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OPTIMAL TRACKING FOR AUTOMOTIVE DRY CLUTCH ENGAGEMENT

Franco Garofalo *, **Luigi Glielmo** **, **Luigi Iannelli** * and **Francesco Vasca** **

* *Università di Napoli Federico II - Dipartimento di Informatica e Sistemistica - via Claudio, 21 - 80125 Napoli, Italy - e-mail: franco.garofalo, luiannel@unina.it*
** *Università del Sannio - Facoltà di Ingegneria - C.so Garibaldi, 107 - 82100 Benevento, Italy - e-mail: glielmo, vasca@unisannio.it*

Abstract: Based on a state space dynamic model of a typical automotive driveline, a new control technique for the dry clutch engagement process is proposed. The feedback controller is designed following an optimal control approach by using the crankshaft speed and the clutch disk speed as state variables: a tracking problem is formulated and solved by using the engine torque and the clutch torque as control variables. The controller guarantees fast engagement, minimum slipping losses and comfortable lock-up. The critical standing start operating conditions are considered. Numerical results show the good performance obtained with the proposed controller. *Copyright ©2002 IFAC*

Keywords: Automotive Control, Transmission Systems, Optimal Control, Tracking Systems

1. INTRODUCTION

Recently, the engagement control of automotive dry clutches is becoming more and more important, due to the increasing use of Automated Manual Transmissions (Link *et al.*, 2001), mainly because it represents an inexpensive add-on solution on classical (in European and Latin countries) manual transmission systems. Several problems related to AMT have been investigated in the literature, such as gear-shift selection, AMT actuators, parameter dependence, clutch engagement strategies in hybrid vehicles (Fredriksson and Egardt, 2000; Bader, 1990; Lee *et al.*, 1998). In particular, the engagement of dry clutches is a very important process both to ensure small facing wear and good powertrain performance. The engagement must be controlled in order to satisfy different and sometimes conflicting objectives: small friction losses, minimum time needed for the engagement, preservation of driver comfort. These goals must be reached by applying a suitable normal force to the clutch driven disk and by suitably regulating the engine torque during the engagement phase.

In this framework, one of the most critical operating conditions for an AMT engagement phase is the start-up. In fact, during this phase the clutch disk starts from stop and, in order to reduce fuel consumption and emissions, it is desired to maintain the engine speed as closer as possible to the idle speed. Moreover, the engagement phase during the vehicle launch is very critical since driveline oscillations can be simply generated thus drastically influencing the driver comfort.

To this aim, some model-based control strategies have been recently proposed in the literature (Szadkowski and Morford, 1992; Ercole *et al.*, 1999; Slicker and Loh, 1996; Glielmo and Vasca, 2000; Garofalo *et al.*, 2001; Bemporad *et al.*, 2001; Zanasi *et al.*, 2001).

In this work a finite horizon Linear Quadratic tracking controller is proposed as an effective solution for the dry clutch engagement control problem. Numerical results, carried out by a Simulink/Stateflow simulation scheme and a realistic set of parameters, show the good performance of the tracking system, also through comparison with controllers previously proposed in the literature.

2. DRIVELINE DYNAMIC MODEL

2.1 *Sixth order model*

A typical scheme of the driveline is reported in Figure 1.

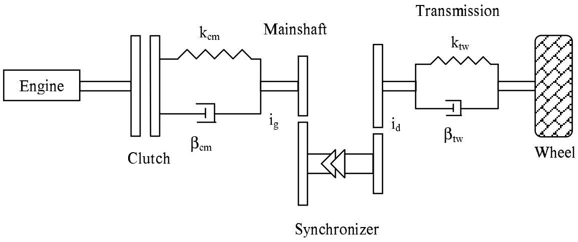

Fig. 1. Driveline scheme.

