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ENERGY FUNCTIONS AND BALANCING FOR NONLINEAR DISCRETE-TIME SYSTEMS

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Abstract: This paper presents the nonlinear discrete-time versions of the controllability and observability functions, its properties and algorithms to find them. Furthermore, since the resulting energy functions are continuousfunctions of the initial state, nonlinear balancing techniques can be directly used. Linear and nonlinear examples are presented to illustrate these algorithms. *Copyright 2001 ©IFAC*

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1. INTRODUCTION

The study of systematic tools for model reduction of dynamic systems has been an early topic of interest in the systems and control fields. Model approximation based on the Hankel norm and the balancing method (Moore, 1981) have shown to be useful tools for model reduction for linear systems. Today singular values-based balancing, LQG balancing and $\mathcal{H}_\infty$ balancing are important tools for linear model reduction. Therefore the study of model reduction for linear systems can be considered a mature topic.

For nonlinear systems, there has been important progress with the continuousnonlinear extensions of systematic methods of balancing (singular-value-based, LQG and $\mathcal{H}_\infty$), mainly based on the controllability and observability functions (Scherpen, 1994), (Scherpen, 1993). Rougly speaking, in such procedure a Hamilton-Jacobi equation and a Lyapunov-like partial differential equation have to be solved in order to determine the energy functions. Then a nonlinear transformation transforms the system in balanced form. The mathematical complexity in solving such partial differental equations has stimulated the search for alternative methods to determine the energy functions (Newman *et al.*, 1998).

In this paper energy functions for stable nonlinear discrete-time systems are discussed with the purpose of extending the continuous-time theory discussed in (Scherpen, 1994), (Scherpen, 1993). Since the determination of such energy functions are a fundamental condition for nonlinear balancing and model reduction, the importance of this results lies on the establishment of firm steps towards a methodology suitable for computer implementation for the reduction of nonlinear discrete-time systems. Notice that in contrast with (Verriest *et. al.*, 2001), this approach does not assume any linearization procedure at all.

The paper is organized as follows. After fixing the notation used, the discrete-time energy functions are presented in Section 2. In Section 3, the observability function and its properties are then discussed. In Section 4 the properties of the controllability function and an optimization-based solution are discussed and commented. Section 5 presents the balancing method. In Sections 6 and 7, in order to illustrate the previous methods, linear and nonlinear examples are shown and briefly discussed. Finally, some conclusions are presented.

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Notation: The set of nonnegative and nonpositive integers are denoted as $\mathbb{Z}^+ \stackrel{\text{def}}{=} \{0, 1, 2, ...\}$ and $\mathbb{Z}^- \stackrel{\text{def}}{=} \{0, -1, -2, ...\}$ respectively. Discrete-time vector variables are denoted for instance as x_k or $x(k)$. Where convenient for clarity of exposition a function of several variables $f(x_k, u_k)$ may be denoted simply as f_k in the understanding that the corresponding state depends on k. Given a function f_k its inverse function (map) is denoted as f_k^{-1}. A successive application of a step variant linear map Φ_k for a discrete interval $k \in [m, n]$, is denoted as $\Phi_{[m,n]} \stackrel{\text{def}}{=} \Phi_m \Phi_{m+1} \cdots \Phi_n$. A successive composition of functions is denoted as, $f_{[m,n]} \stackrel{\text{def}}{=} f_m \circ f_{m+1} \circ ... \circ f_n$, with $f_{[n,n]} \stackrel{\text{def}}{=} f_n$. An optimal variable at time k is denoted as $v_k^\star$. The zero observation space is denoted as O_0. Finally, denote by $x_{k_2} = \phi(k_2, k_1, x_1, u_k)$ the solution of the system $x_{k+1} = f(x_k, u_k)$ with initial condition $x(k_1) = x_1$ and input $u_k \in \ell_2(0, \infty)$.

