

B-SPLINES AND CONTROL THEORY

Hiroyuki Kano * **Magnus Egerstedt** **,1 **Hiroaki Nakata** *
Clyde F. Martin ***

* *Department of Information Sciences, Tokyo Denki University,
Saitama 350-0394, Japan*

** *Department of Electrical and Computer Engineering, Georgia
Institute of Technology, Atlanta, GA 30332, USA*

*** *Department of Mathematics and Statistics, Texas Tech
University, Lubbock, TX 79409, USA*

Abstract: In this paper some of the relationships between B-splines and linear control theory is examined. In particular, the controls that produce the B-spline basis is constructed and compared to the basis elements for dynamic splines.

Keywords: Splines, control theory, smoothing, interpolation

1. INTRODUCTION

In this paper the connections between the theory of B-splines and control theoretic or dynamics splines are examined. The theory of B-splines is a well developed area of applied numerical analysis and interpolation theory, and the use of B-splines rivals that of Bezier curves in applicability to computer graphics and approximation theory. (See for example (de Boor, 1968), (de Boor, 1978).) On the other hand, the idea of dynamic splines was first used by Crouch and his colleagues in the determination of aircraft trajectories (Crouch and Jackson, 1991). Quite independently Martin, Egerstedt, and their colleagues began exploiting the properties of controlled linear systems to solve interpolation and approximation problems.

Since the introduction of splines by Shoenberg, (Shoenberg, 1958), (Shoenberg and Whitney, 1953), it has been recognized that they are extremely powerful tools both in application and theory. Many variants have been introduced over the years and this paper is an attempt to show how some of these variations are related.

In (Sun *et al.*, 2000), (Martin *et al.*, 2001) it was recognized that the dynamics splines generalized the classical concepts of splines and that many

applications were easy to formulate and solve using the control theoretic approach. The idea is to find a control that drives a linear, single-input, single-output control system of the form

$$\dot{x} = Ax + bu, \quad y = cx$$

through, in the case of interpolation, a series of way points or close to a series of way points, in the case of smoothing. Here, $x \in \mathbb{R}^n$, $u, y \in \mathbb{R}$, and A, b, c are matrices and vectors of compatible dimensions. When adopting this control theoretic point of view, the goal becomes that of constructing the control directly rather than the actual spline.

Given a set of data of the form

$$\{(t_i, \alpha_i) : i = 1, \dots, N\},$$

where $\alpha_i \in \mathbb{R}$, $i = 1, \dots, N$, and $0 < t_1 < \dots < t_N < T$ for some final time T , we generate two optimal control problems that produce the desired controls. The first problem, the problem of interpolating splines, is as follows.

Problem 1.

$$\min_{u \in L_2[0, T]} \int_0^T u^2(t) dt$$

subject to the constraints

$$y(t_i) = \alpha_i, \quad i = 1, \dots, N.$$

¹ Partially supported by the US Army Research Office, Grant number DAAG 5597-1-0114, and by the Sweden-America Foundation 2000 Research Grant.

As shown in (Sun *et al.*, 2000), this problem can be solved by reducing it to the problem of finding the point of minimal norm on an affine linear variety in the Hilbert space, $L_2[0, T]$.

The second problem, the problem of smoothing or approximation, is formulated as follows.

Problem 2.

$$\min_{u \in L_2[0, T]} \left(\int_0^T u^2(t) dt + \sum_{i=1}^N w_i (y(t_i) - \alpha_i)^2 \right),$$

where the weights satisfy $w_i \geq 0$, $i = 1, \dots, N$.

The main goal of this paper is to understand the extent to which these two problems can be applied to the theory and application of B-splines.

2. A BASIS FOR B-SPLINES

We consider the standard basis for normalized, uniform B-splines. We take two approaches. The first is a modification of the general approach of de Boor (de Boor, 1978) and is inspired by (Takayama and Kano, 1995). The second is a more geometric approach, where we determine the geometric properties of the basis in order to gain an understanding of its relationship to optimal control.

