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CHANGING SUPPLY RATES FOR INPUT-OUTPUT TO STATE STABLE DISCRETE-TIME SYSTEMS

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Abstract: We present results on changing supply rates for input-output to state stable (IOSS) discrete-time nonlinear systems. Our results can be used to combine two Lyapunov functions, none of which can be used to verify that the system has a certain property, into a new Lyapunov function from which the property of interest can be concluded. We present two applications of our results to discrete-time systems, i.e. a LaSalle criterion for input to state stability (ISS) and input to state stability with positive semidefinite Lyapunov functions.

Keywords: Discrete-time systems; Sampled-data systems; Lyapunov function; Nonlinear control; Stability; Cascade control.

1. INTRODUCTION

The Lyapunov method is one of the most important and useful methods in stability analysis and design of nonlinear control systems (see for example (Khalil, 1996), (Rouche *et al.*, 1977)). Lyapunov functions, which are the main tool in this method, can be used to characterize various properties of control systems, such as stability, detectability and passivity. Unfortunately, there is no general systematic way of finding a Lyapunov function for a system. Hence, methods for constructing Lyapunov functions are of utmost importance.

A very useful method for partially constructing Lyapunov functions was introduced in (Sontag and Teel, 1995) where it was shown how it is possible to combine two Lyapunov functions, none of which can be used to conclude a property of interest, into a new composite Lyapunov function from which the desired property follows. Results in (Sontag and Teel, 1995) apply to analysis of input to state stability (ISS) property of continuous-time cascade-connected systems. In (Angeli, 1999) a similar proof technique was used to combine a Lyapunov function whose derivative is negative semidefinite and another Lyapunov function that characterizes a detectability property, which is called input-output to state stability (IOSS) (see (Sontag and Wang, 1997)), into a new Lyapunov function from which ISS property of a continuous-time system follows. A discrete-time counterpart of results in (Sontag and Teel, 1995) was presented in (Nešić and Teel, 2001*b*). These results were useful, for instance in discrete-time backstepping (Nešić and Teel, 2001*a*) and in stabilization of robot manipulators (Angeli, 1999).

The purpose of this paper is twofold. First, we present discrete-time counterpart of results in (Angeli, 1999), to prove the lemma on changing supply rates for IOSS discrete-time systems (see Lemma 3.1) that generalizes the results of (Nešić and Teel, 2001*b*). Second, we pose and solve the problem in a more general framework (see Definition 2.1) than in (Angeli, 1999), (Nešić and Teel, 2001*b*), (Sontag and Teel, 1995), so that the results in (Nešić and Teel, 2001*b*) and discrete-time counterparts of results in (Angeli, 1999) are generalized. In this way, we are able to apply the

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original technique of (Sontag and Teel, 1995) to several new problems. While the statements of our main results are very similar to (Angeli, 1999), (Sontag and Teel, 1995), the proof technique is notably different from the one used in continuous-time results. It requires a judicious use of the Mean Value Theorem (see Lemma 3.1), which is similar to the discrete-time results of (Nešić and Teel, 2001*b*). We consider families of discrete-time systems parameterized by T (sampling period), which naturally arise when using approximate discrete-time model to design a digital controller for a nonlinear sampled-data system (see (Nešić *et al.*, 1999), (Nešić and Laila, 2001)).

Our main results (Theorems 3.1 and 3.2) are based on changing supply rates for discrete-time IOSS systems. They present two constructions of composite IOSS Lyapunov functions. We show the usefulness of our results by applying them to two different problems. In particular, we present a LaSalle criterion for ISS property and testing ISS via two positive semidefinite Lyapunov functions.

2. PRELIMINARIES

The set of real numbers is denoted by $\mathbb{R}$. $\mathcal{SN}$ denotes the class of all smooth nondecreasing functions $q : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, which satisfy $q(t) > 0$ for all $t > 0$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{G}$ if it is continuous, nondecreasing and zero at zero. It is of class $\mathcal{K}$ if it is of class $\mathcal{G}$ and strictly increasing; and it is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. Functions of class $\mathcal{K}_\infty$ are invertible. $|x|$ denotes the 1-norm of a vector $x \in \mathbb{R}^n$, where $|x| := \sum_{i=1}^{n} |x_i|$. We consider a parameterized family of discrete-time nonlinear systems of the following form:

$$\begin{aligned} x_{k+1} &= F_T(x_k, u_k) \\ y_k &= h_T(x_k) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ are respectively the state, input and output of the system. The function h_T is a continuous function, with $h_T(0) = 0$. T is the sampling period, which parameterizes the system and can be arbitrarily assigned. Parameterized discrete-time systems (1) commonly arise when an approximate discrete-time model is used for designing a digital controller for a nonlinear sampled-data system. We use the following definition.