2. ENERGY FUNCTIONS

Consider the following discrete-time nonlinear system,

$$\begin{aligned} x_{k+1} &= f(x_k, u_k), \\ y_k &= h(x_k), \end{aligned} \qquad k \in \mathbb{Z} \tag{1}$$

where $u_k = (u_1, ..., u_p)_k \in \mathbb{R}^p$, $y_k = (y_1, ..., y_q)_k \in \mathbb{R}^q$ and $x_k = (x_1, ..., x_n)_k \in \mathbb{R}^n$ are local coordinates for a smooth state space manifold M. Moreover f and h are class C^∞ in a neighborhood $D \subset \mathbb{R}^n$ around an equilibrium point in $x = 0$ such that $f(0, 0) = 0$ and $h(0) = 0$. In this work it is assumed that $f(\cdot, u)$ is a diffeomorphism, (*i.e.* invertible in the spirit of (Fliess, 1992)). The following definitions of energy functions are discrete equivalents of the continuous versions presented in (Scherpen, 1993).

Definition 2.1. The controllability and observability functions of the system (1) are defined respectively as,

$$L_c(x_0) = \min_{\substack{u \in \ell_2(-\infty,0), \\ x(-\infty)=0,\ x(0)=x_0}} \frac{1}{2} \sum_{k=-\infty}^{0} \| u_k \|^2, \tag{2}$$

$$L_o(x_0) = \frac{1}{2} \sum_{k=0}^{\infty} \| y_k \|^2, x(0) = x_0, u_k = 0, \tag{3}$$

$\forall k \in \mathbb{Z}^+$.
In the following sections further properties of these functions restricted to system (1) are discussed.

3. OBSERVABILITY FUNCTION

In this section a recursive procedure to find the observability function is provided along with some properties. Also a Lyapunov-like difference equation analog to that found in (Scherpen, 1994), is presented.

Lemma 3.1. Consider the following recursive equation

$$\mathcal{L}_o(x_{i+1}) = \mathcal{L}_o(x_i) + \frac{1}{2} h^T[f(x_i, 0)] h[f(x_i, 0)], \tag{4}$$

for $i = 0, 1, 2...$ and $\mathcal{L}_o(x_0) = \frac{1}{2} h^T(x_0) h(x_0)$ as initial condition. Then $L_o(x_0)$ can be found from the solution of (4) as follows

$$L_o(x_0) = \lim_{i \to \infty} \mathcal{L}_o(x_i). \tag{5}$$

Proof. By eq. (1) and by definition of L_o in eq. (3), one obtains

$$\begin{aligned} L_o(x_0) &= \frac{1}{2} \sum_{i=0}^{\infty} h^T(x_i) h(x_i) \\ &= \frac{1}{2} h^T(x_0) h(x_0) \\ &\quad + \frac{1}{2} \sum_{i=0}^{\infty} h^T[f(x_i, 0)] h[f(x_i, 0)], \end{aligned} \tag{6}$$

for $i \in \mathbb{Z}^+$. Noting that

$$\mathcal{L}_o(x_{i+1}) = \mathcal{L}_o(x_0) + \frac{1}{2} \sum_{i=0}^{\infty} h^T[f(x_i, 0)] h[f(x_i, 0)],$$

the result is obtained. ■

Theorem 3.1. Consider the discrete-time nonlinear system (1). Assume that $dim(dO_0) = n$, then the system is locally zero-state observable at 0.

Proof. Similar to (Scherpen, 1993). ■

Theorem 3.2. Assume that (1) with $f(\cdot, 0)$ is asymptotically stable on a neighborhood D of $x = 0$. If the system is zero-state observable and L_o exists and is smooth on M, then $L_o(x_0) > 0, \forall x_0 \in M, x_0 \neq 0$.

Proof. Recall eq. (3), then, if $x_0 \neq 0$, zero state observability implies that for some $\bar{K} \in \mathbb{Z}^+ \backslash \{0\}$ we have $h(\phi(\bar{k}, 0, x_0, 0)) \neq 0$ for some $0 \leq \bar{k} < \bar{K}$. Therefore if $x_0 \neq 0$, $L_o(x_0) > 0$. ■

Proposition 3.1. Assume that the observability function L_o exists and is positive definite. Then L_o as defined in eq. (3) is a Lyapunov function for system (1). Furthermore, if the system is locally asymptotically stable at $x(0) = x_0$ for $u_k = 0$, then the system is dissipative and L_o is a storage function, with supply rate $\frac{1}{2} h^T(x_k) h(x_k)$.