The following recursive algorithm for the computation of the basis elements of the B-splines is taken from (Takayama and Kano, 1995):

Algorithm 1. Let $N_{0,0}(s) = 1$ and compute

$$\begin{cases} N_{0,k}(s) = \frac{1-s}{k} N_{0,k-1}(s) \\ N_{k,k}(s) = \frac{s}{k} N_{k-1,k-1}(s) \\ N_{j,k}(s) = \frac{k-j+s}{k} N_{j-1,k-1}(s) \\ \quad + \frac{1+j-s}{k} N_{j,k-1}(s), \quad j = 1, \dots, k-1. \end{cases}$$

We then define the basis element $B_k(s)$ as

$$B_k(s) = \begin{cases} N_{k-j,k}(s-j) & j \leq s \leq j+1 \\ & j = 0, \dots, k \\ 0 & s \leq 0, k+1 \leq s. \end{cases} \quad (1)$$

The spline function is given by a weighted sum of shifted B-splines for a fixed value of k , i.e. the spline function becomes

$$S_k(t) = \sum_{i=1}^M c_i B_k(t-i+1). \quad (2)$$

For now our primary objects of interest are the basis element $B_k(s)$. We first note that

$$B_k^{(l)}(0) = 0 \quad \text{and} \quad B_k^{(l)}(k+1) = 0$$

Table 1. $N_{j,k}$ for $k = 0, 1, 2, 3$.

$k : j$	0	1	2	3
0	1			
1	$(1-s)$	s		
2	$\frac{(1-s)^2}{2}$	$\frac{1+2s-2s^2}{2}$	$\frac{s^2}{2}$	
3	$\frac{(1-s)^3}{3!}$	$\frac{4-6s^2+3s^3}{3!}$	$\frac{1+3s+3s^2-3s^3}{3!}$	$\frac{s^3}{3!}$

for $l = 0, 1, \dots, k-1$, where $B_k^{(l)}(\cdot)$ denotes the l th derivative.

We furthermore observe that $B_k(s)$ is a piecewise polynomial of degree k and that it is $k-1$ times continuously differentiable. These are of course just the properties that make it a polynomial spline. We are, however, particularly concerned with the characterization of the k th derivative. This function is piecewise constant and if we use a piecewise constant input to the controlled differential equation

$$\frac{d^k}{dt^k} y(t) = u(t)$$

we can generate the function $B_k(t)$. It is tedious to compute the derivatives of the general $B_k(t)$ so in the remainder of this paper we restrict ourselves to the cubic case.

In Table 1 we calculate the first few of the elements, using Algorithm 1.

3. A GEOMETRIC APPROACH

We know that the B-spline, B_3 , should have the property that the $B_3(0) = B_3^{(1)}(0) = B_3^{(2)}(0) = 0$ and $B_3(T) = B_3^{(1)}(T) = B_3^{(2)}(T) = 0$, where the spline is defined on the interval $[0, T]$, in order to ensure that it has two continuous derivatives over the entire real line.

We first observe that

$$B_3(t) = \frac{a}{3} t^3, \quad 0 \leq t \leq 1$$

$$B_3(t) = \frac{d}{3} (t-T)^3, \quad T-1 \leq t \leq T$$

for some a and d . Now, on the interval $[1, 2]$

$$B_3^{(1)}(t) = b(t-1)^2 + \alpha(t-1) + \gamma$$

and in order for $B_3^{(1)}(t)$ and $B_3^{(2)}(t)$ to be continuous we must have $\gamma = a$ and $\alpha = 2a$ respectively. Thus we have that

$$B_3^{(1)}(t) = b(t-1)^2 + 2a(t-1) + a.$$

We can show in a similar fashion that on the interval $[T-2, T-1]$ we must have

$$B_3^{(1)}(t) = c(t-T+1)^2 - 2d(t-T+1) + d.$$

We now have four free parameters a, b, c, d that need to be determined.

