular velocity of the arm, x_3 is the angle on the motor side, x_4 is the angular velocity on the motor side, the input signal u is the driving torque from the motor, the constants $M,\ b_1,\ k_1,\ k_2,\ J$ are all positive.

It is supposed that only angular rotation x_1 of the arm is available for measurements. For this system our problem is to design a feedback control law that uses only values of angular coordinate x_1 and makes the equilibrium $x = 0$ globally asymptotically stable.

One can show that conditions of theorem 2.1 are satisfied with $p(\tau) \equiv 1$ for system (23) and assumptions 2.1 and 2.2 hold since the change of coordinates $x = \Phi(\chi), \Phi(0) = 0$

$$\begin{aligned} &x_1 = \chi_1, \ \ x_2 = \chi_2 - b_1\chi_1, \\ &x_3 = \chi_3/k_1 - b_1\chi_2/k_1 + (b_1^2 - k_2)\chi_1/k_1, \\ &x_4 = \chi_4/k_1 - b_1\chi_3/k_1 + (b_1^2 - k_2)\chi_2/k_1 \\ &\qquad + (2b_1k_2 - b_1^3)\chi_1/k_1 \end{aligned} \quad (24)$$

transforms (23) into the form

$$\begin{aligned} \dot{\chi}_1 &= \chi_2 - b_1\chi_1, \\ \dot{\chi}_2 &= \chi_3 - M_1 \sin\chi_1 - \chi_1(k_1 + k_2), \\ \dot{\chi}_3 &= \chi_4 - b_1M_1 \sin\chi_1 - b_1k_1\chi_1, \\ \dot{\chi}_4 &= -k_2M_1 \sin\chi_1 + k_1u/J, \\ y &= \chi_1 = [1,0,0,0]\chi = C\chi \end{aligned} \quad (25)$$

and generates diffeomorphism $R^4 = \{\chi\}$ onto $R^4 = \{x\}$. An exponentially stable observer for system (25) is readily constructed as

$$\dot{\hat{\chi}} = D\hat{\chi} + GC(\hat{\chi} - \chi) + \psi(\chi_1) + Bu, \quad (26)$$

where matrices D, C and vectors $B, \psi(\chi_1)$ refer to vector form of system (25), vector G is chosen such that $D + GC$ is Hurwitz. State estimation error dynamics is given by (8). Therefore the state estimation error doesn't depend on control and converges to zero exponentially fast.

System (25) is in strict-feedback form. Let us use feedback linearization procedure to construct a state feedback law that makes the equilibrium $\chi = 0$ of (25) globally exponentially stable. In new variables

$$
\begin{aligned}
z_1 &= \chi_1, \quad z_2 = \dot{z}_1 = \chi_2 - b_1\chi_1, \\
z_3 &= \dot{z}_2 = \chi_3 - M_1 \sin\chi_1 \\
&\quad - \chi_1(k_1+k_2) - b_1\chi_2 + b_1^2\chi_1, \\
z_4 &= \dot{z}_3 = \chi_4 - M_1\chi_2\cos\chi_1 \\
&\quad + b_1 M_1 \chi_1 \cos\chi_1 - (k_1+k_2)\chi_2 \\
&\quad + b_1 k_2 \chi_1 + b_1(k_1+k_2)\chi_1 \\
&\quad - b_1\chi_3 + b_1^2\chi_2 - b_1^3\chi_1
\end{aligned} \tag{27}
$$

system (25) can be written in the form

$$
\begin{aligned}
&\dot{z}_1 = z_2, \quad \dot{z}_2 = z_3, \quad \dot{z}_3 = z_4, \\
&\dot{z}_4 = f(z) + k_1 u/J, \quad y = z_1,
\end{aligned} \tag{28}
$$

where $f(z) = -k_2 M_1 \sin z_1 - M_1 \cos z_1(z_3 + b_1 z_2)$ $+M_1 z_2^2 \sin z_1 - (k_1+k_2)(z_3+b_1z_2) + b_1k_2z_2$ $-b_1z_4$. Obviously $z = \mu^{-1}(\chi)$, $\mu^{-1}(0) = 0$ (27) is ivertible as $\chi = \mu(z)$ and generates diffeomorphism $R^4 = \{\chi\}$ onto $R^4 = \{z\}$. A continously differentiable state feedback $u^*(z)$ that makes the equilibrium $z = 0$ of (28) globally exponentially stable is the following

$$
u^*(z) = Jk_1^{-1}(-f(z) - \sum_{i=0}^{3} \kappa_i z_{i+1}),
$$

where positive constants κ_i, $i = \overline{0,3}$ are chosen such that the closed-loop system (28)

$$
\begin{aligned}
&\dot{z}_1 = z_2, \quad \dot{z}_2 = z_3, \quad \dot{z}_3 = z_4, \\
&\dot{z}_4 = -\kappa_0 z_1 - \kappa_1 z_2 - \kappa_2 z_3 - \kappa_3 z_4, \\
&y = z_1
\end{aligned}
$$

is globally asymptotically stable. Moreover notice that $z = \mu^{-1}(\chi)$ and $\chi = \mu(z)$ are such that

$$
\begin{aligned}
|z| &= |\mu^{-1}(\chi)| \le L_1|\chi| \quad \forall \chi \in R^n, \\
|\chi| &= |\mu(z)| \le L_2|z| \quad \forall z \in R^n,
\end{aligned}
$$

where L_1, L_2 are some positive constants. The latter implies that the map $u^*(\mu^{-1}(\chi))$ globally exponentially stabilizes system (25) at $\chi = 0$. Since the change of variables (24) is linear, the state feedback law $u_0(x) = u^*(\mu^{-1}(\Phi^{-1}(x)))$ globally exponentially stabilizes system (23) without output at $x = 0$. So, according to theorem 2.4 the equilibrium $x = 0$ of (23) in closed-loop form with the output feedback $u_0(\hat{x}) = u^*(\mu^{-1}(\Phi^{-1}(\hat{x})))$, $\hat{x} = \Phi(\hat{\chi})$ is globally asymptotically stable.

5. CONCLUSIONS

In this paper for affine systems that can be transformed into a form in which the nonlinearities do not depend on unmeasured states and admit an exponentially stable observer we presented global output feedback stabilization results and showed that the global separation principle holds. The global separation principle for output feedback stabilization of this class of nonlinear systems was not treated before in the literature. Our stabilization results are also presented for general nonlinear systems.

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