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A GLOBAL OPTIMIZATION APPROACH FOR THE PROBLEM OF CONTROLLER ORDER REDUCTION

Edvaldo Assunção [*,1] **Pedro L. D. Peres** [**]

[*] *State University of São Paulo*
DEE/FEIS/UNESP
CP 31 — 15.385-000 — Ilha Solteira — SP — BRAZIL
e-mail: edvaldo@dee.feis.unesp.br
[**] *University of Campinas*
DT/FEEC/UNICAMP
CP 6101 — 13.081-970 — Campinas — SP — BRAZIL
e-mail: peres@dt.fee.unicamp.br

Abstract: A novel formulation for the problem of optimal controller order reduction for continuous-time linear systems is presented. The $\mathcal{H}_2$ or the $\mathcal{H}_\infty$-norms of the error realization between the full and the reduced order controlled systems are used as optimization criteria. Linear matrix inequality optimization problems provide the upper and the lower bounds required for the branch and bound algorithm used to solve the order reduction problem. The convergence in finite time to the global optimum is assured. An example illustrates the results.

Keywords: Model Reduction, Controller Order Reduction, LMI, Optimization, $\mathcal{H}_2$-norm, $\mathcal{H}_\infty$-norm.

1. INTRODUCTION

The problem of model reduction has been deeply investigated in the last three decades. As a matter of fact, a low order model which accurately represents a higher order plant can be very useful for analysis and synthesis of control systems. Moreover, low order controllers can be easily implemented in practical situations (Joshi and Kelkar, 1998).

In the literature, there are several model reduction methods such as the truncated balanced realization (Moore, 1981) and the Hankel norm model reduction (Glover, 1984). Essentially, these methods eliminate less significant states (those associated to small singular values) to obtain the reduced order system.

More recently, the $\mathcal{H}_2$ and the $\mathcal{H}_\infty$-norms of the transfer function realization of the error between the reduced and the original system have been used as optimization criteria for model reduction. These problems can be formulated in terms of Linear Matrix Inequalities (LMIs) with some coupling variables, being solved by means of iterative procedures with no convergence assured (Assunção and Peres, 1999*b*), (Helmersson, 1994), (Valentin and Duc, 1997). Only very recently a global optimization approach based on a branch-and-bound algorithm has been proposed to solve the $\mathcal{H}_2$-norm model reduction problem (Assunção and Peres, 1999*a*).

On the other hand, LMI and other optimization methods have been extensively used to design optimal and robust controllers with several different performance specifications (Boyd and Barratt, 1991). However, the resulting optimal controller order can sometimes be very high (in most cases the controller order is equal or higher than

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the order of the plant). It would be of great importance and enormous practical interest to obtain a similar controller (representing minor losses in the closed-loop performance) with lower order. Standard model reduction techniques could be used to provide a low order model to the controller, but no guarantee can be done with respect to the overall performance of the controlled system.

In this paper, the problem of controller order reduction is formulated in terms of the optimal behavior of the closed-loop controlled system. First, it is assumed that a (possibly higher order) controller has been previously determined to cope with the design specifications. Then, the low order controller is obtained through an optimization procedure which minimizes the $\mathcal{H}_2$ or the $\mathcal{H}_\infty$-norms of the transfer function realization of the error between the full order controlled system and the reduced order one. The optimization problem is formulated in the state space representation by means of Bilinear Matrix Inequalities (BMIs), being solved by a branch-and-bound algorithm. This class of global optimization methods has been used in control system problems ((VanAntwerp *et al.*, 1997), (VanAntwerp and Braatz, 2000)) and in BMI optimization problems involving scalar variables (Goh *et al.*, 1994). Although exhibiting sometimes slow convergence, the branch-and-bound algorithm attains the global optimum in finite time. The main idea is to divide iteratively the parameter space while lower and upper bounds to the objective function are derived from convex LMI optimization problems (this algorithm has been applied for model reduction purposes in (Assunção and Peres, 1999*a*)). An example illustrates the proposed method.

