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A DESCRIPTION OF ALL STABILIZING CONTROLLERS AND ITS APPLICATIONS

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Abstract: This paper present a description of all stabilizing controllers by polynomial matrix approach. As the applications to proposed description, a model matching problem for a plant with measurement noise and a pole placement for a plant with perturbation will be considered. Some simulations studies will be shown to confirm the validity of the methods. *Copyright©2002 IFAC*

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1. INTRODUCTION

Since the analysis and synthesis of control systems are originally discussed based on the transfer function, the approach of the transfer function as a fraction of polynomial matrices, i.e., polynomial matrix approach, is developed. This approach is convenience especially for the design of adaptive control system, since it becomes clear the degrees of controller and the number of parameters to be estimated (or calculated). The polynomial matrix approach is then an important strategy for analysis and synthesis of control systems. However, it is pointed out that some operations of large dimensional real matrices are necessary, and it is hard to confirm the properness of the controllers in polynomial matrix approach.

There seems to be some advantages in the control system design by using a proper and stable rational matrices (Vidyasagar 1985) for both calculation and properness confirmation. This comes from the fact that it can be easy to calculate the all stabilizing controllers (Youla *et al*. 1976) by using state space representation (Nett *et al*. 1984). In other words, there are insufficient literatures on calculations by using state space representation for polynomial matrix approach, although there are some important papers (Wolovich 1984, Wolovich and Antsaklis 1984).

From the above point of view, we proposed a solution of Diophantine equation, which will appear in stabilizing controller design by polynomial matrix approach, and showed the proposed solution gave another description of all stabilizing controllers. However, in some applications such as considered in this paper, there exists more preferred description. In this paper, a slight modified form of the previous proposed description will be shown. Then, two application of the description will be shown.

2. PROBLEM STATEMENT

The problem discussed in this paper is finding a polynomial matrix pair $X_d(z) \in \boldsymbol{R}^{m\times p}[z]$ and $Y_d(z) \in \boldsymbol{R}^{m\times m}[z]$ satisfying

$$X_d(z)D(z) + Y_d(z)N(z) = H_d(z)[D(z) - D_*(z)] \quad (1)$$

for given polynomial matrices $D(z) \in \boldsymbol{R}^{m\times m}[z]$, $N(z) \in \boldsymbol{R}^{p\times m}[z]$ and $H_d(z)D_*(z) \in \boldsymbol{R}^{m\times m}[z]$. It is assumed that the given polynomial matrices are satisfies the following assumptions:
A1 $D(z)$ is nonsingular and column proper.
A2 $N(z)D^{-1}(z)$ is strictly proper.
B1 Column degrees and leading column coefficient matrix of $D_*(z)$ is equivalent to that of $D(z)$ (thus $\lim_{z\to\infty} D_*(z)D^{-1}(z) = I_m$).
B2 $H_d(z)$ is row proper with all row degrees are $d-1$.

Eqn.(1) is called Diophantine equation which appears in the design of control system like Fig.1, where $C(z) = [H_d(z)-X_d(z)]^{-1}Y_d(z)$. In the next section, it will be given an algebraic equation any solution of which gives the parameters of eqn.(1).

3. ALL SOLUTION OF DIOPHANTINE EQUATION

To derive a solution of eqn.(1), it is convenient to use the extended division algorithm (Kase 1999a) which original version was presented by Wolovich (1984).

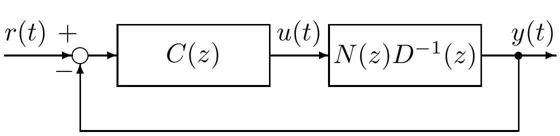

Fig. 1. A unity feedback system

Lemma 1. Let $Q(z) \in \boldsymbol{R}^{q\times m}[z]$ and $R(z) \in \boldsymbol{R}^{q\times m}[z]$ denote the polynomial matrices satisfying

$$\begin{aligned} &P(z)N(z) = Q(z)D(z) + R(z), \\ &R(z)D^{-1}(z) \text{ is strictly proper,} \end{aligned} \quad (2)$$

