15th Triennial World Congress, Barcelona, Spain

DISTURBANCE ASPECTS OF HIGH ORDER ITERATIVE LEARNING CONTROL [1]

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Abstract: Some initial results on the disturbance properties of high order iterative learning control (ILC) algorithms are presented. Of particular interest is to investigate how high order ILC algorithms cope with measurement and load disturbances of different character. Some results are obtained by assuming statistical properties of the involved disturbances and deriving equations for the covariance matrix of the control error vector. The results are illustrated by analytic derivation of the covariance matrix for a second order ILC algorithm with a particular choice of design variables.

Keywords: Learning control, linear systems, disturbance rejection, measurement noise, iterative methods

1. INTRODUCTION

High order ILC algorithms have received an increasing attention during the last years. Although most contributions on ILC have been on the first order case, the idea of utilizing the measurements from more than the previous iteration has been covered in many articles. In (Liang and Looze, 1993) two dimensional transforms are used to analyze the behavior of the system in both the time and the iteration directions. In (Arimoto, 1991) the errors from previous iterations are used in an indirect way. Chen et al. have also investigated the use of high order ILC and the main reference is (Chen *et al.*, 1998), but the issue is also discussed in (Chen *et al.*, 1997*a*; Chen *et al.*, 1997*b*). High order ILC has also been covered in e.g., (Bien and Huh, 1989), (Bien and Xu, 1998) and (Norrlöf and Gunnarsson, 1999). In (Norrlöf, 2000) a framework for dealing with high order ILC algorithms is presented and some conditions for convergence are presented. The purpose of this paper is to present some initial results on the disturbance properties of high order ILC algorithms. These aspects has not been addressed before for this class of ILC algorithms, as the authors know about. It is also clear that this is a very important aspect to study from a practical point of view since, in real cases, disturbances will always be present.

The paper is organized as follows. Section 2 gives an introduction to linear iterative systems which is the mathematical framework that will be used, and in Section 3 the type of systems considered are described. A framework for representing high order ILC-algorithms is presented in Section 4, and Section 5 presents the error equations that are obtained when the high order algorithms are applied to the systems. The influence of measurement and load disturbances are investigated in Sections 6 and 7 respectively. Section 8 presents an example and finally some conclusions are given in Section 9.

2. LINEAR ITERATIVE SYSTEMS

To analyze the higher order ILC algorithms a system structure called linear iterative systems will be used. A general linear iterative system can be described by

$$\boldsymbol{z}_{k+1} = \boldsymbol{F}\boldsymbol{z}_k + \boldsymbol{F}_r\boldsymbol{r} \tag{1}$$

[1] This work was supported by ISIS and CENIIT at Linköpings universitet.

where for a first order linear iterative system

$$z_k = \big(z_k(0), \dots, z_k(n-1)\big)^T \tag{2}$$

and

$$r = \big(r(0), \dots, r(n-1)\big)^T \tag{3}$$

For a high order (Nth order) linear iterative system,

$$z_k = \big[\zeta_k^T, \dots, \zeta_{k-N+1}^T\big]^T \tag{4}$$

where ζ_k are defined as in (2).

As pointed out in (Norrlöf, 2000) this class of systems can be analyzed easily with standard linear discrete time systems theory (Kailath, 1980; Rugh, 1996). Two different measures of the size of a matrix are necessary for this analysis. The first is the spectral radius which is defined as,

$$\rho(F) = \max_{i=1,\dots,n} |\lambda_i(F)| \tag{5}$$

where $\lambda_i(F)$ is the ith eigenvalue of the matrix $F \in \mathbb{R}^{n\times n}$. The second is the maximum singular value, defined as,

$$\bar{\sigma}(F) = \sqrt{\rho(F^T F)} \tag{6}$$

The maximum singular value gives a bound of the gain of a matrix by the fact that,

$$\|Fx\| \le \bar{\sigma}(F)\|x\|$$

If the maximum singular value is less than one it is clear that the norm of the result decreases every time x is mapped by F. This is an important observation that is used in many of the stability results for linear iterative systems. The first result concerns bounded input bounded output (BIBO) stability.