Definition 2.1. The system (1) is $(V_T, \underline{\alpha}, \overline{\alpha}, \alpha, \lambda, \sigma)$-input-output to state stable ($(V_T, \underline{\alpha}, \overline{\alpha}, \alpha, \lambda, \sigma)$-IOSS) with measuring functions, if there exist functions $\underline{\alpha}$, $\overline{\alpha}$, $\alpha \in \mathcal{K}_\infty$, and λ, $\sigma \in \mathcal{G}$, functions $w_{\underline{\alpha}} : \mathbb{R}^n \to \mathbb{R}^{n_1}$, $w_{\overline{\alpha}} : \mathbb{R}^n \to \mathbb{R}^{n_2}$, $w_\alpha : \mathbb{R}^n \to \mathbb{R}^{n_3}$, $w_\lambda : \mathbb{R}^n \to \mathbb{R}^{\nu}$, $w_\sigma : \mathbb{R}^m \to \mathbb{R}^{\mu}$, which are zero at zero, $T^* > 0$ and for all $T \in (0, T^*)$ there exists a smooth function $V_T : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that:

$$\underline{\alpha}(|w_{\underline{\alpha}}(x)|) \leq V_T(x) \leq \overline{\alpha}(|w_{\overline{\alpha}}(x)|) \tag{2}$$

$$\begin{aligned} V_T(F_T(x,u)) - V_T(x) &\leq -T\alpha(|w_\alpha(x)|) \\ &\quad + T\lambda(|w_\lambda(x)|) + T\sigma(|w_\sigma(u)|) \ , \end{aligned} \tag{3}$$

for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $T \in (0, T^*)$. The functions $w_{\underline{\alpha}}$, $w_{\overline{\alpha}}$, w_α, w_λ, w_σ are called measuring functions; $\underline{\alpha}$, $\overline{\alpha}$, α, λ, σ are called bounding functions; α, λ, σ are called supply functions; and V_T is called an IOSS Lyapunov function. ■

Often, when all functions are clear from the context, we refer to the property defined in Definition 2.1 simply as IOSS. Moreover, if the system is IOSS with $\lambda = 0$ then we say that the system is ISS. IOSS with measuring function is quite a general notion that covers a range of different properties of nonlinear discrete-time systems, such as stability, detectability, output to state stability, etc. The reason for introducing such a general property in Definition 2.1 is that we will apply our results to a range of its different special cases (see Section 4) for particular choices of bounding functions and measuring functions. Hence, Definition 2.1 is a very compact way of defining a range of different properties to which our results apply.

The following two lemmas and remark are used in proving our main results (Theorems 3.1 and 3.2).

Lemma 2.1. (Sontag and Teel, 1995) Assume that the functions β, $\beta' \in \mathcal{K}$ are such that $\beta'(s) = O[\beta(s)]$ as $s \to 0^+$. Then there exists a function $q \in \mathcal{SN}$ so that $\beta'(s) \leq q(s)\beta(s), \ \ \forall s \geq 0$. ■

Lemma 2.2. (Sontag and Teel, 1995) Assume that the functions β, $\beta' \in \mathcal{K}$ are such that $\beta(r) = O[\beta'(r)]$ as $r \to +\infty$. Then there exists a function $q \in \mathcal{SN}$ so that $q(r)\beta(r) \leq \beta'(r), \ \ \forall r \geq 0$. ■

Remark 1. Since for any $\alpha \in \mathcal{K}$ we have $\alpha(s_1 + s_2) \leq \alpha(2s_1) + \alpha(2s_2)$ for all $s_1 \geq 0, s_2 \geq 0$, then for any $\alpha_1, \alpha_2 \in \mathcal{K}$, there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}$ such that the following holds for all $s_1 \geq 0, s_2 \geq 0$:

$$\underline{\alpha}(s_1 + s_2) \leq \alpha_1(s_1) + \alpha_2(s_2) \leq \overline{\alpha}(s_1 + s_2) \ , \tag{4}$$

where we can take $\underline{\alpha}(s) := \min\{\alpha_1(\frac{s}{2}), \alpha_2(\frac{s}{2})\}$ and $\overline{\alpha}(s) := \max\{2\alpha_1(s), 2\alpha_2(s)\}$. ■

3. MAIN RESULTS

In this section, we present our main results (Theorems 3.1 and 3.2), in which we show the construction of an IOSS Lyapunov function from two auxiliary IOSS Lyapunov functions. We first present Lemma 3.1, which is instrumental in proving our main results. The lemma is a discrete-time version, as well as a generalization, of the lemma on changing supply rates for IOSS systems in (Angeli, 1999). We introduce the following construction that follows the one used in

(Angeli, 1999), (Sontag and Teel, 1995). Denote $V_T(F_T) := V_T(F_T(x,u))$ and $V_T := V_T(x)$. Given an arbitrary $q \in \mathcal{SN}$, we define:

$$\rho(s) := \int_0^s q(\tau)d\tau \ , \tag{5}$$

where it is easy to see that $\rho \in \mathcal{K}_\infty$ and ρ is smooth. Suppose that we have an IOSS Lyapunov function V_T, and then consider a new composite function $\rho(V_T)$. In Lemma 3.1, we state conditions under which the new function is also an IOSS Lyapunov function for the system.