2. PROBLEM STATEMENT

Consider the following linear time-invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) \\ y(t) &= C_1 x(t) + Du(t) \\ z(t) &= C_2 x(t) \end{aligned} \tag{1}$$

with $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times r}$, $B_2 \in \mathbb{R}^{n\times m}$, $C_1 \in \mathbb{R}^{s\times n}$, $D \in \mathbb{R}^{s\times m}$ and $C_2 \in \mathbb{R}^{p\times n}$. It is assumed that a k-order linear controller has been determined to cope with some design specifications. The controller state space realization is given by

$$\begin{aligned} \dot{x}_k(t) &= A_k x_k(t) + B_k z(t) \\ u(t) &= C_k x_k(t) \end{aligned} \tag{2}$$

with $A_k \in \mathbb{R}^{k\times k}$, $B_k \in \mathbb{R}^{k\times p}$ and $C_k \in \mathbb{R}^{m\times k}$. The closed-loop system (assumed asymptotically stable) composed by the controller connected to the plant is given by

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ \dot{x}_k(t) \end{bmatrix} &= \begin{bmatrix} A & B_2 C_k \\ B_k C_2 & A_k \end{bmatrix} \begin{bmatrix} x(t) \\ x_k(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ \mathbf{0} \end{bmatrix} w(t) \\ y(t) &= \begin{bmatrix} C_1 & DC_k \end{bmatrix} \begin{bmatrix} x(t) \\ x_k(t) \end{bmatrix} \end{aligned} \tag{3}$$

being denoted as

$$\left[\begin{array}{cc|c} A & B_2 C_k & B_1 \\ B_k C_2 & A_k & \mathbf{0} \\ \hline C_1 & DC_k & \mathbf{0} \end{array}\right] \triangleq \left[\begin{array}{c|c} A_f & B_f \\ \hline C_f & \mathbf{0} \end{array}\right] \tag{4}$$

The controller order reduction problem can be formulated as follows. Given the system (1) and the full order controller (2) (that is, the closed-loop matrices A_f, B_f and C_f), find a r-order controller ($r < k$) with the state space realization

$$\begin{aligned} \dot{x}_r(t) &= A_r x_r(t) + B_r z(t) \\ u(t) &= C_r x_k(t) \end{aligned} \tag{5}$$

where $A_r \in \mathbb{R}^{r\times r}$, $B_r \in \mathbb{R}^{r\times p}$ and $C_r \in \mathbb{R}^{m\times r}$ such that the closed-loop system feedback by the reduced order controller given by

$$\left[\begin{array}{cc|c} A & B_2 C_r & B_1 \\ B_r C_2 & A_r & \mathbf{0} \\ \hline C_1 & DC_r & \mathbf{0} \end{array}\right] \tag{6}$$

exhibits a dynamical behavior as close as possible to the original one.

In terms of the norms of the transfer function error realization, the problem can be formulated as: find matrices (A_r, B_r, C_r) such that the $\mathcal{H}_2$ or the $\mathcal{H}_\infty$-norms of the transfer function

$$\left[\begin{array}{ccc|c} A_f & \mathbf{0} & \mathbf{0} & B_f \\ \mathbf{0} & A & B_2 C_r & B_1 \\ \mathbf{0} & B_r C_2 & A_r & \mathbf{0} \\ \hline C_f & -C_1 & -DC_r & \mathbf{0} \end{array}\right] \triangleq \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & \mathbf{0} \end{array}\right] \tag{7}$$

is minimized. The above transfer function represents the difference between the full order and the reduced order controlled systems (see the block diagram in figure 1).

In the sequel, the problems of optimal $\mathcal{H}_2$ and $\mathcal{H}_\infty$-norms controller order reduction are formulated in terms of coupled LMIs. For details concerning the state space computation of $\mathcal{H}_2$ and $\mathcal{H}_\infty$-norms via LMIs and also about Schur's complement see, for instance, (Boyd *et al.*, 1994) and (Palhares *et al.*, 1997).