Theorem 1. (BIBO stability). If $\rho(F) < 1$ in the linear iterative system,

$$z_{k+1} = F z_k + F_r r$$

then the system is bounded-input, bounded-output stable.

This result is a straight-forward application of results from linear systems theory, see e.g., (Rugh, 1996). BIBO stability means that an input r with bounded norm generates a bounded output z_k for all k.

3. SYSTEM DESCRIPTION

The system description that will be used in this paper is given by

$$z_k = T_r r + T_u u_k + T_w w_k \tag{7a}$$
$$y_k = z_k + T_v v_k \tag{7b}$$

where r, u_k, w_k and v_k denote reference signal, ILC input signal, load and measurement disturbances respectively. Furthermore z_k and y_k denote the controlled and measured output signal. The signals z_k, r, u_k, w_k and v_k in equation (7) are represented by vectors containing the signal values in the sampling points as in (2).

For a time and iteration invariant and causal system the relationship between e.g., the reference signal and the controlled output, T_r, is described by a lower diagonal Toeplitz matrix where the elements are the impulse coefficients. If the system is linear time variant, the matrix T_r does not become a lower triangular Toeplitz matrix but instead a general lower triangular matrix. The matrices T_u, T_w, and T_v are given in the same way.

When analyzing the high order ILC algorithms the effect of u_{k-j+1} for $j \in [1, N]$ is necessary to study for stability. Defining Z_k as

$$Z_k = \big[z_k^T \; z_{k-1}^T \; \dots \; z_{k-N+1}^T\big]^T \tag{8}$$

and R, U_k, Y_k, W_k, and V_k analogous to (8), the system can be described by

$$Z_k = \mathcal{T}_r R + \mathcal{T}_u U_k + \mathcal{T}_w W_k \tag{9}$$

and

$$Y_k = Z_k + \mathcal{T}_v V_k \tag{10}$$

In equation (9) the matrix $\mathcal{T}_r$ is a block diagonal matrix where the N blocks are given by T_r. The other matrices are defined similarly.

4. HIGH ORDER ILC ALGORITHMS

A general N-th order ILC algorithm can be written according to,

$$\begin{aligned} u_{k+1} = {} & H_1^u u_k + \dots + H_N^u u_{k-N+1} \\ & + H_1^e e_k + \dots + H_N^e e_{k-N+1} \end{aligned} \tag{11}$$

where e_k is the difference between the desired output and the measured output, i.e.

$$e_k = r - y_k \tag{12}$$

The matrices H_j^e and H_j^u are design variables determining the properties of the ILC algorithm. Defining U_k and $\mathbf{E}_k$ as in (8) the N-th order ILC algorithm can be written as

$$U_{k+1} = \mathcal{H}^u U_k + \mathcal{H}^e E_k \tag{13}$$

where

$$\mathcal{H}^u = \begin{bmatrix} H_1^u & H_2^u & \dots & H_N^u \\ I & 0 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & \dots & I & 0 \end{bmatrix} \quad \mathcal{H}^e = \begin{bmatrix} H_1^e & H_2^e & \dots & H_N^e \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \tag{14}$$

From the general theory on linear iterative systems presented in Section 2 it is straightforward to find a stability condition, formulated in the following corollary.

Corollary 2. (Stability, disturbance free case). The system

$$\boldsymbol{y}_k = \boldsymbol{T}_r \boldsymbol{r} + \boldsymbol{T}_u \boldsymbol{u}_k$$

controlled with the ILC updating equation from (13) is stable if the spectral radius fulfills

$$\rho\left(\begin{bmatrix} \boldsymbol{F}_1 & \boldsymbol{F}_2 & \dots & \boldsymbol{F}_N \\ I & 0 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & \dots & I & 0 \end{bmatrix}\right) < 1$$

where $\boldsymbol{F}_j = \boldsymbol{H}_j^u - \boldsymbol{H}_j^e \boldsymbol{T}_u$.

This corollary is a direct application of Theorem 1 when the system and the updating formula are put on the linear iterative systems form. How this is done will be shown in the next section. An even stronger result than Corollary 2 is achieved when the spectral radius is replaced by a condition on the singular value. If this stronger condition is fulfilled then the norm of the difference between the current control signal and the control signal that the ILC algorithm ultimately will converge to is monotonously decreasing.