Lemma 3.1. Let the following conditions be satisfied:

(1) System (1) is $(V_T, \underline{\alpha}, \overline{\alpha}, \alpha, \lambda, \sigma)$-IOSS with measuring functions $w_{\underline{\alpha}}$, $w_{\overline{\alpha}}$, w_α, w_λ and w_σ.
(2) There exist $\underline{\kappa}, \overline{\kappa} \in \mathcal{K}_\infty$ such that $\underline{\kappa}(|w_\alpha(x)|) \leq |w_{\underline{\alpha}}(x)|$ and $|w_{\overline{\alpha}}(x)| \leq \overline{\kappa}(|w_\alpha(x)|)$ for all $x \in \mathbb{R}^n$.

Then for any $q \in \mathcal{SN}$ and $\rho \in \mathcal{K}_\infty$ defined by (5) there exist $\underline{\alpha}'$, $\overline{\alpha}'$, α', λ', σ' such that the system (1) is $(\rho(V_T), \underline{\alpha}', \overline{\alpha}', \alpha', \lambda', \sigma')$-IOSS with the same measuring functions, where $\underline{\alpha}'(s) = \rho \circ \underline{\alpha}(s)$, $\overline{\alpha}'(s) = \rho \circ \overline{\alpha}(s)$, $\alpha'(s) = \frac{1}{4} q \circ \frac{1}{2} \underline{\alpha} \circ \underline{\kappa}(s) \cdot \alpha(s)$, $\lambda'(s) = 2q \circ \theta_\lambda(s) \cdot \lambda(s)$, $\sigma'(s) = 2q \circ \theta_\sigma(s) \cdot \sigma(s)$, and

$$\theta_\sigma(s) := \overline{\alpha} \circ \overline{\kappa} \circ \alpha^{-1} \circ 4\sigma(s) + 2T^*\sigma(s) \tag{6}$$

$$\theta_\lambda(s) := \overline{\alpha} \circ \overline{\kappa} \circ \alpha^{-1} \circ 4\lambda(s) + 2T^*\lambda(s) \tag{7}$$

■

We present in Theorems 3.1 and 3.2 two constructions that can be used (under different conditions on the bounding and measuring functions) to produce a new IOSS Lyapunov function V_T from two IOSS Lyapunov functions V_{1T} and V_{2T}. We note that Theorem 3.1 is a discrete-time version, as well as generalization, of the continuous-time results in (Angeli, 1999), while Theorem 3.2 has appeared in a simpler form in (Nešić and Teel, 2001*b*), which is a discrete-time version of (Sontag and Teel, 1995), when $\lambda = 0$, $w_{\underline{\alpha}}(x) = w_{\overline{\alpha}}(x) = w_\alpha(x) = x$, and $w_\sigma(u) = u$.

Theorem 3.1. Suppose that:

(1) the system (1) is $(V_{1T}, \underline{\alpha}_1, \overline{\alpha}_1, \alpha_1, \lambda_1, \sigma_1)$-IOSS with measuring functions $w_{\underline{\alpha}_1}$, $w_{\overline{\alpha}_1}$, w_{α_1}, w_{λ_1}, w_{σ_1}, where $\lambda_1 = 0$;
(2) the system (1) is $(V_{2T}, \underline{\alpha}_2, \overline{\alpha}_2, \alpha_2, \lambda_2, \sigma_2)$-IOSS with measuring functions $w_{\underline{\alpha}_2}$, $w_{\overline{\alpha}_2}$, w_{α_2}, w_{λ_2}, w_{σ_2} and there exist $\underline{\kappa}_2, \overline{\kappa}_2 \in \mathcal{K}_\infty$, such that the second condition of Lemma 3.1 holds;
(3) $|w_{\lambda_2}(x)| \leq |w_{\alpha_1}(x)|$ for all $x \in \mathbb{R}^n$;
(4) $\limsup_{r \to +\infty} \frac{\lambda_2(r)}{\alpha_1(r)} < +\infty$.

Then there exists $\rho \in \mathcal{K}_\infty$ such that the system (1) is $(V_T, \underline{\alpha}, \overline{\alpha}, \alpha, \lambda, \sigma)$-IOSS with $\lambda = 0$ and new measuring functions $w_{\underline{\alpha}}$, $w_{\overline{\alpha}}$, w_α, w_σ, where

$$V_T = V_{1T} + \rho(V_{2T}) \ , \tag{8}$$

and the new measuring functions are $w_{\underline{\alpha}}(s) := |w_{\underline{\alpha}_1}(s)| + |w_{\underline{\alpha}_2}(s)|$, $w_{\overline{\alpha}}(s) := |w_{\overline{\alpha}_1}(s)| + |w_{\overline{\alpha}_2}(s)|$, $w_\alpha(s) := |w_{\alpha_2}(s)|$ and $w_\sigma(s) := |w_{\sigma_1}(s)| + |w_{\sigma_2}(s)|$. ■

Note that the second condition of Lemma 3.1 does not hold for V_{1T} in Theorem 3.1. Hence, we need an extra condition on the bounding functions. In the next result, we consider a stronger condition for the Lyapunov function V_{1T}, so that we can relax the condition 4 of the Theorem 3.1.