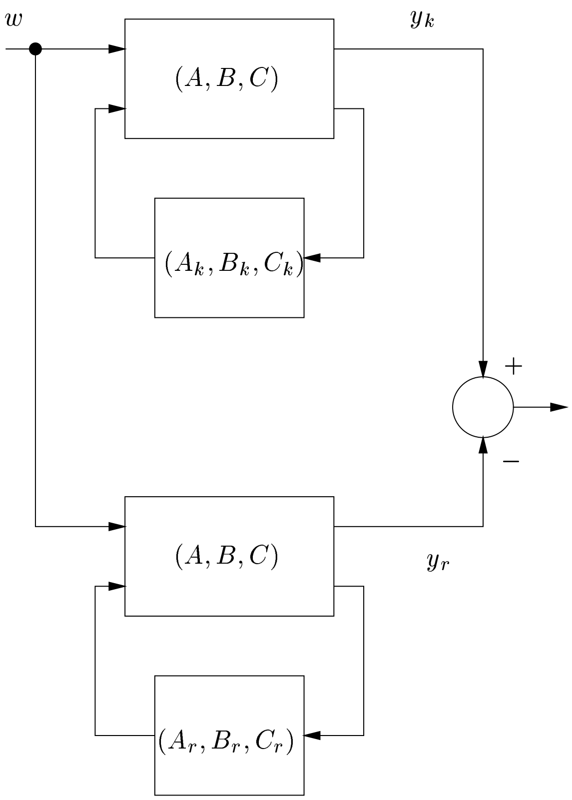

Fig. 1. Error between the full order and the reduced order controlled systems.

3. $\mathcal{H}_2$-NORM CONTROLLER ORDER REDUCTION

The $\mathcal{H}_2$-norm of a state space realization $(\tilde{A}, \tilde{B}, \tilde{C})$ can be calculated through the following optimization problem:

$$\begin{aligned} \Phi = \min \ & Tr(R) \\ \text{s.t.} \ & \begin{bmatrix} R & \tilde{C}P \\ P\tilde{C}' & P \end{bmatrix} \geq \mathbf{0} \\ & \begin{bmatrix} \tilde{A}P + P\tilde{A}' & \tilde{B} \\ \tilde{B}' & -\mathbf{I} \end{bmatrix} \leq \mathbf{0} \\ & P > \mathbf{0} \end{aligned} \qquad (8)$$

Partitioning matrix $P \in \mathbb{R}^{(n+k+n+r)\times(n+k+n+r)}$ as follows

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12}' & P_{22} & P_{23} \\ P_{13}' & P_{23}' & P_{33} \end{bmatrix} \qquad (9)$$

with $P_{11} \in \mathbb{R}^{(n+k)\times(n+k)}$, $P_{22} \in \mathbb{R}^{n\times n}$, $P_{33} \in \mathbb{R}^{r\times r}$ and taking into account the definition of $(\tilde{A}, \tilde{B}, \tilde{C})$ in (7), one gets

$$\Phi = \min \ Tr(R)$$

$$\text{s.t.} \begin{bmatrix} R & * & * & * \\ P_{11}C_f' - P_{12}C_1' - P_{13}C_r'D' & P_{11} & * & * \\ P_{12}'C_f' - P_{22}C_1' - P_{23}C_r'D' & P_{12}' & P_{22} & * \\ P_{13}'C_f' - P_{23}'C_1' - P_{33}C_r'D' & P_{13}' & P_{23}' & P_{33} \end{bmatrix} \geq \mathbf{0}$$