5. ERROR EQUATIONS

From a performance viewpoint the quantity of main interest is the error between the desired output and the controlled output, i.e.

$$\epsilon_k = \boldsymbol{r} - \boldsymbol{z}_k \tag{15}$$

This is different from $\boldsymbol{e}_k$ in (12) because $\boldsymbol{e}_k$ is based upon the measured output while ϵ_k depends on the true output (not corrupted by measurement noise). The size of ϵ_k will in the next sections be calculated when considering the impact from measurement and load disturbances.

Now some equations necessary for analyzing the properties of the error will be presented for the high order ILC case. Using (9) and and the ILC algorithm in (11) the error equation is given by

$$\begin{aligned} \mathcal{E}_{k+1} = (I - \mathcal{T}_r)\boldsymbol{R} - \mathcal{T}_u \mathcal{H}^e \mathcal{E}_k - \mathcal{T}_u \mathcal{H}^u \boldsymbol{U}_k \\ + \mathcal{T}_u \mathcal{H}^e \mathcal{T}_v \boldsymbol{V}_k - \mathcal{T}_w \boldsymbol{W}_{k+1} \end{aligned} \tag{16}$$

Furthermore, using

$$\mathcal{A} = \mathcal{H}^u - \mathcal{T}_u \mathcal{H}^e \tag{17}$$

the input signal is given by the difference equation

$$\begin{aligned} \boldsymbol{U}_{k+1} = \mathcal{A}\boldsymbol{U}_k + \mathcal{H}^e (I - \mathcal{T}_r)\boldsymbol{R} \\ - \mathcal{H}^e \mathcal{T}_w \boldsymbol{W}_k - \mathcal{H}^e \mathcal{T}_v \boldsymbol{V}_k \end{aligned} \tag{18}$$

or equivalently

$$\begin{aligned} \boldsymbol{U}_{k+1} = \begin{bmatrix} \boldsymbol{F}_1 & \boldsymbol{F}_2 & \dots & \boldsymbol{F}_N \\ I & 0 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & \dots & I & 0 \end{bmatrix} \boldsymbol{U}_k + \begin{bmatrix} \sum_{j=1}^{N} \boldsymbol{H}_j^e (I - \boldsymbol{T}_r) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \boldsymbol{r} \\ - \begin{bmatrix} \boldsymbol{H}_1^e & \boldsymbol{H}_2^e & \dots & \boldsymbol{H}_N^e \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T}_w \boldsymbol{w}_k + \boldsymbol{T}_v \boldsymbol{v}_k \\ \boldsymbol{T}_w \boldsymbol{w}_{k-1} + \boldsymbol{T}_v \boldsymbol{v}_{k-1} \\ \vdots \\ \boldsymbol{T}_w \boldsymbol{w}_{k-N+1} + \boldsymbol{T}_v \boldsymbol{v}_{k-N+1} \end{bmatrix} \end{aligned} \tag{19}$$

with $\boldsymbol{F}_j = \boldsymbol{H}_j^u - \boldsymbol{H}_j^e \boldsymbol{T}_u$. If it is assumed that the matrices $\mathcal{T}_u$ and $\mathcal{H}^u$ commute the error equation becomes

$$\begin{aligned} \mathcal{E}_{k+1} = (I - \mathcal{H}^u)(I - \mathcal{T}_r)\boldsymbol{R} + \mathcal{A}\mathcal{E}_k + \mathcal{B}\boldsymbol{V}_k \\ + (\mathcal{H}^u \mathcal{T}_w \boldsymbol{W}_k - \mathcal{T}_w \boldsymbol{W}_{k+1}) \end{aligned} \tag{20}$$

with $\mathcal{A}$ from (17) and

$$\mathcal{B} = \mathcal{T}_u \mathcal{H}^e \mathcal{T}_v \tag{21}$$

$\mathcal{T}_u$ and $\mathcal{H}^u$ commute when the matrices T_u and H_i^u commute for $i = 1, \dots, N$. Although this equation is obtained using the assumption that the matrices $\mathcal{T}_u$ and $\mathcal{H}^u$ commute it will be the basis for the following discussion. An obvious extension to the results achieved below is to relax this assumption.