Theorem 3.2. Suppose that:

(1) the system (1) is $(V_{1T}, \underline{\alpha}_1, \overline{\alpha}_1, \alpha_1, \lambda_1, \sigma_1)$-IOSS with measuring functions $w_{\underline{\alpha}_1}$, $w_{\overline{\alpha}_1}$, w_{α_1}, w_{λ_1}, w_{σ_1}, where $\lambda_1 = 0$, and there exist $\underline{\kappa}_1, \overline{\kappa}_1 \in \mathcal{K}_\infty$, such that the second condition of Lemma 3.1 holds;
(2) the system (1) is $(V_{2T}, \underline{\alpha}_2, \overline{\alpha}_2, \alpha_2, \lambda_2, \sigma_2)$-IOSS with measuring functions $w_{\underline{\alpha}_2}$, $w_{\overline{\alpha}_2}$, w_{α_2}, w_{λ_2}, w_{σ_2} and there exist $\underline{\kappa}_2, \overline{\kappa}_2 \in \mathcal{K}_\infty$, such that the second condition of Lemma 3.1 holds;
(3) $|w_{\lambda_2}(x)| \leq |w_{\alpha_1}(x)|$ for all $x \in \mathbb{R}^n$.

Then there exist $\rho_1, \rho_2 \in \mathcal{K}_\infty$ such that the system (1) is $(V_T, \underline{\alpha}, \overline{\alpha}, \alpha, \lambda, \sigma)$-IOSS with $\lambda = 0$ and new measuring functions $w_{\underline{\alpha}}$, $w_{\overline{\alpha}}$, w_α, w_σ, where

$$V_T = \rho_1(V_{1T}) + \rho_2(V_{2T}) \ , \tag{9}$$

and the new measuring functions are $w_{\underline{\alpha}}(s) := |w_{\underline{\alpha}_1}(s)| + |w_{\underline{\alpha}_2}(s)|$, $w_{\overline{\alpha}}(s) := |w_{\overline{\alpha}_1}(s)| + |w_{\overline{\alpha}_2}(s)|$, $w_\alpha(s) := |w_{\alpha_1}(s)| + |w_{\alpha_2}(s)|$ and $w_\sigma(s) := |w_{\sigma_1}(s)| + |w_{\sigma_2}(s)|$. ■

4. APPLICATIONS

4.1 *Application 1: a LaSalle criterion for ISS*

We present an application of Theorem 3.1 which is novel. The application illustrates a discrete-time version of the continuous-time result presented in (Angeli, 1999). The results of (Angeli, 1999) were useful in PD control of robot manipulators. Consider the system (1). Let $T^* > 0$ be given, and for all $T \in (0, T^*)$ the following conditions hold:

(1) the system (1) is $(V_{1T}, \underline{\alpha}_1, \overline{\alpha}_1, \alpha_1, \lambda_1, \sigma_1)$-IOSS with measuring functions $w_{\underline{\alpha}_1}(x) = w_{\overline{\alpha}_1}(x) = x$, $w_{\alpha_1}(x) = h(x) = y$, $w_{\sigma_1}(u) = u$, and $\lambda_1 = 0$;
(2) the system (1) is $(V_{2T}, \underline{\alpha}_2, \overline{\alpha}_2, \alpha_2, \lambda_2, \sigma_2)$-IOSS with measuring functions $w_{\underline{\alpha}_2}(x) = w_{\overline{\alpha}_2}(x) = w_{\alpha_2}(x) = x$, $w_{\lambda_2}(x) = h(x) = y$ and $w_{\sigma_2}(u) = u$; the second condition of Lemma 3.1 holds since $w_{\underline{\alpha}_2}(x) = w_{\overline{\alpha}_2}(x) = w_{\alpha_2}(x)$;

(3) $|w_{\lambda_2}(x)| = |w_{\alpha_1}(x)| = h(x) = y, \ \forall x \in \mathbb{R}^n$;
(4) $\limsup_{r\to+\infty} \frac{\lambda_2(r)}{\alpha_1(r)} < +\infty$.

Hence, all conditions of Theorem 3.1 hold. Then, applying Theorem 3.1 and defining the new IOSS Lyapunov function V_T as in (23), we obtain that there exist $\underline{\alpha}, \overline{\alpha}, \alpha, \sigma$ such that

$$\underline{\alpha}(|x|) \leq V_T \leq \overline{\alpha}(|x|) \tag{10}$$
$$V_T(F_T) - V_T \leq -T\alpha(|x|) + T\sigma(|u|) \ . \tag{11}$$

Therefore, the system (1) is IOSS with measuring functions $w_{\underline{\alpha}}(x) = w_{\overline{\alpha}}(x) = w_\alpha(x) = x$ and $w_\sigma(u) = u$, where $\lambda = 0$. By Definition 3.2 and Lemma 3.5 of (Jiang and Wang, 2001), we can conclude that the system (1) is ISS. Note that the property of IOSS with measuring functions stated in conditions 1 and 2 are respectively referred to in (Angeli, 1999) as quasi input to state stability (qISS) and IOSS properties.