Proof. In order to show that the difference $L_o(x_{k+1}) - L_o(x_k)$ is negative semi-definite (and thus a Lyapunov function (LaSalle, 1976)), express $L_o(x_k)$ for an arbitrary state x_k as,

$$\begin{aligned} L_o(x_k) &= \frac{1}{2} h^T(x_k) h(x_k) \\ &\quad + \frac{1}{2} \sum_{i=k}^{\infty} h^T[f(x_i, 0)] h[f(x_i, 0)], \end{aligned}$$

doing the same for x_{k+1}, and taking the difference then

$$L_o(x_{k+1}) - L_o(x_k) = -\frac{1}{2} h^T(x_k) h(x_k), \tag{7}$$

for $k \in \mathbb{Z}^+$, which is negative semidefinite. As can be seen, the *discrete-time dissipation inequality* (see e.g. (Lin *et al.*, 1996), (Willems, 1972)) is preserved and then L_o is a storage function with supply rate $\frac{1}{2}h^T(x_k)h(x_k)$. ■

Remark 3.1. Following the terminology used in (Scherpen, 1993), eq.(7) can be called the discrete-time Lyapunov-like equation.

Theorem 3.3. (Existence of L_o). Let $\|h(x_i)\|_2^2 \leq M_i$, $M_i \in \mathbb{R}$, such that $\sum_{i=0}^{\infty} M_i$ convergesuniformly and absolutely. Then L_o exists as given by (5) and is a smooth solution of (4) for all $x_0 \in D$.

Proof. By Lemma 3.1, eq. (5) is a solution of (4). Existence of the limit (5) for all $x_0 \in D$ is necessary and sufficent for existence of L_o. Since $(\mathbb{R}^n, \|\cdot\|_2)$ is a complete normed space, by Weierstrass' M-Theorem, the series of functions (6) converges uniformly and absolutely. ■

4. CONTROLLABILITY FUNCTION

Before determining some properties of the controllability function (2) of (1), it is useful to transform the definition of L_c into a more adequate representation.

Definition 4.1. Define the following system associated to system (1) as

$$w_{\kappa+1} = f^{-1}(w_\kappa, v_{\kappa+1}), \qquad \kappa \in \mathbb{Z}^+. \tag{8}$$

Where (8) can be obtained by applying two operations on eq.(1):

Backward-time: Inverting the map in eq.(1) and evolving in $k \in \mathbb{Z}^-$.

Flip-time: Defining $w_k \overset{\text{def}}{=} x_{-k}$ and $v_k \overset{\text{def}}{=} u_{-k}$ for $k \in \mathbb{Z}$, and changing the time index as $\kappa = -k$, $\kappa \in \mathbb{Z}^+$.

Remark 4.1. Consider the system (8). Then the definition of L_c from eq. (2), may be expressed as

$$L_c(w_0) = \min_{\substack{v \in \ell_2(0,\infty),\\ w(\infty)=0,\ w(0)=w_0}} \frac{1}{2}\sum_{\kappa=0}^{\infty} v_\kappa^T v_\kappa, \tag{9}$$

for w and v from (8).

Remark 4.2. v_0 does not influence the new state in (8), where it results $w_1 = f^{-1}(w_0, v_1)$. Therefore the value of v_0 which minimizes (9) is $v_0^\star = 0$ and thus $u_0^\star = 0$.

Lemma 4.1. Assume the existence of the optimal sequence $v^\star = \{v_i^\star | i = 0, 1, ...\}$ such that it satisfies (9) and consider the following recursive equation

$$\mathcal{L}_c(w_{i+1}) = \mathcal{L}_c(w_i) + \frac{1}{2}v_i^{\star T} v_i^\star, \tag{10}$$

for $i = 0, 1, 2, ...$ and initial condition $\mathcal{L}_c(w_0) = 0$. Then $L_c(w_0)$ can be found from the solution of (10) as follows

$$L_c(w_0) = \lim_{i \to \infty} \mathcal{L}_c(w_i). \tag{11}$$

Proof. Express (9) as,

$$L_c(w_0) = \frac{1}{2}v_0^{\star T} v_0^\star + \sum_{i=0}^{\infty} v_{i+1}^{\star T} v_{i+1}^\star, \tag{12}$$

which may be written as a recurrence equation with the initial condition $\mathcal{L}_c(w_0) = \frac{1}{2}v_0^{\star T} v_0^\star = 0$ as consequence of Remark 4.2. By solving iteratively (10), $L_c(w_0)$ can be found as i tends to infinity. ■