Let $T = 4$. In order to achieve continuity of the first and second derivatives at $t = 2$ we must have

$$B_3^{(1)}(2) = b + 2a + a = c + 2d + d, \quad (3)$$

$$B_3^{(2)}(2) = 2b + 2a = -2c - 2d. \quad (4)$$

Thus we have used two of the degrees of freedom. We now integrate $B_3^{(1)}(t)$ to obtain $B_3(t)$, and it can be shown that

$$B_3(t) = \int_0^t B_3^{(1)}(s)ds = \frac{1}{3}((7a + b + c + 7d) + d(t-4)^3) \quad 3 \leq t \leq 4.$$

Now, in order for $B_3(4) = 0$ we must have

$$7a + b + c + 7d = 0. \quad (5)$$

Solving equations (3)-(5) in four unknowns we have $b = -3a$, $c = 3a$, and $d = -a$. Thus we may use a as the free parameter to obtain

$$B_3(t) = \begin{cases} a\frac{1}{3}t^3 & 0 \leq t \leq 1 \\ a(\frac{1}{3} - (t-1)^3 + (t-1)^2 + (t-1)) & 1 \leq t \leq 2 \\ a(\frac{1}{3} + (t-3)^3 + (t-3)^2 - (t-3)) & 2 \leq t \leq 3 \\ a\frac{-1}{3}(t-4)^3 & 3 \leq t \leq 4. \end{cases} \quad (6)$$

From this we see that a is just a scaling parameter and the continuity of the derivatives can easily be checked.

Now that the parameters have been chosen we can evaluate the third derivative to obtain

$$B_3^{(3)}(t) = \begin{cases} 2a & 0 \leq t < 1 \\ -6a & 1 \leq t < 2 \\ 6a & 2 \leq t < 3 \\ -2a & 3 \leq t < 4. \end{cases} \quad (7)$$

Letting $u = B_3^{(3)}(t)$ gives us that the spline function can be uniquely generated by the control system

$$\frac{d^3}{dt^3}x = u, \quad x(0) = \dot{x}(0) = \ddot{x}(0) = 0,$$

for a given choice of a .

4. AN OPTIMAL CONTROL APPROACH

A natural question to ask is if the basis element for the B-splines $B_k(t)$ is optimal with respect to some standard optimal control law in the same sense that interpolating and smoothing splines are optimal. Because of the initial and terminal conditions care must be taken in the formulation of the optimization problem. Here we continue to restrict our attention to cubic case for ease of computation. It is natural to use the system

$$\frac{d^3}{dt^3}x = u$$

because in this case we can prescribe the correct boundary values in a natural manner as $x(0) = \dot{x}(0) = \ddot{x}(0) = 0$ and $x(4) = \dot{x}(4) = \ddot{x}(4) = 0$.

We are thus asking for a control that is piecewise constant to generate the B-spline. From the proceeding work we see that $B_3^{(3)}(t)$ in (7) is the desired control. Now in the space of B-splines with nodes at the integers, the B-spline from the previous section is certainly optimal with respect to some optimal control law due to its uniqueness. However, a reasonable question to ask is if it is the solution to the following problem.

Problem 3.

$$\min_{u \in L_\infty[0,4]} \|u\|_{L_\infty}$$

subject to the constraints $\frac{d^3}{dt^3}x = u$, $x(0) = \dot{x}(0) = \ddot{x}(0) = 0$ and $x(4) = \dot{x}(4) = \ddot{x}(4) = 0$.

In other words, are the nodes forced by some choice of the optimal control law? Surprisingly the answer is no. There is a bang-bang control law that does better than the uniform B-spline, as we will see in what follows.

4.1 Dual Optimization

If we assume that the B-spline passes through the point ξ at time $t = 2$, then the augmented optimization constraints become

$$\ddot{x}(4) = \int_0^4 u(t)dt = 0$$

$$\dot{x}(4) = \int_0^4 \int_0^t u(s)dsdt = \int_0^4 (4-t)u(t)dt = 0$$

$$x(4) = \int_0^4 \int_0^t \int_0^s u(r)drdsdt = \frac{1}{2} \int_0^4 (4-t)^2 u(t)dt = 0$$

$$x(2) = \frac{1}{2} \int_0^4 (2-t)_2^2 u(t)dt = \xi,$$

where $(2-t)_2^2 = (2-t)^2$ if $t \leq 2$ and 0 otherwise. These constraints can in turn be rewritten, adopting an inner product notation, as

$$\langle 1, u \rangle = 0, \quad \langle 4-t, u \rangle = 0, \quad \langle (4-t)^2, u \rangle = 0$$

$$\langle (2-t)_2^2, u \rangle = 2\xi,$$

where the inner product is taken between elements in $L_\infty[0,4]$ and $L_1[0,4]$, which is the dual space of $L_\infty[0,4]$.