4.2 *Application 2: Positive semidefinite input to state stable (ISS) Lyapunov functions*

This application deals with systems with positive semidefinite Lyapunov functions. It is motivated by the work on stability with positive semidefinite Lyapunov functions by (Grizzle and Kang, 2001), (Iggidr *et al.*, 1996). Consider the system (1). Let $T^* > 0$ be given, and for all $T \in (0, T^*)$ the following conditions hold:

(1) the system (1) is $(V_{1T}, \underline{\alpha}_1, \overline{\alpha}_1, \alpha_1, \lambda_1, \sigma_1)$-IOSS with measuring functions $w_{\underline{\alpha}_1}(x)$, $w_{\overline{\alpha}_1}(x)$, $w_{\alpha_1}(x)$, $w_{\sigma_1}(u) = u$, where $\lambda_1 = 0$ and $|w_{\underline{\alpha}_1}(x)|$ is positive semidefinite, and there exist $\underline{\kappa}_1, \overline{\kappa}_1 \in \mathcal{K}_\infty$, such that the second condition of Lemma 3.1 holds;
(2) the system (1) is $(V_{2T}, \underline{\alpha}_2, \overline{\alpha}_2, \alpha_2, \lambda_2, \sigma_2)$-IOSS with measuring functions $w_{\underline{\alpha}_2}(x)$, $w_{\overline{\alpha}_2}(x)$, $w_{\alpha_2}(x)$, $w_{\lambda_2}(x)$, $w_{\sigma_2}(u) = u$, where $|w_{\underline{\alpha}_2}(x)|$ is positive semidefinite, and there exist $\underline{\kappa}_2, \overline{\kappa}_2 \in \mathcal{K}_\infty$, such that the second condition of Lemma 3.1 holds;
(3) $|w_{\lambda_2}(x)| \leq |w_{\alpha_1}(x)|$, $\forall x \in \mathbb{R}^n$.
(4) there exists $\kappa_{\underline{\alpha}}, \kappa_{\overline{\alpha}}, \kappa_\alpha \in \mathcal{K}_\infty$, such that $\kappa_{\underline{\alpha}}(|x|) \leq |w_{\underline{\alpha}_1}(x)| + |w_{\underline{\alpha}_2}(x)|$, $\kappa_{\overline{\alpha}}(|x|) \geq |w_{\overline{\alpha}_1}(x)| + |w_{\overline{\alpha}_2}(x)|$ and $\kappa_\alpha(|x|) \leq |w_{\alpha_1}(x)| + |w_{\alpha_2}(x)|$, $\forall x \in \mathbb{R}^n$.

Hence, all conditions of Theorem 3.2 hold. Then, applying Theorem 3.2 and define the new Lyapunov function V_T as in (32), we obtain

$$\begin{aligned}\underline{\alpha} \circ \kappa_{\underline{\alpha}}(|x|) &\leq \underline{\alpha}(|w_{\underline{\alpha}_1}(x)| + |w_{\underline{\alpha}_2}(x)|) \leq V_T \\ &\leq \overline{\alpha}(|w_{\overline{\alpha}_1}(x)| + |w_{\overline{\alpha}_2}(x)|) \leq \overline{\alpha} \circ \kappa_{\overline{\alpha}}(|x|) \ ,\end{aligned} \tag{12}$$

$$\begin{aligned}V_T(F_T) - V_T &\leq T\sigma(|u|) - T\alpha(|w_{\alpha_1}(x)| \\ &\quad + |w_{\alpha_2}(x)|) \\ &\leq T\sigma(|u|) - T\alpha \circ \kappa_\alpha(|x|)\end{aligned} \tag{13}$$

Therefore, the system (1) is IOSS with measuring functions $w_{\underline{\alpha}}(x) = w_{\overline{\alpha}}(x) = w_\alpha(x) = x$, $w_\sigma(u) = u$, where $\lambda = 0$. A special case of this application is the case of cascade-connected systems:

$$\begin{aligned}x_{k+1} &= F_T(x_k, z_k) \\ z_{k+1} &= G_T(z_k, u_k) \ ,\end{aligned} \tag{14}$$

which has been presented in (Nešić and Teel, 2001*b*) to prove ISS property of a cascade-connection of two ISS discrete-time systems. In this case, denote the states of the overall system as $\tilde{x} := (x^T \ z^T)^T$, $\tilde{x} \in \mathbb{R}^n$, we have

(1) the Lyapunov function V_{1T} with measuring functions $w_{\underline{\alpha}_1}(\tilde{x}) = w_{\overline{\alpha}_1}(\tilde{x}) = w_{\alpha_1}(\tilde{x}) = x$, $w_{\lambda_1}(\tilde{x}) = z$ and $\sigma_1 = 0$, so that $\underline{\kappa}_1$, $\overline{\kappa}_1$ exist;
(2) the Lyapunov function V_{2T} with measuring functions $w_{\underline{\alpha}_2}(\tilde{x}) = w_{\overline{\alpha}_2}(\tilde{x}) = w_{\alpha_2}(\tilde{x}) = z$, $w_{\sigma_2}(u) = u$ and $\lambda_2 = 0$, so that $\underline{\kappa}_2$, $\overline{\kappa}_2$ exist;
(3) $|w_{\lambda_1}(\tilde{x})| = |w_{\alpha_2}(\tilde{x})| = z$, $\forall \tilde{x} \in \mathbb{R}^n$;
(4) $\kappa_{\underline{\alpha}}, \kappa_{\overline{\alpha}}, \kappa_\alpha \in \mathcal{K}_\infty$ exist, since $|w_{\underline{\alpha}_1}(\tilde{x})| + |w_{\underline{\alpha}_2}(\tilde{x})| = |w_{\overline{\alpha}_1}(\tilde{x})| + |w_{\overline{\alpha}_2}(\tilde{x})| = |w_{\alpha_1}(\tilde{x})| + |w_{\alpha_2}(\tilde{x})| = |x| + |z| =: |\tilde{x}|$.