4.1 *Properties of L_c*

Proposition 4.1. Assume that the system (1) is asymptotically stable on D, that there exist a solution $v^\star$ to (9) and that the limit (11) exists. Then $L_c(w_0) > 0$ for $w_0 \in D$, $w_0 \neq 0$, if and only if the system

$$w_{\kappa+1} = f^{-1}(w_\kappa, v_{\kappa+1}^\star), \qquad \kappa \in \mathbb{Z}^+, \tag{13}$$

is asymptotically stable on D.

Proof. Assume that there exists $w_0 \in D$, $w_0 \neq 0$ such that $L_c(w_0) = 0$. Since in eq. (12) this is only possible if all $v_{i+1}^\star = 0$, for $i = 0, ..., \infty$, the system (13) is equivalent to the unforced system $w_{\kappa+1} = f^{-1}(w_\kappa, 0)$, for $\kappa \in \mathbb{Z}^+$, but this system cannot be stable since this wouldimply that the flipped system $w_\kappa = f(w_{\kappa+1}, 0)$, for $\kappa \in \mathbb{Z}^-$ is unstable, which contradicts the asymptotic stability of f. ■

Proposition 4.2. Assume that the system (13) is asymptotically stable on D, then the controllability function $L_c(w_0)$ as defined in eq. (9) is a Lyapunov function for system (8). Furthermore the system (8) is dissipative and $L_c(w_\kappa)$ is also a storage function, with supply rate $\frac{1}{2}v_\kappa^{\star T} v_\kappa^\star$.

Proof. That $L_c(w_\kappa)$ is a Lyapunov function for (8), can be shown noticing its nonnegative definitness from eq. (9). Since by assumption (13) is asymptotically stable, by Prop. 4.1 then $L_c(w_0) > 0$ for $w_0 \in D$. In order to show that the difference $L_c(w_{\kappa+1}) - L_c(w_\kappa)$ is negative semi-definite, note that for an arbitrary state w_κ, from (12), L_c can be expressed as

$$L_c(w_\kappa) = \frac{1}{2}v_\kappa^{\star T} v_\kappa^\star + \frac{1}{2}\sum_{i=\kappa}^{\infty} v_{i+1}^{\star T} v_{i+1}^\star, \tag{14}$$

doing the same for $w_{\kappa+1}$, and taking the difference yields,

$$L_c(w_{\kappa+1}) - L_c(w_\kappa) = -\frac{1}{2}v_\kappa^{\star T} v_\kappa^\star, \tag{15}$$

which is negative semidefinite. Since the discrete-time dissipation inequality is preserved $L_c(w_\kappa)$ is a storage function with supply rate $\frac{1}{2}v_\kappa^{\star T} v_\kappa^\star$. Note that $L_c(w_\kappa)$ has a finite value if $v_\kappa^\star$ is bounded and tends to zero as $\kappa \to \infty$. This is a direct consequence of the asymptotic stability of eq. (13). ■

Proposition 4.3. A necessary existence condition of $L_c(w_\kappa)$ in eq. (10), is that $v_\kappa^\star$ is the solution of the following two-point boundary value problem

$$\lambda_\kappa = [\frac{\partial}{\partial w_\kappa} f^{-1}(w_\kappa, v_{\kappa+1})]^T \lambda_{\kappa+1}, \quad (16)$$

$$v_{\kappa+1} = -[\frac{\partial}{\partial v_{\kappa+1}} f^{-1}(w_\kappa, v_{\kappa+1})]^T \lambda_{\kappa+1}, \quad (17)$$

subject to the boundary conditions $w(\infty) = 0$ and $w(0) = w_0$.