Now, in (Luenberger, 1969) the following standard theorem in dual optimization can be found:

Theorem 1. Let X be a Banach space and let X^* be the dual of X . Given $y_i \in X$, $i = 1, \dots, p$, suppose that $D = \{x^* \in X^* \mid \langle y_i, x^* \rangle = c_i, i = 1, \dots, p\}$ is nonempty. Then

$$\min_{x^* \in D} \|x^*\| = \max_{\|Y a\| \leq 1} c^T a,$$

where $c = (c_1, \dots, c_p)^T$ and $Y a = y_1 a_1 + \dots + y_p a_p$. Furthermore, the optimal \hat{a} and \hat{x}^* satisfy

$$\langle Y\hat{a}, \hat{x}^* \rangle = \|Y\hat{a}\| \cdot \|\hat{x}^*\|.$$

By applying Theorem 1 to our problem, the dual maximization problem becomes

$$\max_{\|Ya\|_{L_1} \leq 1} \xi a_4,$$

where $a = (a_1, a_2, a_3, a_4)^T$ and

$$Ya = a_1 + a_2(4-t) + a_3(4-t)^2 + a_4(2-t)_2^2.$$

If $a^* \in L_1[0, 4]$ solves the dual problem then the optimal u^* has to satisfy

$$\langle Ya^*, u^* \rangle = \|Ya^*\|_{L_1} \|u^*\|_{L_\infty}.$$

This directly gives that $|u^*|$ has to be constant on the entire interval and that it only changes sign when Ya changes sign. It is thus a bang-bang controller that solves the problem.

It is not difficult to see that Ya and hence u^* changes sign exactly three times in the interval $(0, 4)$, and thus we assume

$$u^*(t) = \begin{cases} U & 0 \leq t < t_1 \\ -U & t_1 \leq t < t_2 \\ U & t_2 \leq t < t_3 \\ -U & t_3 \leq t < 4 \end{cases} \quad (8)$$

with $0 < t_1 < t_2 < t_3 < 4$. Then it can be shown that the constraints $\ddot{x}(4) = \dot{x}(4) = x(4) = 0$ are expressed as

$$\begin{aligned} r_1 - r_2 + r_3 &= 2, & r_1^2 - r_2^2 + r_3^2 &= 8 \\ r_1^3 - r_2^3 + r_3^3 &= 32, \end{aligned}$$

where $r_1 = 4 - t_1, r_2 = 4 - t_2, r_3 = 4 - t_3$. Solving this system of algebraic equations with $4 > r_1 > r_2 > r_3 > 0$ yields the unique solution

$$r_1 = 2 + \sqrt{2}, \quad r_2 = 2, \quad r_3 = 2 - \sqrt{2},$$

and we obtain optimal switching times as

$$t_1 = 2 - \sqrt{2}, \quad t_2 = 2, \quad t_3 = 2 + \sqrt{2}. \quad (9)$$

On the other hand, the value of U is obtained from the constraint $x(2) = \xi$ with $\xi = 2/3$ as $U = (2 + \sqrt{2})/2$.

Finally, the optimal solution $x(t)$ is obtained by

$$\frac{6}{U}x(t) = \begin{cases} t^3 & 0 \leq t < t_1 \\ t^3 - 2(t - t_1)^3 & t_1 \leq t < 2 \\ (4 - t)^3 - 2(t_3 - t)^3 & 2 \leq t < t_3 \\ (4 - t)^3 & t_3 \leq t < 4. \end{cases}$$

In Figure 1, the solution $x(t)$ is depicted together with the B-spline $B_3(t)$ (dotted line).

4.2 Bang-Bang Control

There is another way in which to approach this problem. We can assume that there is a bang-bang control law and simply ask if the nodes are forced.