Hence, by substitution to (12), (13) we obtain ISS characterization for the system (14).

5. PROOF OF MAIN RESULTS

In this section, we provide the proof of Theorems 3.1 and 3.2. Due to limited space, we only provide the sketch of the proof of Lemma 3.1. Readers are referred to (Laila and Nešić, 2001) for complete results and proofs.

Sketch of proof of Lemma 3.1 Suppose that all conditions in Lemma 3.1 are satisfied. From the Mean Value Theorem and the fact that $q(\cdot) = \frac{d\rho}{ds}(\cdot)$ is nondecreasing, it follows that

$$\rho(a) - \rho(b) \leq q(a)[a-b], \quad \forall a \geq 0, \ b \geq 0 \ . \tag{15}$$

We also use (2) and (3) to arrive at

$$\begin{aligned}V_T \geq \max\{&\underline{\alpha}(|w_{\underline{\alpha}}(x)|), T\alpha(|w_\alpha(x)|) \\ &- T\sigma(|w_\sigma(u)|) - T\lambda(|w_\lambda(x)|)\}\end{aligned} \tag{16}$$

$$\begin{aligned}V_T(F_T) &\leq \overline{\alpha}(|w_{\overline{\alpha}}(x)|) \\ &\quad + T^*\sigma(|w_\sigma(u)|) + T^*\lambda(|w_\lambda(x)|) \ .\end{aligned} \tag{17}$$

Let $T \in (0, T^*)$, where $T^* > 0$ comes from the first condition of the lemma. The proof is carried out by considering the following three cases, where in each of them we consider the sub-cases $V_T(F_T) \leq \frac{1}{2}V_T$ and $V_T(F_T) > \frac{1}{2}V_T$:

Case 1: $\lambda(|w_\lambda(x)|) + \sigma(|w_\sigma(u)|) \leq \frac{1}{2}\alpha(|w_\alpha(x)|)$
Using the fact that $q \in \mathcal{SN}$, (16) and the second condition of the Lemma, then

$$\begin{aligned}&\rho(V_T(F_T)) - \rho(V_T) \\ &\leq -\frac{T}{4} q \circ \frac{1}{2}\underline{\alpha} \circ \underline{\kappa}(|w_\alpha(x)|) \cdot \alpha(|w_\alpha(x)|) \ .\end{aligned} \tag{18}$$

Case 2: $\lambda(|w_\lambda(x)|) + \sigma(|w_\sigma(u)|) > \frac{1}{2}\alpha(|w_\alpha(x)|)$ and $\lambda(|w_\lambda(x)|) \geq \sigma(|w_\sigma(u)|)$

Using (2), the fact that $q \in \mathcal{SN}$, (16), the second condition of the Lemma, $\lambda(|w_\lambda(x)|) > \frac{1}{4}\alpha(|w_\alpha(x)|)$ and (7), then

$$\begin{aligned}\rho(V_T(F_T)) - \rho(V_T) \\ \leq 2Tq \circ \theta_\lambda(|w_\lambda(x)|) \cdot \lambda(|w_\lambda(x)|) \\ - \frac{T}{2} q \circ \frac{1}{2}\underline{\alpha} \circ \underline{\kappa}(|w_\alpha(x)|) \cdot \alpha(|w_\alpha(x)|) \ .\end{aligned} \tag{19}$$

Case 3: $\lambda(|w_\lambda(x)|) + \sigma(|w_\sigma(u)|) > \frac{1}{2}\alpha(|w_\alpha(x)|)$ and $\lambda(|w_\lambda(x)|) < \sigma(|w_\sigma(u)|)$
Using (2), the fact that $q \in \mathcal{SN}$, (16), the second condition of the Lemma, $\sigma(|w_\sigma(u)|) > \frac{1}{4}\alpha(|w_\alpha(x)|)$ and (6), then

$$\begin{aligned}\rho(V_T(F_T)) - \rho(V_T) \\ \leq 2Tq \circ \theta_\sigma(|w_\sigma(u)|) \cdot \sigma(|w_\sigma(u)|) \\ - \frac{T}{2} q \circ \frac{1}{2}\underline{\alpha} \circ \underline{\kappa}(|w_\alpha(x)|) \cdot \alpha(|w_\alpha(x)|) \ .\end{aligned} \tag{20}$$