$$\left[\begin{matrix} A_fP_{11}+P_{11}A_f' & A_fP_{12}+P_{12}A'+P_{13}C_r'B_2' \\ AP_{12}'+B_2C_rP_{13}'+P_{12}'A_f' & AP_{22}+B_2C_rP_{23}'+P_{22}A'+P_{23}C_r'B_2' \\ B_rC_2P_{12}'+A_rP_{13}'+P_{13}'A_f' & B_rC_2P_{22}+A_rP_{23}'+P_{23}'A'+P_{33}C_r'B_2' \\ B_f' & B_1' \end{matrix}\right.$$

$$\left.\begin{matrix} A_fP_{13} + P_{12}C_2'B_r' + P_{13}A_r' & B_f \\ AP_{23} + B_2C_rP_{33} + P_{22}C_2'B_r' + P_{23}A_r' & B_1 \\ B_rC_2P_{23} + A_rP_{33} + P_{23}'C_2'B_r' + P_{33}A_r' & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{matrix}\right] \leq \mathbf{0}$$

$$\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12}' & P_{22} & P_{23} \\ P_{13}' & P_{23}' & P_{33} \end{bmatrix} > \mathbf{0} \qquad (10)$$

where $*$ denote symmetric element.

For a fixed order r, the variables of the problem are: A_r, B_r, C_r and the submatrices of P and R. There are several bilinear terms in the above BMIs such as DC_rP_{13}', DC_rP_{23}', DC_rP_{33}', $P_{13}C_r'B_2'$, $P_{12}C_2'B_r'$, $P_{13}A_r'$, $B_2C_rP_{23}'$, $B_2C_rP_{33}$, $P_{22}C_2'B_r'$, $P_{23}A_r'$, $B_rC_2P_{23}$, A_rP_{33} and their transposes. Although some previous algebraic manipulations could simplify or even discard a few coupling variables, a branch and bound algorithm (very similar to the one proposed in (Assunção and Peres, 1999a) for model reduction) can be applied to solve the problem. Note that the objective function $Tr(R)$ depends on the optimization variables (A_r, B_r, C_r, P, R) which belong to the parameter space denoted by $\mathcal{Q}$. If A_r, B_r and C_r are given, $\Phi = \min \ Tr(R)$ under the constraints of (10) equals the $\mathcal{H}_2$-norm square of the error transfer function realization (7).

4. $\mathcal{H}_\infty$-NORM CONTROLLER ORDER REDUCTION

The $\mathcal{H}_\infty$-norm of a transfer function with a state space realization $(\tilde{A}, \tilde{B}, \tilde{C})$ is given by γ and can be computed through the following optimization procedure:

$$\begin{aligned} \gamma^2 = \min \ & \mu \\ \text{s.t.} \ & \begin{bmatrix} \tilde{A}P + P\tilde{A}' & P\tilde{C}' & -\tilde{B} \\ \tilde{C}P & -\mathbf{I} & \mathbf{0} \\ -\tilde{B}' & \mathbf{0} & -\mu\mathbf{I} \end{bmatrix} \leq \mathbf{0} \\ & P > \mathbf{0} \end{aligned} \qquad (11)$$

where $P \in \mathbb{R}^{(n+k+n+r)\times(n+k+n+r)}$ is partitioned as in (9). Taking into account the definition of the error transfer function (7) and the submatrices of P, the equation (11) can be rewritten as

$$\Phi = \min \ \mu$$

s.t.

$$\left[\begin{matrix} A_fP_{11}+P_{11}A_f' & * \\ AP_{12}'+B_2C_rP_{13}'+P_{12}'A_f' & AP_{22}+B_2C_rP_{23}'+P_{22}A'+P_{23}C_r'B_2' \\ B_rC_2P_{12}'+A_rP_{13}'+P_{13}'A_f' & B_rC_2P_{22}+A_rP_{23}'+P_{23}'A'+P_{33}C_r'B_2' \\ C_fP_{11}-C_1P_{12}'-DC_rP_{13}' & C_fP_{12} - C_1P_{22} - DC_rP_{23}' \\ -B_f' & -B_1' \end{matrix}\right.$$

$$
\left.\begin{matrix} * & * & * \\ & * & * \\ B_r C_2 P_{23} + A_r P_{33} + P_{23}' C_2' B_r' + P_{33} A_r' & * & * \\ C_f P_{13} - C_1 P_{23} - D C_r P_{33} & -\mathbf{I} & * \\ \mathbf{0} & \mathbf{0} & -\mu \mathbf{I} \end{matrix}\right] \leq \mathbf{0}
$$

$$
\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12}' & P_{22} & P_{23} \\ P_{13}' & P_{23}' & P_{33} \end{bmatrix} > \mathbf{0} \tag{12}
$$