As a measure of the size of the error $\mathcal{E}_k$ the following property will be used

$$\mathcal{P}_k = E[\mathcal{E}_k \mathcal{E}_k^T] \tag{22}$$

In the two next sections the stationary value of $\mathcal{P}_k$ will be used and denoted $\mathcal{P}$.

6. MEASUREMENT DISTURBANCES

In most applications it is realistic to assume that the load and measurement disturbances are uncorrelated, and hence both types of disturbances could be handled at the same time. For clarity in the presentation the two disturbances will however be handled separately, and in this section the influence of measurement disturbances is investigated. Consider therefore equation (20) in the case when the reference signal is zero, i.e. $\boldsymbol{R} = 0$, and there are no load disturbances, i.e. $\boldsymbol{W}_k = 0 \;\forall\, k$. The equation then becomes

$$\mathcal{E}_{k+1} = \mathcal{A}\mathcal{E}_k + \mathcal{B}\boldsymbol{V}_k \tag{23}$$

By assuming some statistical properties of the measurement disturbance the effects on the control error can be found by computing the covariance matrix of the error vector as in (22). In stationarity $\mathcal{P}$ is a block matrix where the diagonal blocks equal $E[\epsilon_j \epsilon_j^T]$ with $j = k - N + 1, \dots, k$. Since it is assumed that $\mathcal{E}_k$ has reached stationarity all diagonal blocks will be the same. Introduce therefore

$$P = E[\epsilon_k \epsilon_k^T] \tag{24}$$

Using e.g., a second order ILC algorithm $\mathcal{P}$ will be a 2×2 block matrix where the two diagonal blocks are given by P and the off-diagonal blocks represent the quantity $E[\epsilon_k \epsilon_{k-1}^T]$. In the next step it is assumed that the covariance matrix of the measurement noise vectors in each iteration is given by

$$R_v = E[\boldsymbol{v}_k \boldsymbol{v}_k^T] \tag{25}$$

and that the measurement disturbance vectors in different iterations are uncorrelated. It is then possible to introduce the covariance matrix

$$\mathcal{R}_v = E[\boldsymbol{V}_k \boldsymbol{V}_k^T] \tag{26}$$

which, using the assumptions above, is a block diagonal matrix with the N diagonal blocks given by R_v.

Recall now the equation (23). Take both sides times its transpose, apply the expectation operator and assume that stationarity is reached. This gives that the covariance matrix is found as the positive semi definite and symmetric solution to the Lyapunov equation

$$\begin{aligned} \mathcal{P} &= \mathcal{A}\mathcal{P}\mathcal{A}^T + \mathcal{B}\mathcal{R}_v\mathcal{B}^T \\ &\quad + \mathcal{A}E[\mathcal{E}_k \boldsymbol{V}_k^T]\mathcal{B}^T + \mathcal{B}E[\boldsymbol{V}_k \mathcal{E}_k^T]\mathcal{A}^T \end{aligned} \tag{27}$$

The cross terms are due to the fact that the vectors $\mathcal{E}_k$ and $\boldsymbol{V}_k$ are not uncorrelated. From the assumption that $E[\boldsymbol{v}_k \boldsymbol{v}_{k-j}] = 0$ for all $j \neq 0$ and Equation (15) it is clear that $E[\mathcal{E}_k \boldsymbol{V}_k^T]$ is a block matrix with zeros in and below the diagonal. This follows from the fact that $E[\epsilon_{l-j} \boldsymbol{v}_l] = 0$ for $j \geq 0$.

An immediate use of equation (27) is to compute the covariance matrix for some proposed ILC algorithm. This can be solved numerically using a standard Lyapunov equation solver. An analytic solution can only be expected in special cases, like the one presented in Section 8.