We obtain through these three cases that the following holds:

$$\begin{aligned}\rho(V_T(F_T)) - \rho(V_T(x)) \\ \leq T\Big[2q \circ \theta_\sigma(|w_\sigma(u)|) \cdot \sigma(|w_\sigma(u)|) \\ + 2q \circ \theta_\lambda(|w_\lambda(x)|) \cdot \lambda(|w_\lambda(x)|) \\ - \frac{1}{4} q \circ \frac{1}{2}\underline{\alpha} \circ \underline{\kappa}(|w_\alpha(x)|) \cdot \alpha(|w_\alpha(x)|)\Big] \ ,\end{aligned} \tag{21}$$

which completes the proof of Lemma 3.1. ■

Proof of Theorem 3.1 Let all conditions of the theorem be satisfied. Let $T_1^* > 0$ and $T_2^* > 0$ respectively come from conditions 1 and 2, and define $T^* := \min\{T_1^*, T_2^*\}$. Consider arbitrary $T \in (0, T^*)$. Let V_{1T}, α_1, σ_1 come from the condition 1 and V_{2T}, α_2, λ_2, σ_2 come from the condition 2. Define $\tilde{q}$ as:

$$\tilde{q}(r) := \inf_{r \leq s} \frac{\alpha_1(s)}{2(1+\lambda_2(s))} \ . \tag{22}$$

Notice that $\tilde{q}$ is by definition a nondecreasing function. Condition 4 of the theorem implies $\tilde{q}(r) > 0$ for all $r > 0$. Let $q(s) := \tilde{q} \circ \theta_{\lambda_2}^{-1}(s)$, where θ_{λ_2} is defined in (7). By using q to obtain ρ via (5), we define V_T as:

$$V_T := V_{1T} + \rho(V_{2T}) \ . \tag{23}$$

We claim that the system (1) is $(V_T, \underline{\alpha}, \overline{\alpha}, \alpha, \lambda, \sigma)$-IOSS with $\lambda = 0$ and the measuring functions given in the statement of Theorem 3.1.

It follows from the definition of V_T that

$$\begin{aligned}\underline{\alpha}_1(|w_{\underline{\alpha}_1}(x)|) + \rho \circ \underline{\alpha}_2(|w_{\underline{\alpha}_2}(x)|) \leq V_T \\ \leq \overline{\alpha}_1(|w_{\overline{\alpha}_1}(x)|) + \rho \circ \overline{\alpha}_2(|w_{\overline{\alpha}_2}(x)|) \ .\end{aligned} \tag{24}$$

By Remark 1, there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that

$$\begin{aligned}\underline{\alpha}(|w_{\underline{\alpha}_1}(x)| + |w_{\underline{\alpha}_2}(x)|) \leq V_T \\ \leq \overline{\alpha}(|w_{\overline{\alpha}_1}(x)| + |w_{\overline{\alpha}_2}(x)|) \ .\end{aligned} \tag{25}$$

In particular, we can take $\underline{\alpha}(s_1+s_2) := \underline{\alpha}_1(s_1) + \rho \circ \underline{\alpha}_2(s_2)$ and $\overline{\alpha}(s_1+s_2) := \overline{\alpha}_1(s_1) + \rho \circ \overline{\alpha}_2(s_2)$.

Moreover, using (22), condition 3 of the theorem and let $\alpha_2', \lambda_2', \sigma_2'$ come from Lemma 3.1, the dissipation inequality for V_T becomes:

$$\begin{aligned}V_T(F_T) - V_T \\ \leq T\Big[\sigma_1(|w_{\sigma_1}(u)|) + \sigma_2'(|w_{\sigma_2}(u)|) \\ - \alpha_1(|w_{\alpha_1}(x)|) + \lambda_2'(|w_{\alpha_1}(x)|) \\ - \alpha_2'(|w_{\alpha_2}(x)|)\Big] \\ \leq T\Big[\sigma_1(|w_{\sigma_1}(u)|) + \sigma_2'(|w_{\sigma_2}(u)|) \\ + \frac{\alpha_1(|w_{\alpha_1}(x)|)\lambda_2(|w_{\alpha_1}(x)|)}{1+\lambda_2(|w_{\alpha_1}(x)|)} \\ - \alpha_1(|w_{\alpha_1}(x)|) - \alpha_2'(|w_{\alpha_2}(x)|)\Big] \ .\end{aligned} \tag{26}$$

Since

$$\frac{\lambda_2(|w_{\alpha_1}(x)|)}{1+\lambda_2(|w_{\alpha_1}(x)|)} \leq 1 \ ,$$

by monotonicity of q and using Remark 1, there exist $\alpha \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ so that we can write

$$\begin{aligned}V_T(F_T) - V_T \leq -T\alpha(|w_{\alpha_1}(x)| + |w_{\alpha_2}(x)|) \\ + T\sigma(|w_{\sigma_1}(u)| + |w_{\sigma_2}(u)|) \ .\end{aligned} \tag{27}$$

In particular, we can take $\alpha(s_1+s_2) := \alpha_2'(s_2)$ and $\sigma(s_1+s_2) := \sigma_1(s_1) + \sigma_2'(s_2)$. This completes the proof of Theorem 3.1. ■