Proof. In order to find $L_c(w_\kappa)$ given by eq. (9), applying standard tools of the discrete optimal control theory (see for instance (Lewis *et al.*, 1995), (Bryson, 1999)) results in the following Hamiltonian,

$$H_\kappa = \frac{1}{2} v_{\kappa+1}^T v_{\kappa+1} + \lambda_{\kappa+1}^T f^{-1}(w_\kappa, v_{\kappa+1}), \quad (18)$$

resulting in

$$\frac{\partial H_\kappa}{\partial w_\kappa} = \lambda_{\kappa+1}^T \frac{\partial}{\partial w_\kappa} f^{-1}(w_\kappa, v_{\kappa+1}) = \lambda_\kappa^T,$$
$$\frac{\partial H_\kappa}{\partial v_{\kappa+1}} = v_{\kappa+1}^T + \lambda_{\kappa+1}^T \frac{\partial}{\partial v_{\kappa+1}} f^{-1}(w_\kappa, v_{\kappa+1}) = 0,$$

from which eqs. (16) and (17) follow. ■

As can be observed from eq. (17), the input $v_{\kappa+1}$ may appear implicitly. Therefore the analytical solution of this problem may be difficult to find.

Theorem 4.1. (Existence of L_c). Assume that $v^\star$ satisfies eq. (9) with $L_c(w_0)$ smooth for all $x \in D$ and such that eq. (13) is asymptotically stable. Let $\|v_i^\star\|_2^2 \leq M_i$, $M_i \in \mathbb{R}$ such that $\sum_{i=0}^{\infty} M_i$ converges uniformly and absolutely. Then $L_c(w_0)$ exists as given by (11) and is a smooth solution of (10) for all $w_0 \in D$.

Proof. By Remark 4.1 existence of $L_c(x_0)$ is equivalent to existence of $L_c(w_0)$. By Lemma 4.1, eq. (11) is a solution of (10). $L_c(w_0)$ exists if the series of functions (11) converges. Since $(\mathbb{R}^n, \|\cdot\|_2)$ is a complete normed space, by Weierstrass' M-Theorem, the series (11) converges uniformly and absolutely.

■

4.2 *About the structure of* $v_\kappa^\star$

In order to study the structure of $v_\kappa^\star$ in (16)-(17), the corresponding boundary value problem is addressed. Define the following functions,

$$\Phi_\kappa = \frac{\partial}{\partial w_\kappa} f^{-1}(w_\kappa, v_{\kappa+1}), \quad (19)$$

$$\Gamma_\kappa = -\frac{\partial}{\partial v_{\kappa+1}} f^{-1}(w_\kappa, v_{\kappa+1}), \quad (20)$$

then the solution of (16), given an initial λ_N, with $0 \leq \kappa \leq N$ can be expressed as, $\lambda_\kappa = \Phi_{[\kappa,N-1]}^T \lambda_N$, and in consequence the possibly implicit input $v_{\kappa+1}$ can be obtained from the following expression,

$$v_{\kappa+1} = \Gamma_\kappa^T \Phi_{[\kappa+1,N-1]}^T \lambda_N. \quad (21)$$

Consider the following composition operations for the map $f_{[i,N]} \stackrel{\text{def}}{=} f_{i+1} \circ f_{i+2} \circ \ldots \circ f_N$, and for the inverse map $f_{[i,0]}^{-1} \stackrel{\text{def}}{=} f_i^{-1} \circ f_{i-1}^{-1} \circ \ldots \circ f_0^{-1}$, as well[2]. Then eq. (8) and the backward-time system $w_\kappa = f(w_{\kappa+1}, v_{\kappa+1})$, $\kappa \in \mathbb{Z}^-$, in terms of equation (21) can be expressed as,

$$w_{\kappa+1} = f^{-1}(w_\kappa, \Gamma_\kappa^T \Phi_{[\kappa+1,N-1]}^T \lambda_N),$$
$$w_\kappa = f(w_{\kappa+1}, \Gamma_\kappa^T \Phi_{[\kappa+1,N-1]}^T \lambda_N),$$

At the boundary for $\kappa = 0$, $w(0) = w_0$,

$$w_0 = f(w_1, \Gamma_0^T \Phi_{[1,N-1]}^T \lambda_N) = f_{[0,N]},$$

and for $\kappa = N$, $w_N = 0$,

$$0 = f^{-1}(w_{N-1}, \Gamma_{N-1}^T \Phi_{N-1}^T \lambda_N) = f_{[N,0]}^{-1}, \quad (22)$$

and its inverse map is,

$$w_{N-1} = f(0, \Gamma_{N-1}^T \Phi_{N-1}^T \lambda_N) = f_{[N,N]}.$$

In the last equation, we have a nonlinear relation between w_{N-1}, and λ_N. Notice also that $\Gamma_{N-1} = \Gamma(w_{N-1}, v_N)$, with v_N inserted possibly in implicit form. In the linear case this never occurs and thus it is always solvable. In the general case this problem is difficult to solve in closed form. However, optimization algorithms can be used in order to solve it. This is presented in the next subsection.