7. LOAD DISTURBANCES

The influence of load disturbances are here analyzed assuming that the reference signal is zero and that there are no measurement disturbances. The error equation (20) is then given by

$$\mathcal{E}_{k+1} = \mathcal{A}\mathcal{E}_k + \Delta_k \tag{28}$$

where

$$\Delta_k = (\mathcal{H}^u \mathcal{T}_w \boldsymbol{W}_k - \mathcal{T}_w \boldsymbol{W}_{k+1}) \tag{29}$$

Even though it is a standard assumption in ILC that the operating conditions are repeatable it is realistic to assume that there can be iteration dependent changes in the load disturbance. This assumption motivates the iteration index on the vector $\boldsymbol{W}_k$. The simplest case to analyze is however when the load disturbance is iteration invariant, i.e. $w_k = w_{k+1} = w$, which implies

$$\Delta = (\mathcal{H}^u - I)\mathcal{T}_w \boldsymbol{W} \tag{30}$$

where $\boldsymbol{W}$ now is iteration independent column vector consisting of the vector w repeated N times. Assuming $\mathcal{T}_w$ to be a unit matrix and using that

$$\mathcal{H}^u - I = \begin{bmatrix} \boldsymbol{H}_1^u - I & \boldsymbol{H}_2^u & \ldots & \boldsymbol{H}_N^u \\ I & -I & \ldots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & \ldots & I & -I \end{bmatrix} \tag{31}$$

the vector Δ will be zero provided that

$$\sum_{k=1}^{N} \boldsymbol{H}_k^u = I \tag{32}$$

In order to have zero asymptotic error due to an iteration invariant load disturbance (and reference signal) it is necessary that condition (32) is satisfied. This result is shown for the special case, a second order ILC algorithm, in (Norrlöf, 2000) and (Bien and Huh, 1989).

When considering an iteration variant load disturbance some assumption concerning the properties of the disturbance have to be made. Several different assumptions are possible and the one that will be utilized here is that the load disturbance consists of one iteration invariant term and one randomly varying term, i.e.

$$\boldsymbol{w}_k = \boldsymbol{w} + \delta_{w,k} \tag{33}$$

where the random vectors $\delta_{w,k}$ from different iterations are uncorrelated. Furthermore it is assumed that

$$E[\delta_{w,k}\delta_{w,k}^T] = R_\delta \tag{34}$$

This implies

$$\boldsymbol{W}_k = \boldsymbol{W} + \Delta_{w,k} \tag{35}$$

where $\Delta_{w,k} = \begin{bmatrix} \delta_{w,k}^T & \delta_{w,k-1}^T & \ldots & \delta_{w,k-N+1}^T \end{bmatrix}^T$ and

$$R_{\Delta,w} = E[\Delta_{w,k}\Delta_{w,k}^T] \tag{36}$$

is a block diagonal matrix having R_δ as diagonal elements. Equation (35) implies that, provided that (32) is satisfied,

$$\Delta_k = \mathcal{H}^u \mathcal{T}_w \Delta_{w,k} - \mathcal{T}_w \Delta_{w,k+1} \tag{37}$$

The covariance matrix of Δ_k is now given by

$$R_\Delta = \mathcal{H}^u \mathcal{T}_w R_{\Delta,w} (\mathcal{H}^u \mathcal{T}_w)^T \tag{38}$$

$$+ \mathcal{T}_w R_{\Delta,w} \mathcal{T}_w^T + \text{cross terms} \tag{39}$$

where the cross terms occur due to the fact that $\Delta_{w,k}$ and $\Delta_{w,k+1}$ are correlated. When e.g. $N = 2$

$$E[\Delta_{w,k}\Delta_{w,k+1}^T] = \begin{bmatrix} 0 & R_\delta \\ 0 & 0 \end{bmatrix}$$

and the cross terms are in this case given by

$$-\mathcal{H}^u \mathcal{T}_w \begin{bmatrix} 0 & R_\delta \\ 0 & 0 \end{bmatrix} \mathcal{T}_w^T - \mathcal{T}_w \begin{bmatrix} 0 & 0 \\ R_\delta & 0 \end{bmatrix} (\mathcal{H}^u \mathcal{T}_w)^T \tag{40}$$

Similar to (27) an equation for the covariance of the error $\mathcal{E}_k$ is given by

$$\begin{aligned} \mathcal{P} &= \mathcal{A}\mathcal{P}\mathcal{A}^T + R_\Delta \\ &\quad + \mathcal{A}E[\mathcal{E}_k\Delta_k^T] + E[\Delta_k\mathcal{E}_k^T]\mathcal{A}^T \end{aligned} \tag{41}$$

which can be solved once the cross terms have been decided.