The torque produced by the internal combustion engine is transmitted to the driveline by means of the clutch. During the gear up-shift and down-shift the force applied to the clutch disk separates the engine disk and the clutch disk and the engine torque is no more transmitted to the powertrain until a new engagement starts. Elastic and friction elements in Figure 1 are used to model the mechanical behavior of the driveline. The whole dynamic model can be obtained by applying the equilibrium torque condition at the different nodes of the structure presented in Figure 1. The dynamic equation of the crankshaft speed (also called engine speed) ω_e can be written as

$$J_e\dot{\omega}_e = T_e - \beta_e\omega_e - T_{cl}, \tag{1}$$

where the subscripts ‘e’ and ‘c’ are used for the engine and the clutch, respectively, J_e is the engine inertia, T_e is the engine torque, β_e is the crankshaft friction coefficient, T_{cl} is the torque transmitted by the clutch (acting as a load torque for the engine dynamic subsystem) and, for notational simplicity, the explicit time dependence has been omitted.

Analogously, the dynamic equation of the clutch disk speed ω_c can be written as

$$J_c\dot{\omega}_c = T_{cl} - k_{cm}(\theta_c - \theta_m) - \beta_{cm}(\omega_c - \omega_m), \tag{2}$$

where the subscript ‘m’ indicates the mainshaft and the variables θ are used for the angular positions of the shafts.

By applying the torque equilibrium condition at the mainshaft one can write

$$\begin{aligned} J_{eq}(i_g,i_d)\dot{\omega}_m &= k_{cm}(\theta_c - \theta_m) + \beta_{cm}(\omega_c - \omega_m) \\ &\quad - \frac{1}{i_g i_d}\left[k_{tw}\left(\frac{\theta_m}{i_g i_d} - \theta_w\right)\right. \\ &\quad \left. + \beta_{tw}\left(\frac{\omega_m}{i_g i_d} - \omega_w\right)\right] \end{aligned} \tag{3}$$

where $J_{eq}(i_g,i_d) = J_m + \frac{1}{i_g^2}(J_{s1} + J_{s2} + \frac{J_t}{i_d^2})$, the subscripts ‘s’ and ‘t’ are used for the synchronizer and the transmission shafts, respectively, J_{s1} and J_{s2} are the inertia of the two disks connected to the synchronizer, i_g is the gear ratio, i_d is the differential gear ratio, i.e. $\omega_m = i_g i_d \omega_w$. Finally, the following torque equilibrium condition holds at the wheel

$$\begin{aligned} J_w\dot{\omega}_w &= k_{tw}\left(\frac{\theta_m}{i_g i_d} - \theta_w\right) \\ &\quad + \beta_{tw}\left(\frac{\omega_m}{i_g i_d} - \omega_w\right) - \beta_w\omega_w - T_{\mathrm{load}} \end{aligned} \tag{4}$$

where the subscript ‘w’ is used for the wheel, J_w is the inertia which takes into account the wheels and the remaining parts of the vehicle, β_w is the friction coefficient and T_{load} is the load torque.

2.2 *Piecewise LTI model*

When the clutch is engaged the engine speed ω_e and the clutch disk speed ω_c are equal since elastic forces lock the clutch disk to the crankshaft. In order to model this situation one can add (1) and (2) thus obtaining

$$\begin{aligned} (J_e + J_c)\dot{\omega}_c &= T_e - \beta_e\omega_c - k_{cm}(\theta_c - \theta_m) \\ &\quad - \beta_{cm}(\omega_c - \omega_m), \end{aligned} \tag{5}$$

since engine and clutch speeds are equal. The switch from the *slipping model* (1)-(4) to the *engaged model* (5), (3), (4) is determined by the condition $\omega_e = \omega_c$ and the constraint that the clutch torque is smaller than the static friction torque, so that further slipping is avoided (we’ll assume that no slipping is possible after the engagement). In other words, the powertrain can be modeled as a piecewise linear time-invariant model whose configurations correspond to the slipping phase and the engaged phase. In order to represent the whole system in a compact form, a switching variable can be defined, similarly to the procedure classically adopted in order to model switching power electronics converters (Kassakian *et al.*, 1991). By introducing the switching variable d, equal to 1 when the system is in the slipping phase and 0 otherwise, the powertrain model (1)-(4), (5) can be written as

$$\begin{aligned} \dot{z} &= [A_{sl}(i_g,i_d)\cdot d + A_{eng}(i_g,i_d)\cdot(1-d)]z \\ &\quad + [B_{sl}\cdot d + B_{eng}\cdot(1-d)]v + \Gamma T_{\mathrm{load}}, \end{aligned} \tag{6}$$