where

$$\begin{aligned} P(z) &= P_0 + zP_1 + \cdots + z^{f-1}P_{f-1} + z^fP_f \\ &= \boldsymbol{P}S^f_{I_p}(z) \in \boldsymbol{R}^{q\times p}[z] \\ \boldsymbol{P} &:= \begin{bmatrix} P_0 & P_1 & \cdots & P_{f-1} & P_f \end{bmatrix} \\ S^f_{I_p}(z) &:= \begin{bmatrix} I_p & zI_p & \cdots & z^fI_p \end{bmatrix}^T \end{aligned}$$

and let $(A,\ B,\ C)$ denote the controllability companion form (Wolovich 1974) of $N(z)D^{-1}(z)$. Then, the polynomial matrix pair $Q(z)$ and $R(z)$ satisfying eqn.(2) is given by

$$\begin{aligned} Q(z) &= \boldsymbol{P}\boldsymbol{T}_{f-1}(A,B,C)S^{f-1}_{I_m}(z) \\ R(z) &= \boldsymbol{O}_f(C,A)S^{\mu_i-1}(z) \end{aligned} \quad (3)$$

where μ_i is the i-th column degree of $D(z)$ and

$$S^{\mu_i}(z) = block\ diag \left\{ \begin{matrix} 1 \\ z \\ \vdots \\ z^f \end{matrix} \right\},$$

$$\boldsymbol{O}_f(C,A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^f \end{bmatrix},$$

$$\boldsymbol{T}_f(A,B,C) = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ CB & 0 & \cdots & 0 \\ CAB & CB & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^fB & CA^{f-1}B & \cdots & CB \end{bmatrix}$$

The above Lemma shows that calculation method of quotient $Q(z)$ and remainder $R(z)$ when $P(z)N(z)$ divides $D(z)$.

Let $\boldsymbol{X}_d$, $\boldsymbol{Y}_d$ and $\boldsymbol{H}_d$ denote the parameters satisfying

$$\begin{aligned} X_d(z) &= \boldsymbol{X}_dS^{d-2}_{I_p}(z), \\ Y_d(z) &= \boldsymbol{Y}_dS^{d-1}_{I_m}(z), \\ H_d(z) &= \boldsymbol{H}_dS^{d-1}_{I_m}(z). \end{aligned} \quad (4)$$

Right multiplying $D^{-1}(z)$ to both sides in eqn.(1),

$$\begin{aligned} &Y_d(z)N(z)D^{-1}(z) + X_d(z) \\ &= H_d(z)[D(z) - D_*(z)]D^{-1}(z). \end{aligned} \quad (5)$$

The above relation means that the remainder of $Y_d(z)N(z)D^{-1}(z)$ equals to that of $H_d(z)[D(z) - D_*(z)]D^{-1}(z)$. Thus, the following Theorem holds:

Theorem 1. Let $(A,\ B,\ C)$ denote any observable realization of $N(z)D^{-1}(z)$, and $(A,\ B,\ F)$ denote the realization of $[D(z) - D_*(z)]D^{-1}(z)$. Then, there exist $X_d(z)$ and $Y_d(z)$ which satisfy
1) Diophantine equation (1).
2) $H_d^{-1}(z)X_d(z)$ is strictly proper.
3) $H_d^{-1}(z)Y_d(z)$ is proper.
if and only if the following relation holds:

$$\left[\ \boldsymbol{X}_d\ \boldsymbol{Y}_d\ \right]\begin{bmatrix} I_m(d-1) & 0 \\ \boldsymbol{T}_{d-2}(A,B,C) & \boldsymbol{O}_{d-1}(C,A) \end{bmatrix} = \left[\ \boldsymbol{H}_d\boldsymbol{T}_{d-2}(A,B,F)\ \ \boldsymbol{H}_d\boldsymbol{O}_{d-1}(F,A)\ \right]. \tag{6}$$

Basically, the above Theorem is same as was given in Kase (1999a). But the expression given in eqn.(6) is more general and useful as will be shown in the following.