As in the $\mathcal{H}_2$-norm case, there are several bilinear products in the above optimization procedure, which could perhaps be avoided by previous algebraic manipulation. Nevertheless, the branch and bound algorithm proposed in (Assunção and Peres, 1999*a*) can be applied to solve the problem.

5. THE BRANCH AND BOUND ALGORITHM

Branch and bound algorithms are global optimization techniques that can be applied to NP-hard problems, finding the global minimum of a possibly nonconvex function $f : \mathbb{R}^l \rightarrow \mathbb{R}$ over an l-dimensional rectangle $\mathcal{Q}$. The rectangle is iteratively gridded until the global optimum is attained within a given ϵ precision. As discussed in (Goh *et al.*, 1994), the branch and bound method can be viewed as a clever way of gridding that uses upper and lower bounds to progressively refine the areas of interest (thus avoiding the need of an extensive grid covering the entire domain). The procedure ends when the difference between the upper and the lower bounds is less than ϵ.

A branch and bound algorithm for this kind of problem has been presented in (Assunção and Peres, 1999*a*), with a discussion about the convergence issues. Details concerning branch and bound algorithms can be found in (Ryoo and Sahinidis, 1995).

The determination of upper and lower bounds can be achieved through LMI (convex) optimization problems, in both $\mathcal{H}_2$ and $\mathcal{H}_\infty$ cases.

5.1 *Upper bound*

Consider the controller order reduction problems presented in sections 3 and 4. An upper bound to the error between the original controlled system and the reduced order one can be obtained by fixing $A_r = \hat{A}_r$, $B_r = \hat{B}_r$ and $C_r = \hat{C}_r$, that is, imposing a constraint on $\mathcal{Q}$ such that

$$
\Phi_U(\hat{A}_r, \hat{B}_r, \hat{C}_r) \geq \Phi_{(A_r, B_r, C_r) \in \mathcal{Q}} \tag{13}
$$

It is important to stress that if the triple (A_r, B_r, C_r) is fixed, problems (10) and (12) are LMI optimization problems which can be efficiently solved by means of polynomial type procedures (Gahinet *et al.*, 1995).

5.2 *Lower bound*

A relaxation technique can be used to provide a lower bound Φ_L to the transfer function error realization. This can be done by simply creating new variables $W_1 = C_r P_{13}'$, $W_2 = C_r P_{23}'$, $W_3 = C_r P_{33}$, $W_4 = P_{12} C_2' B_r'$, $W_5 = P_{22} C_2' B_r'$, $W_6 = P_{23} C_2' B_r'$, $W_7 = P_{13} A_r'$, $W_8 = P_{23} A_r'$, $W_9 = P_{33} A_r'$ in problems (10) and (12). Clearly, the optimal solution Φ_L of the relaxed problems is such that

$$
\Phi_{(A_r, B_r, C_r) \in \mathcal{Q}} \geq \Phi_L \tag{14}
$$

As in the upper bound case, the above LMI optimization problem can be efficiently solved in polynomial time.