8. EXAMPLE

To illustrate the derived expressions this section will contain an example dealing with a second order ILC-algorithm. Consider therefore a problem where the matrix $\boldsymbol{T}_u$ is invertible such that the design variables $\boldsymbol{H}_i^e = \mu_i \boldsymbol{T}_u^{-1} \quad i = 1, 2$ can be used. Furthermore the matrices $\boldsymbol{H}_1^u = r \cdot I$ and $\boldsymbol{H}_2^u = (1 - r) \cdot I$ are chosen. The choice of $\boldsymbol{H}_1^u$ and $\boldsymbol{H}_2^u$ is motivated by the condition in equation (32). It can be notice that a first order ILC algorithm is obtained as the special case $r = 1$ and $\mu_2 = 0$. The design variables imply

$$\mathcal{A} = \mathcal{H}^u - \mathcal{T}_u \mathcal{H}^e = \begin{pmatrix} (r - \mu_1) \cdot I & (1 - r - \mu_2) \cdot I \\ I & 0 \end{pmatrix} \tag{42}$$

The aim is now to investigate how the choice of the design variables μ_1, μ_2 and r affects the properties of the second order ILC algorithm.

8.1 *Stability*

A first requirement on the design variables is that the algorithm is stable, and the condition for this is, see Corollary 2 that $\rho(\mathcal{A}) < 1$. Using the rule, see for example (Kailath, 1980), for determinants of block matrices

$$\det \begin{pmatrix} A & D \\ C & B \end{pmatrix} = \det A \cdot \det(B - CA^{-1}D) \tag{43}$$

it is found that the eigenvalues of $\mathcal{A}$ are determined by the equation

$$\lambda^2 + (\mu_1 - r)\lambda + (r + \mu_2 - 1) = 0 \tag{44}$$

It is difficult to get immediate insight into how the roots depend on the design variables, and therefore the special case $\mu_1 = \mu_2 = \mu$ is treated. Figure 1 shows a plot of how the maximum eigenvalue of $\mathcal{A}$ depends on μ and r.

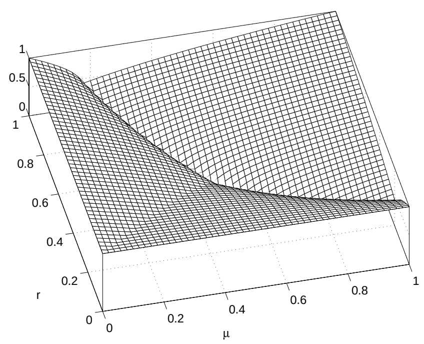

Fig. 1. Maximum eigenvalue of $\mathcal{A}$ as function of μ and r.

The plot verifies the observation in (44) that all the all eigenvalues are placed in the origin for the choice $\mu = r = 0.5$. From a pure transient viewpoint this is a desirable choice. Note that this is also achieved for all designs where $\mu_1 = r$ and $r + \mu_2 = 1$.

8.2 *Measurement disturbances*

The design variables selected above and an assumption that $\mathcal{T}_v$ equals the identity matrix imply that

$$\mathcal{B} = \mathcal{T}_u \mathcal{H}^e \mathcal{T}_v = \begin{pmatrix} \mu_1 \cdot I & \mu_2 \cdot I \\ 0 & 0 \end{pmatrix} \tag{45}$$

The cross term in the equation for the covariance matrix contains the factor

$$E[\mathcal{E}_k \boldsymbol{V}_k^T] = E \begin{bmatrix} \epsilon_k \boldsymbol{v}_k^T & \epsilon_k \boldsymbol{v}_{k-1}^T \\ \epsilon_{k-1} \boldsymbol{v}_k^T & \epsilon_{k-1} \boldsymbol{v}_{k-1}^T \end{bmatrix} \tag{46}$$

Using equation (20) it is found that all elements, except $E[\epsilon_k \boldsymbol{v}_{k-1}^T]$, are zero. For the non-zero element equation (20) gives, assuming $R_v = I$, that

$$E[\epsilon_k \boldsymbol{v}_{k-1}^T] = \boldsymbol{T}_u \boldsymbol{H}_1^e \boldsymbol{T}_v R_v = \mu_1 \cdot I \tag{47}$$