It is worth to note that the controllers $X_d(z)$ and $Y_d(z)$ have more flexibility by increasing the degree d of observer polynomial matrix $H(z)$. Let $[\boldsymbol{X}_i^{opt}\ \ \boldsymbol{Y}_i^{opt}]$ denote the optimal controller parameters which minimizes a certain norm $||\cdot||$. When $d = i+1$, consider the controller parameters $[\boldsymbol{X}_{i+1}\ \ \boldsymbol{Y}_{i+1}] = [\boldsymbol{X}_i^{opt}\ 0_{m\times p}\ \boldsymbol{Y}_i^{opt}\ 0_{m\times m}]$. It is easy to see this controller parameters satisfy eqn(6) for $d = i+1$. Therefore,

$$\begin{aligned} ||[\boldsymbol{X}_{i+1}^{opt}\ \ \boldsymbol{Y}_{i+1}^{opt}]|| &\le ||[\boldsymbol{X}_{i+1}\ \ \boldsymbol{Y}_{i+1}]|| \\ &= ||[\boldsymbol{X}_i^{opt}\ 0_{m\times p}\ \boldsymbol{Y}_i^{opt}\ 0_{m\times m}]|| \\ &= ||[\boldsymbol{X}_i^{opt}\ \ \boldsymbol{Y}_i^{opt}]||. \end{aligned} \tag{7}$$

4. APPLICATIONS

4.1 *EMM Control for a Plat with White Noise*

Consider the plant

$$y(t) = \frac{N(z)}{D(z)}\{u(t) + w_1(t)\} \tag{8}$$

where $u(t)$ is an input, $y(t)$ is an output, $w_1(t)$ is an input noise. $D(z)$ and $N(z)$ are polynomial of degree n and less than m respectively. It is assumed without loss of generality that $D(z)$ is monic. In the above plant, it is supposed that the measured output signal $\bar{y}(t)$ is the sum of the real output signal $y(t)$ and the measurement noise $w_2(t)$, i.e.,

$$\bar{y}(t) = y(t) + w_2(t). \tag{9}$$

Input noise $w_1(t)$ and measurement noise $w_2(t)$ are assumed to satisfy the following properties:

$$\begin{aligned} &E[w_i(t)] = 0, \\ &E[w_i^2(t)] = \sigma_i^2, \\ &E[w_i(t)w_i(t+k)] = 0, \quad k \neq 0\ (integer) \end{aligned} \tag{10}$$

for $i = 1,\ 2$, where $E[\cdot]$ denotes the expectation operator.

The purpose of the design is to determine the exact model matching (EMM) controller which makes the closed-loop system be internally stable and also minimizes the variance of $y(t)$. If $w_i(t) \equiv 0$, it is well known that the EMM can be attained by

$$u(t) = \frac{X_d(z)}{z^{d-1}}u(t) + \frac{Y_d(z)}{z^{d-1}}y(t) \tag{11}$$

where $X_d(z)$ and $Y_d(z)$ are the controller polynomials to be determined so as to satisfy the following Diophantine equation (Fig.2):

$$\begin{aligned} &X_d(z)D(z) + Y_d(z)N(z) \\ &= z^{d-1}[D(z) - N_m^{-1}z^{\delta}N(z)] \end{aligned} \tag{12}$$

where N_m is the highest degree coefficient of $N(z)$, and $\delta := n - m$. Let $(A,\ b,\ c)$ denote an observable realization of $N(z)/D(z)$ (minimality is not assumed). For the above, the equation corresponding to eqn.(5) is

$$X_d(z) + Y_d(z)\frac{N(z)}{D(z)} = z^{d-1} - N_m^{-1}z^{d+\delta-1}\frac{N(z)}{D(z)} \tag{13}$$

and the result of Theorem 1 becomes

$$\left[\ \boldsymbol{X}_d\ \boldsymbol{Y}_d\ \right]\begin{bmatrix} I_m & 0 \\ \boldsymbol{T}_{d-2}(A,b,c) & \boldsymbol{O}_{d-1}(c,A) \end{bmatrix} = \left[\ cA^{d+\delta-2}b\ \cdots\ cb\ cA^{d+\delta-1}\ \right]. \tag{14}$$

In the case where $w_i(t) \neq 0$, the control input is given by

$$u(t) = \frac{X_d(z)}{z^{d-1}}u(t) + \frac{Y_d(z)}{z^{d-1}}\bar{y}(t). \tag{15}$$

From eqns.(12) and (15), the input-output relation from $[w_1(t)\ w_2(t)]^T$ to $y(t)$ becomes

$$y(t) = N_m\left\{\frac{z^{d-1} - X_d(z)}{z^{d+\delta-1}}w_1(t) + \frac{Y_d(z)}{z^{d+\delta-1}}w_2(t)\right\}. \tag{16}$$