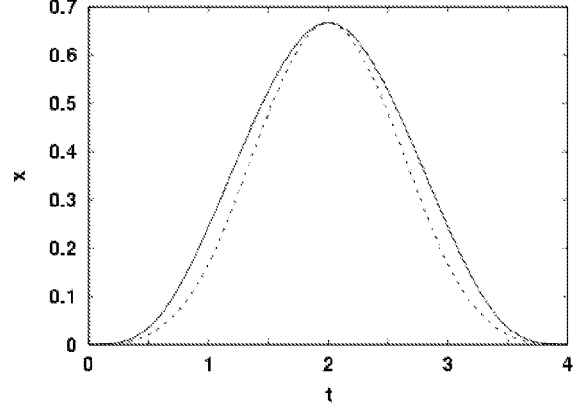


Fig. 1. The optimal bang-bang solution.

Let

$$u(t) = \begin{cases} 1 & 0 \leq t < t_1 \\ -1 & t_1 \leq t < t_2 \\ 1 & t_2 \leq t < t_3 \\ -1 & t_3 \leq t < 4. \end{cases} \quad (10)$$

We only assume that $0 \leq t_1 \leq t_2 \leq t_3 \leq 4$. Using

$$\begin{aligned} \ddot{x}(t) &= \int_0^t u(s)ds, & \dot{x}(t) &= \int_0^t (t-s)u(s)ds \\ x(t) &= \frac{1}{2} \int_0^t (t-s)^2 u(s)ds, \end{aligned}$$

we can show that, in order to have continuities of $\ddot{x}(t), \dot{x}(t), x(t)$ at $t = t_2$, we must have

$$\begin{aligned} a - b &= d - c \\ a^2 + 2ab - b^2 &= -d^2 - 2cd + c^2 \\ a^3 + 3a^2b + 3ab^2 - b^3 &= d^3 + 3d^2c + 3dc^2 - c^3, \end{aligned}$$

where $a = t_1, b = t_2 - t_1, c = t_3 - t_2, d = 4 - t_3$. Solving these equations together with a fourth equation $a + b + c + d = 4$ gives a positive solution $c = b = \sqrt{2}, a = d = 2 - \sqrt{2}$. Thus we get

$$t_1 = 2 - \sqrt{2}, \quad t_2 = 2, \quad t_3 = 2 + \sqrt{2},$$

which is consistent with the solution to the optimal control problem in the previous subsection.

5. THE CONSTRUCTION OF OPTIMAL SPLINES USING B-SPLINES

As we saw in the preceding section the basis elements for uniform B-splines are not optimal in any usual sense. However we can find in the class of all B-splines optimal choices. We will construct two types of splines in analogy with dynamic splines, interpolating and approximating.

Let a data set D be given in \mathbb{R} as

$$D = \{\alpha_i \in \mathbb{R} : i = 1, \dots, N\}.$$

Consider the system

$$\frac{d^k}{dt^k} x(t) = u(t)$$

and a restricted set of controls

$$C = \{u(t) : u(t) = \sum_{i=1}^M \tau_i \frac{d^k}{dt^k} B_k(t-i+1), \tau_i \in \mathbb{R}\}.$$

We now choose the two cost functions from the theory of dynamic splines, and pose two related problems:

Problem 4. [Interpolation]

$$\min_{u \in C} J(u), \quad J(u) = \int_{-\infty}^{\infty} u^2(t) dt \quad (11)$$

subject to the constraints $x(t_i) = \alpha_i, i = 1, \dots, N$.