Proof of Theorem 3.2 Let all conditions of the theorem be satisfied. Let $T_1^* > 0$ and $T_2^* > 0$ respectively come from conditions 1 and 2, and define $T^* := \min\{T_1^*, T_2^*\}$. Consider arbitrary $T \in (0, T^*)$. Let V_{1T}, α_1, σ_1 come from the condition 1 and V_{2T}, α_2, λ_2, σ_2 come from the condition 2. Define a function $\alpha_1' \in \mathcal{K}_\infty$ as follows

$$\alpha_1'(s) := \begin{cases} \alpha_1(s) & \text{for small } s, \\ \lambda_2(s) & \text{for large } s. \end{cases} \tag{28}$$

It is clear that $\alpha_1'(s) = O[\alpha_1(s)]$ for $s \to 0^+$. Hence, by Lemma 2.1 there exists $\tilde{q}_1 \in \mathcal{SN}$ such that $\tilde{q}_1(s) \cdot \alpha_1(s) \geq \alpha_1'(s)$. Further, define a function $\lambda_2' \in \mathcal{K}$:

$$\lambda_2'(s) := \frac{1}{2}\alpha_1'(s) \ , \tag{29}$$

and it is clear that $\lambda_2(s) = O[\lambda_2'(s)]$ for $s \to +\infty$. Then by Lemma 2.2, there exists $\tilde{q}_2 \in \mathcal{SN}$ such that $\tilde{q}_2(s) \cdot \lambda_2(s) \leq \lambda_2'(s)$. Let

$$q_1(s) := 4\tilde{q}_1 \circ \underline{\kappa}_1^{-1} \circ \underline{\alpha}_1^{-1}(2s) \tag{30}$$

$$q_2(s) := \frac{1}{2}\tilde{q}_2 \circ \theta_{\lambda_2}^{-1}(s) \ . \tag{31}$$

Hence, by using q_1, q_2 to obtain $\rho_1, \rho_2 \in \mathcal{K}_\infty$ via (5), we define V_T as:

$$V_T := \rho_1(V_{1T}) + \rho_2(V_{2T}) \ , \tag{32}$$

Using condition 3 of the theorem, we claim that the system (1) is $(V_T, \underline{\alpha}, \overline{\alpha}, \alpha, \lambda, \sigma)$-IOSS with $\lambda = 0$ and the measuring functions given in the statement of Theorem 3.2.

It follows from the definition of V_T that

$$\rho_1 \circ \underline{\alpha}_1(|w_{\underline{\alpha}_1}(x)|) + \rho_2 \circ \underline{\alpha}_2(|w_{\underline{\alpha}_2}(x)|) \leq V_T \leq \rho_1 \circ \overline{\alpha}_1(|w_{\overline{\alpha}_1}(x)|) + \rho_2 \circ \overline{\alpha}_2(|w_{\overline{\alpha}_2}(x)|) \ . \quad (33)$$

By Remark 1, there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}(|w_{\underline{\alpha}_1}(x)| + |w_{\underline{\alpha}_2}(x)|) \leq V_T \leq \overline{\alpha}(|w_{\overline{\alpha}_1}(x)| + |w_{\overline{\alpha}_2}(x)|) \ . \quad (34)$$

In particular, we can take $\underline{\alpha}(s_1 + s_2) := \rho_1 \circ \underline{\alpha}_1(s_1) + \rho_2 \circ \underline{\alpha}_2(s_2)$ and $\overline{\alpha}(s_1 + s_2) := \rho_1 \circ \overline{\alpha}_1(s_1) + \rho_2 \circ \overline{\alpha}_2(s_2)$. Also, from the choice of ρ_1 and ρ_2, by Lemma 3.1 we have the new supply functions for $\rho_1(V_{1T})$ and $\rho_2(V_{2T})$ are respectively α_1', σ_1' and α_2', λ_2', σ_2'. Then by condition 3 of the theorem and using (29), we have

$$\begin{aligned} V_T(F_T) - V_T &\leq T\Big[\sigma_1'(|w_{\sigma_1}(u)|) + \sigma_2'(|w_{\sigma_2}(u)|) \\ &\quad - \alpha_2'(|w_{\alpha_2}(x)|) - \frac{1}{2}\alpha_1'(|w_{\alpha_1}(x)|)\Big] \ . \end{aligned} \quad (35)$$

Finally, using Remark 1, there exist $\sigma \in \mathcal{K}$ and $\alpha \in \mathcal{K}_\infty$ that

$$\begin{aligned} V_T(F_T) - V_T \leq T\Big[&\sigma(|w_{\sigma_1}(u)| + |w_{\sigma_2}(u)|) \\ &- \alpha(|w_{\alpha_1}(x)| + |w_{\alpha_2}(x)|)\Big] \ . \end{aligned} \quad (36)$$

In particular, we can take $\alpha(s_1 + s_2) := \frac{1}{2}\alpha_1'(s_1) + \alpha_2'(s_2)$ and $\sigma(s_1 + s_2) := \sigma_1'(s_1) + \sigma_2'(s_2)$. This completes the proof of Theorem 3.2. ■

6. CONCLUSIONS

We have presented results on changing supply rates for discrete-time IOSS systems. The results are important for a range of problems in analysis and controller design for discrete-time systems, such as a LaSalle criterion for ISS and testing ISS via positive semidefinite Lyapunov functions. We have generalized our results and have shown a practical application in (Laila and Nešić, 2001).

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