Therefore, the variance of $y(t)$ becomes

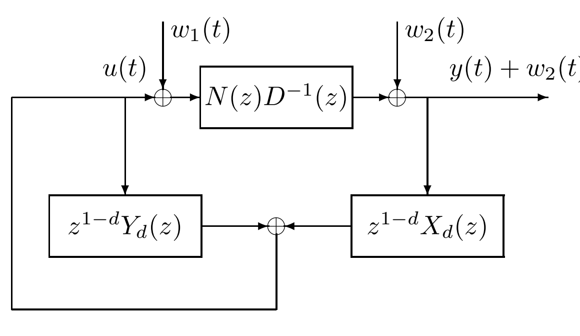

Fig. 2. The EMM system with input and output noise

$$y^2(t) = N_m\{\sigma_1^2(1+||\boldsymbol{X}_d||^2)+\sigma^2||\boldsymbol{Y}_d||^2\}. \tag{17}$$

Thus, the problem becomes the following simple optimization:

$$\min ||[\sigma_1\boldsymbol{X}_d \;\; \sigma_2\boldsymbol{Y}_d]|| \tag{18}$$

$$\text{s.t. } [\sigma_1\boldsymbol{X}_d \;\; \sigma_2\boldsymbol{Y}_d]\begin{bmatrix} \sigma_1^{-1}I_m & 0 \\ \sigma_2^{-1}\boldsymbol{T}_{d-2} & \sigma_2^{-1}\boldsymbol{O}_{d-1}\end{bmatrix} = \begin{bmatrix} cA^{d+\delta-2}b & \cdots & cb & cA^{d+\delta-1}\end{bmatrix} \tag{19}$$

The solution of the above optimization problem can be obtained by using Moore-Penrose pseudoinverse. Unlike Mutoh and Nikiforuk (1994) or Kase and Mutoh (2000), both measurement and input noises are considered. More tight evaluation can be possible than Kase and Mutoh (1999) where non-minimum phase plant is considered.

4.2 *Simple Robust Controller Design*

Although multi-input, multi-output case was discussed in the previous section, single-input, single-output case will be considered in the followings. Consider the transfer function of linear discrete-time plant:

$$G_0(z) := \frac{N_0(z)}{D_0(z)} \tag{20}$$

where $D_0(z)$ and $N_0(z)$ are polynomials. Assume that the mathematical model of the above plant is presented by

$$G(z) := \frac{N(z)}{D(z)}. \tag{21}$$

The purpose of the design is to construct a controller for the plant model (21) with an arbitrary close-loop pole locations, which also stabilizes the plant (20). For this purpose, we make some assumptions as follows:

A1 A desired closed-loop denominator polynomial $D_*(z)$ is Hurwitz.

Note that minimum phase assumption was made in the early report (Kase and Mutoh, 2001). For the plant model (21), it is well known that an arbitrary pole placement can be attained by

$$u(t) = \frac{X_d(z)}{z^{d-1}}u(t) + \frac{Y_d(z)}{z^{d-1}}y(t) + r(t) \tag{22}$$

where $X_d(z)$ and $Y_d(z)$ satisfy the following Diophantine equation

$$X_d(z)D(z) + Y_d(z)N(z) = z^{d-1}\{D(z) - D_*(z)\} \tag{23}$$

(for simplicity, we only consider the case where $H_d(z) = z^{d-1}$). Applying the control input (22) to the plant (20), the closed-loop transfer function $G_{cl}(z)$ is given by

$$G_{cl}(z) = \frac{z^{d-1}N_0(z)}{z^{d-1}D_0(z) - X_d(z)D_0(z) - Y_d(z)N_0(z)} \tag{24}$$

Therefore, the closed-loop system is stable if the denominator polynomial of the above transfer function is Hurwitz.

Define the following rational function:

$$F(z) = \frac{z^{d-1}D_0(z) - X_d(z)D_0(z) - Y_d(z)N_0(z)}{z^{d-1}D_*(z)} \tag{25}$$

where zeros are the closed-loop poles. This function is analytic outside the unit circle due to the assumption 1. Using eqs.(20), (21) and (23), $F(z)$ can be rewritten as

$$F(z) = 1 + \begin{bmatrix} z^{d-1} - X_d(z) & Y_d(z)\end{bmatrix}\frac{\begin{bmatrix}\widetilde{D}(z) \\ \widetilde{N}(z)\end{bmatrix}}{z^{d-1}D_*(z)} \tag{26}$$

where

$$\begin{aligned}\widetilde{D}(z) &:= D_0(z) - D(z), \\ \widetilde{N}(z) &; = N_0(z) - N(z).\end{aligned} \tag{27}$$