4.3 *Optimization-based search of* $v_\kappa^\star$

Proposition 4.4. Assume that the conditions of Theorem 4.1 are satisfied. Let $N, \epsilon \in \mathbb{Z}^+$ be such that $\|w_N\| \leq \epsilon$ for ϵ small enough. Assume that N is known. Then $L_c(w_\kappa)$ in eq. (10) can be determined depending on the solvability of the following optimization problem

$$\min_{\{v_i | i=1,\ldots,N\}} \mathcal{L}_c(w_{N+1}), \quad (23)$$

with equality constraints

$$w_{i+1} = f^{-1}(w_i, v_{i+1}), \quad (24)$$

$$\mathcal{L}_c(w_{i+1}) = \mathcal{L}_c(w_i) + \frac{1}{2} v_i^T v_i, \quad (25)$$

$$w_{N+1} = 0, \quad (26)$$

$$w(0) = w_0, \quad (27)$$

with (suitable) initial conditions $\{v_{0j} | j = 0, \ldots, N\}$ and with $\mathcal{L}_c(w_0) = 0$, determining $v_i^\star$.

Proof. Define the finite set $\{v_i | i = 0, \ldots, N\} \subset \{v_i | i = 0, \ldots, \infty\}$ such that eq. (9) is satisfied. Then by using an optimization approach (Bryson, 1999), the optimization problem takes the form

$$\min_{\{v_i | i=1 \ldots N\}} \frac{1}{2} \sum_{i=0}^{N} v_i^T v_i,$$

with equality constraints (26) and (27). Recasting this problem into the Mayer form (see e.g.(Bryson, 1999)), yields the presented form. ■

[2] With a slight abuse of notation, $f_i^{-1} = f^{-1}(x_i, v_{i+1})$.

The solvability of this nonlinear optimization problem depends, of course, on the optimization methods and the closeness of the guess of the initial conditions used for this purpose. Two drawbacks of this approach can be pointed out. Though for an asymptotically stable system N can be approximated to be finite, introducing some error in the result, the *best value* of N is unknown prior to the nonlinear optimization process. Furthermore, since each iteration implies a repeated composition of the inverse function in the form $f^{-1}_{[N,0]}$, this method may be inefficient for a computational implementation.

5. BALANCING

Remark 5.1. Despite the discrete nature of (1), the energy functions associated to this system, eq (2) and (3) are continuous functions of the initial state for $x_0 = x$. As a consequence, Morse's Lemma can be applied in order to find a desired transformation for a balanced representation, just as in the continuous time (for details see (Scherpen, 1993)).

Theorem 5.1. (Scherpen, 1993) Consider system (1) and assume that there exists a neighborhood D of $x = 0$ where the system is zero-state observable, f_k is asymptotically stable, and L_o and L_c exist and are smooth. Then there exist a coordinate transformation $x = \phi(\bar{x})$, $\phi(0) = 0$ (in D), such that in the new coordinates $\bar{x} = \phi^{-1}(x)$ the function $L_c(x_0)$ is of the form $L_c(\phi(\bar{x})) = \frac{1}{2}\bar{x}^T\bar{x}$. Moreover, in the new coordinates $\bar{x} = \phi^{-1}(x)$ we can write $L_o(x_0)$ in the form $L_o(\phi(\bar{x})) = \frac{1}{2}\bar{x}^T H(\bar{x})\bar{x}$, where $H(0) = \frac{\partial^2 L_o}{\partial x^2}(0)$ with $H(\bar{x})$ is a $n \times n$ symmetric matrix such that its entries are smooth functions of $\bar{x}$. Furthermore, assume that the number of distinct eigenvalues of $H(\bar{x})$ is constant for $\bar{x} \in D$. Then on D there exists a coordinate transformation $x = \psi(z)$, $\psi(0) = 0$, such that in the new coordinates $z \in B \stackrel{def}{=} \psi^{-1}(D)$ the function L_c is of the form $L_c(z) \stackrel{def}{=} L_c(\psi(z)) = \frac{1}{2}z^T z$, and the function L_o is of the form

$$L_o(z) \stackrel{def}{=} L_o(\psi(z)) = \frac{1}{2}z^T \mathrm{diag}[\tau_1(z) \cdots \tau_n(z)]z,$$

where $\tau_1(z) \geq \cdots \geq \tau_n(z)$ are smooth functions of z, called the singular value functions of the system.