where $z = \left(\omega_e, \omega_c, \theta_c - \theta_m, \omega_m, \frac{\theta_m}{i_g i_d} - \theta_w, \omega_w\right)^T$, $v = (T_e, T_{cl})^T$ the subscripts ‘sl’ and ‘eng’ indicate the slipping and engaged system matrices, respectively, and the matrices can be simply deduced from (1)–(4), (5). Equation (6) is a compact model of the driveline and can be interpreted as an hybrid model. The commutation from the slipping model to the engaged one is obtained at each dry clutch lock-up, which is a state-dependent condition, i.e. $\omega_e = \omega_c$. The opposite commutation occurs whenever the driver asks for an up-shift or down-shift of the gear, which can be considered as an external event.

2.3 *Second order models*

The possibility to use a simplified model is a very important step for the clutch controller design. To this aim, let us assume

$$\omega_c = \omega_m = i_g i_d \omega_w \tag{7}$$

(it is essentially a singular perturbation model reduction by assuming as "fast" variables the differences of the angular displacements). By substituting (7) in (2)-(4) and by adding the resulting equations, the whole driveline from the clutch disk to the wheels can be approximated with the first order system

$$J_v(i_g,i_d)\dot{\omega}_c = T_{cl} - \beta_v(i_g,i_d)\omega_c - T_L \tag{8}$$

where $J_v = J_c + J_{eq}(i_g,i_d) + \frac{J_w}{i_g^2 i_d^2}$, $\beta_v(i_g,i_d) = \frac{\beta_w}{i_g^2 i_d^2}$, $T_L = \frac{T_{\text{load}}}{i_g i_d}$. Equation (1) models the rotation of the crankshaft, whereas (8) models the rotation of the clutch disk, after assuming a rigid driveline (see (7)). Though equations (1), (8) do not model in detail the whole powertrain, they capture the main dynamics of the system under investigation and are simple enough to design a controller through analytical procedures.

3. CONTROLLER DESIGN

3.1 *Control objectives*

As already mentioned, the controller must satisfy different objectives (Szadkowski and Morford, 1992). The fundamental constraint of the clutch engagement process is the so called no-kill conditions, i.e. one must avoid the engine stall. This condition can be modeled by imposing that

$$\omega_e(t) \geq \omega_e^{min}, \quad \forall t. \tag{9}$$

A further important condition to be satisfied during the engagement is the so called no-lurch condition. A non-smooth engagement process determines a mechanical oscillation of the powertrain which should be avoided in order to preserve the driver comfort. It can be shown that the driveline oscillations depend on the time derivative of the slip speed $\omega_{sl} = \omega_e - \omega_c$ at the engagement. In fact, by using (1)-(4) and assuming that the engine torque and the load torque are continuous at the lock-up time instant, say $\bar{t}$, the discontinuity of the clutch speed acceleration at $\bar{t}$ can be written as (see the Appendix Afor the algebraic manipulations)

$$\dot{\omega}_c(\bar{t}^+) - \dot{\omega}_c(\bar{t}^-) = \frac{J_e}{J_e + J_c}\dot{\omega}_{sl}(\bar{t}^-). \tag{10}$$

By using similar algebraic manipulations, it can be easily shown that for the second order model (1), (8), the discontinuity of the clutch acceleration at lock-up can be written so as (10), after replacing J_c with J_v.

In order to obtain a smooth engagement process, the controller should try to maintain as small as possible the discontinuity of the clutch acceleration at lock-up, otherwise undesired oscillations can be excited.

Finally, the energy dissipated during the engagement

$$E_d = \int_0^{\bar{t}} \omega_{sl}(t) \cdot T_{cl}(t)dt$$

should be maintained as well as possible.

3.2 *LQ controller*

In order to maintain as simpler as possible the controller design, the optimal controller will be designed by considering the driveline second order model (1), (8). The effectiveness of the controller will be then checked through numerical experiments on the hybrid sixth order model (6).