The unstable region $\mathbf{D}$ is given by

$$\mathbf{D} = \{z : \; |z| \geq 1\}. \tag{28}$$

From the assumption A1, $F(z)$ is analytic in $\mathbf{D}$. Therefore, from Rouche's theorem, $F(z)$ and 1 have the same number of zeros in $\mathbf{D}$ if the following inequality holds:

$$1 > |F(z) - 1|. \tag{29}$$

Since 1 does not have any zeros in $\mathbf{D}$, $F(z)$ also has no zeros in $\mathbf{D}$ if eqn.(29) holds. Thus, the closed-loop system given by eqn.(24) is stable if eqn.(29) holds.

Eq.(29) can be written by

$$\left|\begin{bmatrix} z^{d-1} - X_d(z) & Y_d(z)\end{bmatrix}\begin{bmatrix}\widetilde{D}(z) \\ \widetilde{N}(z)\end{bmatrix}\right| < \left|z^{d-1}D_*(z)\right|. \tag{30}$$

The above inequality holds if the following holds:

$$\left|\begin{bmatrix} z^{d-1} - X_d(z) & Y_d(z)\end{bmatrix}\right| < \left|\frac{z^{d-1}D_*(z)}{\begin{bmatrix}\widetilde{D}(z) & \widetilde{N}(z)\end{bmatrix}}\right|. \tag{31}$$

The right hand side of the above equation is determined priori, and cannot be designed. So we will pay our attention to the design of $X_d(z)$ and $Y_d(z)$, which make the left hand side of eqn.(31) be smaller.

Define the cost function J_d as follow:

$$J_d = \sum_{i=0}^{d-2} |X_i| + \sum_{i=0}^{d-1} |Y_i|. \quad (32)$$

Then,

$$\begin{aligned} & |\left[\, z^{d-1} - X_d(z) \;\; Y_d(z) \,\right]| \\ & \le |z^{d-1}| + \sum_{i=0}^{d-2} |X_i z^i| + \sum_{i=0}^{d-1} |Y_i z^i| \\ & \le J_d + 1 \end{aligned} \quad (33)$$

holds on the unit circle.

Therefore, the closed-loop stability may be confirmed by solving the following optimization problem:

$$\min\ J_d = \sum_{i=0}^{d-2} |X_i| + \sum_{i=0}^{d-1} |Y_i| \quad (34)$$

$$\text{s.t.}\ \left[\, \boldsymbol{X}_d \ \boldsymbol{Y}_d \,\right] \begin{bmatrix} I_m & 0 \\ \boldsymbol{T}_{d-2}(A,b,c) & \boldsymbol{O}_{d-1}(c,A) \end{bmatrix} = \left[\, fA^{d-2}b \ \cdots \ fb \ fA^{d-1} \,\right], \quad (35)$$

and the above optimization problem can be solved by using Linear Programming technique.

5. NUMERICAL EXAMPLES

Consider the following discrete-time transfer function:

$$G_0(z) = \frac{z + 1.1}{z^2 - 0.4z - 0.05}.$$

A mathematical model for the above transfer function is assumed as follow:

$$G(z) = \frac{1.1z + 1.2}{z^2 + 0.2z - 0.48}.$$

A pulse train function is used as a reference input. Fig.3 shows the output and reference signal of the simulation result when $d = 2$, which means that there is no degree of freedom in the controller parameters. The output of this case grows unboundedly. Fig.4 and 5 show the output and reference signal of the simulation result when $d = 4$ and $d = 8$ respectively. The outputs of both case are bounded. The transit time is short in Fig.5 rather than Fig.4. Fig.6 show the poles location of each cases. In the case where $d = 2$, there is a pole outside the unit circle. So the output is unbounded in this case.