Proof. See (Scherpen, 1993). ■

6. LINEAR SYSTEMS

As an example, consider the following linear, stable, minimal, discrete-time system

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ y_k &= Cx_k, \end{aligned} \tag{28}$$

where $u \in \mathbb{R}^p$, $y \in \mathbb{R}^q$ and $x \in \mathbb{R}^n$. The following result is known (Pernebo *et al.*, 1982).

Corollary 6.1. Consider the system (28). Then L_c and L_o, as defined in eq.(2)-(3), are given by,

$$L_c(x_0) = \frac{1}{2}x_0^T P^{-1} x_0, \tag{29}$$

$$L_o(x_0) = \frac{1}{2}x_0^T Q x_0, \tag{30}$$

with Gramians $P = \sum_{k=0}^{\infty} A^k BB^T A^{kT}$ and $Q = \sum_{k=0}^{\infty} A^{kT} C^T C A^k$.

Alternative proof. Use recurrent eq. (4) for the system (28) resulting in the following difference equation

$$\mathcal{L}_o(x_{i+1}) = \mathcal{L}_o(x_i) + \frac{1}{2}x_i^T A^T C^T C A x_i, \tag{31}$$

with initial condition $\mathcal{L}_o(x_0) = \frac{1}{2}x_0^T C^T C x_0$. Then the solution of (31) yields

$$L_o(x_0) = \lim_{i\to\infty} \mathcal{L}_o(x_i) = \sum_{k=0}^{\infty} x_0^T A^{kT} C^T C A^k x_0,$$

which is eq. (30). In order to find L_c, assume the existence of A^{-1} and consider the system from Def. 4.1 associated to eq. (28), given as

$$w_{\kappa+1} = A^{-1}w_\kappa - A^{-1}Bv_{\kappa+1}, \tag{32}$$

whose general solution can be expressed as

$$w_\kappa = A^{-\kappa}w_0 - \sum_{i=0}^{\kappa-1}(A^{-1})^{\kappa-i}Bv_{i+1}. \tag{33}$$

Using (16) and (17), results in

$$\lambda_\kappa = A^{-T}\lambda_{\kappa+1}, \tag{34}$$

$$v_{\kappa+1} = B^T A^{-T}\lambda_{\kappa+1}. \tag{35}$$

Sustitution of (35) in (32) yields,

$$w_{\kappa+1} = A^{-1}w_k - A^{-1}BB^T A^{-T}\lambda_{\kappa+1}. \tag{36}$$

Solving eq. (34) explicitly in *backward time*, results in

$$\lambda_\kappa = (A^{-T})^{N-\kappa}\lambda_N. \tag{37}$$

Then the solution of (36) with input $\lambda_{\kappa+1}$ given by (37) is

$$w_\kappa = A^{-\kappa}w_0 - \sum_{i=0}^{\kappa-1} A^{i-\kappa}BB^T(A^T)^{i-N}\lambda_N. \tag{38}$$

For $w_N = 0$, eq. (38) implies that, $w_0 = P(A^T)^{-N}\lambda_N$ where $P = \sum_{i=0}^{N-1} A^i BB^T (A^T)^i$, which can be expressed as $\lambda_N = (A^T)^N P^{-1}x_0$, which in eq.(37) for $\lambda_{\kappa+1}$ and this result in eq.(35), yields $v^\star_{\kappa+1} = B^T(A^T)^\kappa P^{-1}w_0$ which after substitution in eq.(10) results in eq.(29). ■