5.3 *Construction of $\mathcal{Q}$*

The controller order reduction problems presented in sections 3 and 4 do not impose natural constraints to the variables (which are in fact the elements of matrices A_r, B_r, C_r, R, P, W_j, $j = 1, \ldots, 9$). The hyperectangle $\mathcal{Q}$ can be constructed by solving $2N_t$ LMI optimization problems, where N_t is the total number of variables. Denoting by $x \in R^{N_t}$ the vector of all variables, solve:

$$
\begin{aligned} & \min \; x_i \\ & \text{s.t. the relaxed LMIs hold} \\ & \qquad i = 1, \ldots, N_t \end{aligned}
$$

$$
\begin{aligned} & \max \; x_i \\ & \text{s.t. the relaxed LMIs hold} \\ & \qquad i = 1, \ldots, N_t \end{aligned}
$$

As a result,

$x_i^{min} \leq x_i \leq x_i^{max}$, x_i the i^{th} element of x,

is obtained.

5.4 *Convergence Analysis*

A branch and bound algorithm minimizes Φ if there exist two functions Φ_L and Φ_U over the region $\mathcal{Q}$ such that

C1. Φ_L yields a lower bound and Φ_U an upper bound on Φ, i.e.

$$
\Phi_L \leq \Phi \leq \Phi_U
$$

for every hyperectangle $\mathcal{Q}$.

C2. Let Size($\mathcal{Q}$) denote the length of the longest side of the hyperectangle $\mathcal{Q}$, then as Size($\mathcal{Q}$) $\to 0$, $\Phi_U(\mathcal{Q}) - \Phi_L(\mathcal{Q}) \to 0$ uniformly, i.e. $\forall\ \epsilon > 0$ $\exists\ \delta > 0$, such that,

$$\text{Size}(\mathcal{Q}) \leq \delta \quad \Rightarrow \quad \Phi_U(\mathcal{Q}) - \Phi_L(\mathcal{Q}) \leq \epsilon$$

As it can be verified, Φ_U and Φ_L previously defined fulfil conditions **C1** and **C2**. Condition **C1** follows immediately from equations (13) and (14); if the largest side of $\mathcal{Q}$ is iteratively reduced by the algorithm, **C2** is also satisfied. With **C1** and **C2**, the algorithm converges to the global optimum in a finite time (see (Balakrishnan *et al.*, 1991)).

The branch and bound algorithm proposed here involves the construction of a search tree with LMI optimization problems at each node. Although the LMIs are efficiently solved, the manipulation of the hyperectangle $\mathcal{Q}$ requires some heuristic which is expected to reduce the total number of calculations to solve the problem, compared to an exhaustive gridding of the parameter domain. In the following, the algorithm is presented (see (Balakrishnan and Boyd, 1992) for details concerning general branch and bound algorithms).

5.5 *The Algorithm*

In the following description, k stands for the iteration index, $\mathcal{L}_k$ denotes the list of active rectangles at k iteration, $\mathcal{Q}_i$ is the ith rectangle belonging to the list $\mathcal{L}_k$ (subdivision of the initial rectangle $\mathcal{Q}$), Φ_{L_k} the lower bound, Φ_{U_k} the upper bound at the end of k iterations, and $\epsilon > 0$ is a fixed precision.

1. *Set:* $k = 0$, $\mathcal{L}_0 = \{\mathcal{Q}\}$, $L_0 = \Phi_L(\mathcal{Q})$, $U_0 = \Phi_U(\mathcal{Q})$.
2. *Repeat* {
R1. pick $\mathcal{Q}_i \in \mathcal{L}_k$ *such that* $\Phi_L(\mathcal{Q}_i) = L_k$;
R2. split $\mathcal{Q}_i$ *along one of its longest edges into* $\mathcal{Q}_I$ *and* $\mathcal{Q}_{II}$;
R3. $\mathcal{L}_{k+1} := (\mathcal{L}_k - \{\mathcal{Q}_i\}) \bigcup \{\mathcal{Q}_I, \mathcal{Q}_{II}\}$;
R4. $L_{k+1} := \min_{\mathcal{Q}_i \in \mathcal{L}_{k+1}} \Phi_L(\mathcal{Q}_i)$;
R5. $U_{k+1} := \min_{\mathcal{Q}_i \in \mathcal{L}_{k+1}} \Phi_U(\mathcal{Q}_i)$;
R6. $k := k + 1$.
} *until* $U_k - L_k < \epsilon$.