Due to these assumptions all blocks in the block matrices involved will be diagonal and for solution of equation (27) one can assign $\mathcal{P}$ to consist of diagonal block matrices. Solving (27) under these conditions gives

$$P = \frac{r(\mu_1 + \mu_2) - \mu_2(\mu_1 - \mu_2)}{(\mu_1 - \mu_2 - 2r)(r + \mu_2 - 2)} \cdot I \tag{48}$$

In the special case $r = 0.5$ and $\mu_1 = \mu_2 = \mu$ one gets

$$P = \frac{\mu}{1.5 - \mu} \cdot I \tag{49}$$

The covariance is a decreasing function of μ which means that a decreasing gain in $\mathcal{H}^e$ gives a reduced magnitude of the error. This is a logical result but it has to be remembered that the choice of μ also affects the transient properties of the ILC algorithm. In Figure 1 it is seen that using $r = 0.5$ the choice $\mu = 0.5$ gives the best transient properties, which yields $P = 0.5 \cdot I$. This can be compared to the covariance matrix obtained for a first order method, i.e. $r = 1$ and $\mu_2 = 0$, which is given by

$$P = \frac{\mu_1}{2 - \mu_1} \tag{50}$$

Choosing $\mu_1 = 1$ which corresponds to the fastest possible transient behavior the resulting covariance is $P = I$. Using the second order algorithm hence gives a reduction of the covariance matrix of the error.

8.3 *Load disturbances*

Recall now equation (41). In order to solve it for the particular design variables studied here the involved matrices have to be derived. The matrix $\mathcal{A}$, which was determined earlier in this section, is given by equation (42). For the type of load disturbances defined by equation (33) one gets

$$\Delta_k = \begin{pmatrix} r\delta_{w,k} + (1 - r)\delta_{w,k-1} - \delta_{w,k+1} \\ 0 \end{pmatrix} \tag{51}$$

using equation (37). This implies

$$E[\Delta_k \Delta_k^T] = \begin{pmatrix} (r^2 + (1 - r)^2 + 1) \cdot I & 0 \\ 0 & 0 \end{pmatrix} \tag{52}$$

where it has been assumed that $R_\delta = I$. The cross terms in (41), e.g. the term

$$\mathcal{A}E[\mathcal{E}_k\Delta_k^T] \tag{53}$$

are somewhat more complicated to compute in this case. After some calculations one obtains

$$E[\mathcal{E}_k\Delta_k^T] = \begin{pmatrix} (\mu_1(1-r)-r)\cdot I & 0 \\ (r-1)\cdot I & 0 \end{pmatrix} \tag{54}$$

Multiplying this matrix with $\mathcal{A}$ all terms in equation (41) are derived. Analogous to the measurement disturbance case one can assign $\mathcal{P}$ to be a block matrix containing diagonal matrices. Solving for the diagonal block P this finally leads to

$$P = \frac{2(\mu_2 + \mu_1(r-1) - r\mu_2 - (r-2)r)}{(\mu_1 - \mu_2 - 2r)(r + \mu_2 - 2)} \cdot I \tag{55}$$

In order to get some intuitive insight into the properties of this matrix it is assumed, similarly to the study of measurement disturbances, that $r = 0.5$, $\mu_1 = \mu_2 = \mu$. This gives

$$P = \frac{1.5}{1.5 - \mu} \cdot I \tag{56}$$

Also in this case it is advantageous to use a small μ in order to make P small. Since the iteration varying part of the load disturbance is independent between the iterations this variation can not be tracked by the ILC algorithm. It is therefore better to keep the updating gain low. Also here there has to be a trade off between the transient properties and the disturbance rejection. Finally it should be noted that these conclusion concerning load disturbances are highly dependent on the particular assumption on the variations of the load disturbances.

9. CONCLUSIONS

Some initial results concerning the disturbance properties of high order ILC algorithms have been derived. This has resulted in some fairly general expressions for the covariance matrix of the control error obtained when a high order ILC algorithm is used for control of a linear time invariant system. While the general expressions are primarily suited for numerical evaluation a second order example has been evaluated analytically. The obtained expression clearly shows the trade off between transient properties and disturbance rejection.

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