7. EXAMPLE OF A NONLINEAR SYSTEM

Consider the following nonlinear system

$$\begin{aligned} x_{1\,k+1} &= -ax_{1\,k}^2 + x_{2\,k} + u_k, \\ x_{2\,k+1} &= bx_{1\,k}, \\ y_{1\,k} &= x_{1\,k}, \quad y_{2\,k} = x_{2\,k} \end{aligned} \tag{39}$$

in which the state is given as $x_k = (x_{1\,k}, x_{2\,k})^T$. This system is locally stable around the origin for

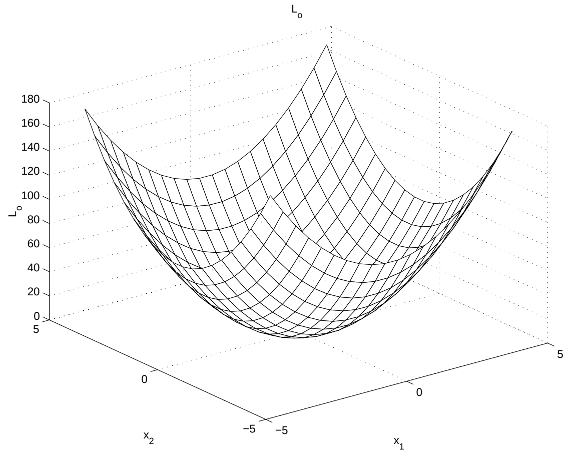

Fig. 1. Observability function L_o.

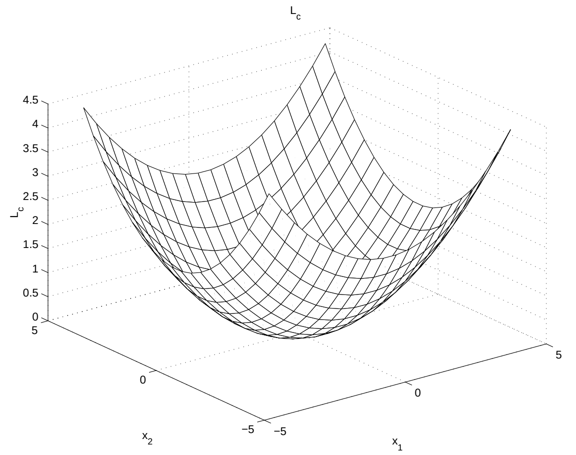

Fig. 2. Controllability function L_c.

$|-ax_{10} \pm \sqrt{a^2 x_{10}^2 + b}| < 1$. To determine L_c and L_o in this case let us take $a = -0.001$ and $b = -0.9$ at the origin.

Observability function: Consider the iterative solution of eq.(4), as $i \to \infty$ for each initial state x_0 within the desired region to plot. The resulting observability function is presented in fig. 1.

Controllability function: The backward-time system (inverse map) associated to (39), is easily obtained and according to Def. 4.1 the transformed system is thus

$$w_{1\,\kappa+1} = \frac{1}{b} w_{2\,\kappa},$$
$$w_{2\,\kappa+1} = w_{1\,\kappa} + \frac{a}{b^2} w_{2\,\kappa}^2 - v_{\kappa+1}.$$

By using the optimization approach of Prop. 4.4 and defining a finite set $\{v_i | i = 1...N\}$, for $N = 20$, the optimization problem stated in eq.(23)-(27) can be solved for each w_0 within the local stability region of system (39) and thus the results can be plotted resulting in fig. 2. The Optimization Toolbox (Matlab) was used to find $v^\star$.

8. CONCLUSIONS

In this paper the discrete-time versions of the controllability and observability energy functions applied to linear and nonlinear discrete-time systems has been presented. Instead of looking for the solution of a Hamilton-Jacobi-Isaacs and a Lyapunov-like partial differential equations as in the continuous-time case, an optimization approach and an iterative algorithm are proposed to find L_c and L_o respectively. Moreover since the resulting energy functions are continuous in its arguments, several tools originally developed for balancing of continuous-time systems are directly applicable to discrete-time systems. The relevance of these results lies on its applicability to model reduction and system identification for discrete-time nonlinear systems. Moreover, with the availability of nonlinear discretization algorithms (Monaco *et al.*, 1986), (Kotta, 1995), the methods presented here may result in alternative algorithms in comparison with (Newman *et al., 1998)* for continuous-time systems.

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