Problem 5. [Approximation]

$$\min_{u \in C} J(u) = \int_{-\infty}^{\infty} u^2(t) dt + \sum_{i=1}^N w_i (x(t_i) - \alpha_i)^2. \quad (12)$$

We can integrate over the entire real line since the B-splines and all of their derivatives vanish outside of a compact set and since the control is allowed to only be a finite sum. Now these problems are both finite dimensional because the space of controls is finite dimensional. It should be noted that they differ from Problem 1 and Problem 2 only in the space of controls and that the number of basis elements is not necessarily the same as the number of data points. This is different that Problem 1 and Problem 2. There the number and the form of the basis elements is determined by the number of data points. Since in this case we have chosen a basis that constraint is lifted. For the cost function in (11) we have

$$J(\tau) = \int_{-\infty}^{\infty} \left(\sum_{i=1}^M \tau_i \frac{d^k}{dt^k} B_k(t-i+1) \right)^2 dt.$$

In the case of $k = 3$, we can show that

$$J(\tau) = \tau^T G \tau,$$

where $\tau = (\tau_1, \tau_2, \dots, \tau_M)^T$, and G is the gram-mian whose ij -element g_{ij} is given by $g_{ij} = \hat{g}_{|i-j|}$ with $\hat{g}_0 = 20$, $\hat{g}_1 = -15$, $\hat{g}_2 = 6$, $\hat{g}_3 = -1$, and $\hat{g}_i = 0$ $i \geq 4$. The matrix G is positive definite since the basis elements are independent functions.

We now calculate the constraints as functions of τ . We have after integrating that

$$x(t) = \sum_{i=1}^M \tau_i B_3(t-i+1)$$

and hence the constraint is given by

$$x(t_j) = \sum_{i=1}^M \tau_i B_3(t_j-i+1) = \alpha_j, \quad j = 1, 2, \dots, N.$$

From the structure of B_3 we see that $B_3(t_j-i+1) \neq 0$ if and only if $i-1 < t_j < i+3$. Let

B denote the matrix such that $B\tau = \alpha$ where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)^T$. Problem 4 then reduces to the following

$$\min_{\tau \in \mathbb{R}^M} \tau^T G \tau, \quad \text{subject to } B\tau = \alpha. \quad (13)$$

Now if $B\tau = \alpha$ is consistent and the matrix B is of row full rank, then the optimal solution τ^* is given explicitly as

$$\tau^* = G^{-1} B^T (B G^{-1} B^T)^{-1} \alpha. \quad (14)$$

Problem 5, the problem of approximation, can be rewritten in a similar manner as

$$J(\tau) = \tau^T G \tau + (B\tau - \alpha)^T W (B\tau - \alpha)$$

where W is the diagonal matrix with the weights w_i on the diagonal. Noting that $G + B^T W B$ is positive-definite, the optimal solution $\hat{\tau}^*$ to this minimization problem is given as

$$\hat{\tau}^* = (G + B^T W B)^{-1} B^T W \alpha. \quad (15)$$

It can be shown that if B is of row full rank then as $w_i \rightarrow +\infty \forall i$, $\hat{\tau}^*$ converges to the solution τ^* for interpolation problem given in (14).

6. CONTROL POINTS, POLYGONS AND AN EXAMPLE OF OPTIMAL CONSTRUCTION

The concepts of control points and control polygons are essential to the application of B-splines and for that matter Bezier curves. A spline, $s(t)$, of degree k in \mathbb{R}^n is constructed using the basis of B-splines $B_k(t)$ as

$$s(t) = \sum_{i=1}^M \tau_i B_k(t-i+1) \quad (16)$$

and the set of points $\{\tau_i \in \mathbb{R}^n : i = 1, \dots, M\}$ is the set of control points. The control polygon is the polygonal line connecting the control points. The control points determine the shape of the spline function.

In the preceding section we constructed optimal weights in the scalar case. By repeating this procedure or by using a more complicated set of dynamics we can produce optimal vector valued weights. Thus given a set of data points

$$\{\alpha_i \in \mathbb{R}^n : i = 1, \dots, N\}$$

we can produce a set of control points optimal for this set of data, either as interpolation or as approximation. To see how this might be done consider a real curve $p(t) \in \mathbb{R}^n$. Our goal is to reproduce this curve using optimal B-splines. If we are precise in our description of the curve choose N points that lie on the curve,

$$D = \{p(t_i) : i = 1, \dots, N\}.$$

We will use these points as data to construct the control points. The designer must choose these points and he must decide on the degree of the spline that he wants to construct. We assume that we are constructing a spline of degree k . Then as in the preceding section let

$$C = \{u(t) : u(t) = \sum_{i=1}^M \tau_i \frac{d^k}{dt^k} B_k(t-i+1), \tau_i \in \mathbb{R}^n\}.$$

The set C consists of all allowable controls that we use in the construction of the optimal spline.