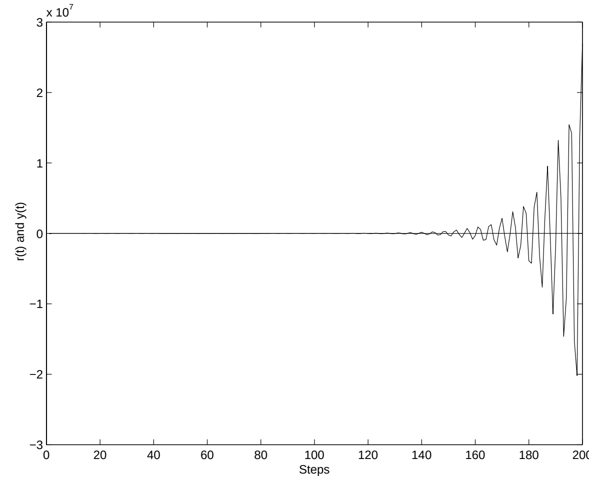

Fig. 3. Simulation of model matching controller with ($d = 2$).

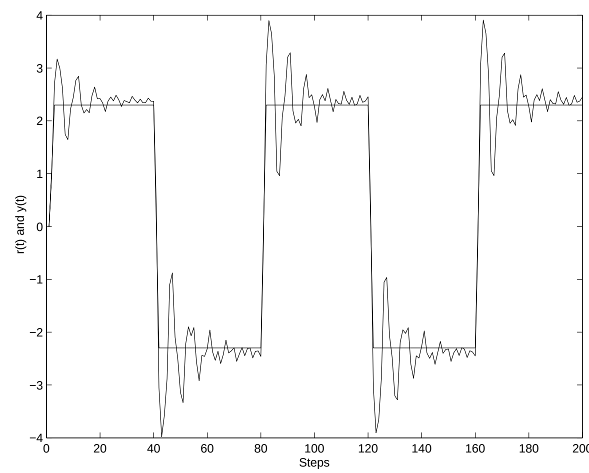

Fig. 4. Simulation result of robust model matching controller ($d = 4$).

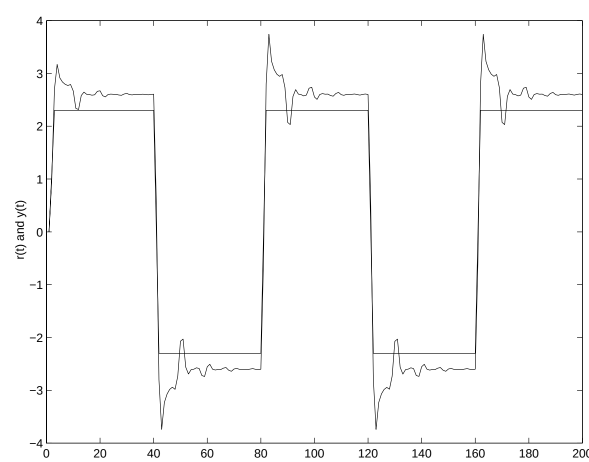

Fig. 5. Simulation result of robust model matching controller ($d = 8$).

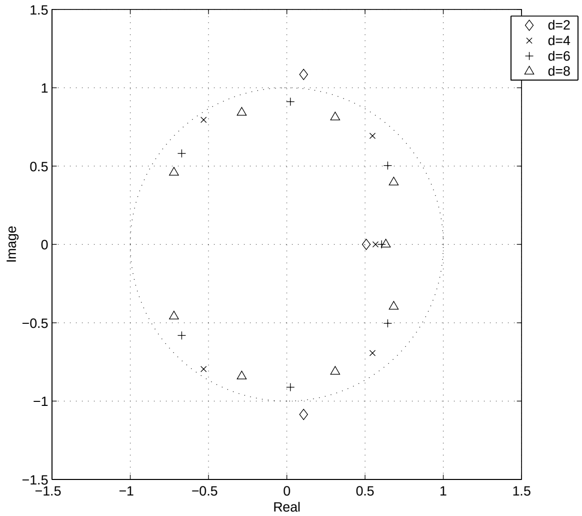

Fig. 6. Pole locations of each simulations.

6. CONCLUSION

In this paper, a parametrization of all stabilizing controller was presented. A feature of the parametrization is that the degree of freedom of stabilizing controller corresponds to the degree of observer polynomial matrix $H(z)$. So by increasing the degree of $H(z)$, more flexibility can be obtained. Using the proposed parametrization, a design of robust pole placement systems for a discrete-time plant with possible unstructured uncertainties was presented. Using a parametrization of all stabilizing controllers, the problem of maximizing robust stability region can be reduced to a simple linear programming problem. This method can be easily extended to the indirect adaptive control.

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