Note that, at step *R3*, the rectangle $\mathcal{Q}_i$ from steps *R1* and *R2* is removed from the list $\mathcal{L}_k$, while the two new partitions $\mathcal{Q}_I$ and $\mathcal{Q}_{II}$ are added. New minimal lower and upper bounds are obtained at steps *R4* and *R5*, respectively. At each iteration, a pruning can be done by eliminating from list $\mathcal{L}_k$ the rectangles $\mathcal{Q}_i \in \mathcal{L}_k$ that satisfy

$$\Phi_L(\mathcal{Q}_i) > U_k$$

thus reducing the storage requirements. The requirement that the rectangles are splited along their longest edges assures a uniform bound to the condition number of the rectangles in the partition (see (Balakrishnan and Boyd, 1992) for details).

6. NUMERICAL EXAMPLE

To illustrate the controller order reduction method[2] proposed in the paper, an example is presented. It consists on the optimal $\mathcal{H}_2$-norm controller order reduction of a third order system given by

$$G(s) = \frac{9}{s^3 + 6s^2 + 11s + 6}$$

feedback by a third order controller

$$C(s) = \frac{20.8}{s^3 + 15s^2 + 74s + 120}$$

yielding the following closed-loop transfer function

$$H(s) = \frac{9s^3 + 135s^2 + 666s + 1080}{s^6 + 21s^5 + 175s^4 + 735s^3 + 1623.99s^2 + \ldots + 1763.99s + 907.2}$$

The aim is to obtain a first order model (i.e. $r = 1$) for the controller such that the $\mathcal{H}_2$ norm of the error between the full order closed-loop system and the system feedback by the reduced order controller is minimized.

The reduced controller determined by the branch and bound method proposed is

$$C_r(s) = \frac{0.6054}{s + 3.3075}$$

The algorithm required 105 iterations to reduce the $\mathcal{H}_2$-norm of the error to 2.92×10^{-2}.

The closed-loop system feedback with the reduced order controller is given by

$$H_r(s) = \frac{9s + 29.76}{s^4 + 9.31s^3 + 30.84s^2 + 42.38s + 25.29}$$

Figure 2 shows the impulse responses of both closed-loop configurations. It can be noted that the system feedback by the reduced order controller performs as good as the full order controlled one.

7. CONCLUSION

The problem of controller order reduction has been formulated in terms of the closed-loop system behavior. A branch and bound algorithm

[2] The branch and bound algorithm has been implemented using the software LMI Control Toolbox (Gahinet *et al.*, 1995).

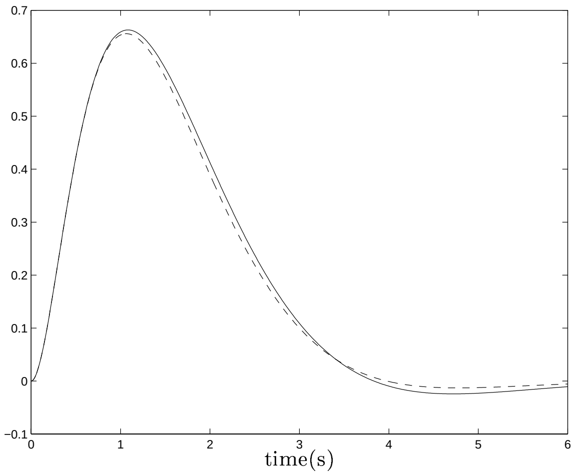

Fig. 2. Impulse responses of the full order controlled system (solid) and the system feedback by the reduced order controller (dashed).

has been applied to the solve the problem using as criteria the $\mathcal{H}_2$ or the $\mathcal{H}_\infty$-norms of the error transfer function. The global optimal solution (with a given ϵ accuracy) is achieved in a finite number of iterations.

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