One of the important applications of splines is in the design of letter fonts, and we show the results as font patterns generated from the curves in \mathbb{R}^3 .

As $p(t)$, we take a cubic spline in \mathbb{R}^3 given by

$$p(t) = \sum_{i=1}^M p_i B_3(t-i+1)$$

where $p_i \in \mathbb{R}^3$ are given control points. Figure 2 shows a Japanese alphabet pronounced ‘ru’ generated from $p(t)$: This is obtained by computing the cross sectional area between a virtual writing device (a cone) moving along $p(t)$ in space $o-xyz$ and a virtual writing plane $o-xy$. In this example, a set of 20 control points counting multiplicities is used and is shown by ‘squares’ together with the control polygon in xy -plane. The figure on the right is the font pattern obtained in this fashion, and it may be considered as a good model of an actual brush-writing alphabet.

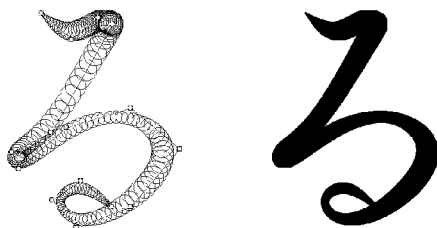


Fig. 2. Japanese character ‘Ru’ generated by cubic spline.

Such a curve or font pattern is then reconstructed using optimal approximation by cubic B-splines (i.e. $k = 3$ in (16)). In order to deal with curves in \mathbb{R}^3 , we apply the method developed in Section 5 for scalar case to each of the three elements independently. $p(t)$ is sampled at ten equally spaced data points $\{3, 5, 7, 9, 11, 13, 15, 17, 19, 20\}$.

Figure 3 show the results for various weighting matrices W . We see that the original font pattern is recovered more accurately as the weights increase. We also verified that by increasing the weights further the pattern approaches the interpolation result, It might be worth noting that there are various degrees of cursive fonts in Japanese brush writing, which may be modeled



Fig. 3. Approximating construction with $W = I$, $W = 10I$, and $W = 100I$ (from left).

by a suitable weight adjustment in the optimal approximation as described here.

7. CONCLUSIONS

In this paper we investigate the connections between B-splines and linear control theory. We show how the B-spline basis functions can be obtained by driving a third order control system with a piecewise constant input. However, we also show that the B-splines are in fact suboptimal with respect to an infinity-norm minimization, and that the solution to this problem is of the bang-bang type. We show that an optimal set control points can be constructed within the space of B-splines. Finally an example is developed to demonstrate the efficacy of this construction.

8. REFERENCES

- Crouch, P. and J. Jackson (1991). Dynamic interpolation and application to flight control. *J. Guidance, Control and Dynamics* **14**, 814–822.
- de Boor, C. (1968). On unifrom approximation by splines. *J. Approximation Theory* **1**, 219–235.
- de Boor, C. (1978). *A practical guide to splines*. Springer-Verlag, New York.
- Luenberger, D. (1969). *Optimization by Vector Space Methods*. John Wiley and Sons, Inc.. New York.
- Martin, C., S. Sun and M. Egerstedt (2001). Optimal control, statistics and path planning. *Math. and Computer Modelling* **33**, 237–253.
- Shoenberg, I. (1958). Spline functions, convex curves, and mechanical quadrature. *Bull Amer. Math. Soc.* **64**, 352–357.
- Shoenberg, I. and A. Whitney (1953). On the poyla frequency function iii. the postivity of translation dterminants with an application to the interpolation problem for spline curves. *Trans. Amer. Math. Soc.* **74**, 246–259.
- Sun, S., M. Egerstedt and C. Martin (2000). Control theoretic smoothing splines. *IEEE Tran. Auto. Cont.* **45**, 2271–2279.
- Takayama, K. and H. Kano (1995). A new approach to synthesizing free motions of robotic manipulators based on the concept of unit motion. *IEEE Tran. SMC* **25**, 453–463.