The engagement process must be performed in a finite time, i.e. at a finite time instant $\bar{t}$ it must be $\omega_e(\bar{t}) = \omega_c(\bar{t})$, or, equivalently, $\omega_{sl}(\bar{t}) = 0$. It is possible to satisfy this final condition by considering it as a *state constraint*. To this aim, the dynamic model (1), (8) can be rewritten as:

$$\dot{\omega}_e = -\frac{\beta_e}{J_e}\omega_e - \frac{1}{J_e}T_{cl} + \frac{1}{J_e}T_e, \tag{11}$$

$$\dot{\omega}_{sl} = \left(-\frac{\beta_e}{J_e} + \frac{\beta_v}{J_v}\right)\omega_e - \frac{\beta_v}{J_v}\omega_{sl} - \left(\frac{1}{J_e} + \frac{1}{J_v}\right)T_{cl} + \frac{1}{J_e}T_e + \frac{1}{J_v}T_L. \tag{12}$$

From (11)-(12) one can write

$$\dot{x}_1 = -\frac{\beta_e}{J_e}x_1 - \frac{1}{J_e}x_3 + \frac{1}{J_e}u_1, \tag{13}$$

$$\dot{x}_2 = \left(-\frac{\beta_e}{J_e} + \frac{\beta_v}{J_v}\right)x_1 - \frac{\beta_v}{J_v}x_2 - \left(\frac{1}{J_e} + \frac{1}{J_v}\right)x_3 + \frac{1}{J_e}u_1 + \frac{1}{J_v}T_L, \tag{14}$$

$$\dot{x}_3 = u_2 \tag{15}$$

where $u_1 = T_e$ and $u_2 = \dot{T}_{cl}$. The time derivative of the clutch torque has been considered as an input in order to avoid discontinuity on the clutch torque. The controller is obtained by applying the Linear Quadratic control theory to the system (13)-(15). A tracking controller problem is formulated to achieve an engine speed reference (which satisfies (9)) and a slip speed reference. The objective function to be minimized is

$$V = \frac{1}{2}\int_0^{\bar{t}} \left[(x(t) - \tilde{x}(t))^T Q(x(t) - \tilde{x}(t)) + u(t)^T Ru(t)\right] dt \tag{16}$$

where $\bar{t}$ is an a priori chosen time instant, and Q and R are two matrices that weight the difference between actual state trajectory $x(t)$ and the desired trajectory $\tilde{x}(t)$; the engagement must be forced through the final time constraint $x_2(\bar{t}) = 0$. The control resulting from the application of this technique has the following expression:

$$u^*(t) = \gamma(t)^T x(t) + \rho(t) \qquad (17)$$

where $\gamma(t)$ is a matrix function obtained by the integration of a differential Riccati equation and $\rho(t)$ is a vector function depending on the initial conditions, the value of the load torque (considered as a disturbance to be rejected), and the reference signal $\tilde{x}(t)$. The derivation of the controller law is reported in Appendix B.

In order to show the effectiveness of the controller and to highlight the undesired oscillatory behaviour of the driveline, the sixth order dynamic model has been used in the simulation experiments. Figure 2 shows the

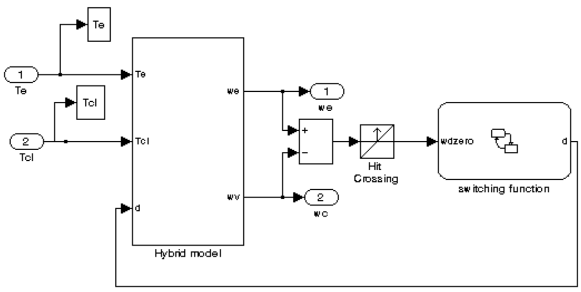

Fig. 2. Simulink scheme implementing the sixth order hybrid driveline model: the Stateflow "switching function" block discriminates between the slipping model and the engaged model.

The reference signals and the weighting matrices Q and R are chosen so that the control objectives presented in the previous section are satisfied.

The following realistic set of parameters for a medium size car have been used to simulate the sixth order model: $J_e = 0.147kgm^2$, $\beta_e = 0.03Nms$, $J_c = 0.004kgm^2$, $k_{cm} = 1500Nm$, $\beta_{cm} = 1Nms$, $J_m = 0.008kgm^2$, $J_{s1} = 0.05kgm^2$, $J_{s2} = 0.015kgm^2$, $J_t = 0.0005kgm^2$, $k_{tw} = 20kNm$, $\beta_{tw} = 250Nms$, $\beta_w = 7.32Nms$ and $J_w = 166kgm^2$. The gear ratio for the five different gear positions can be chosen as $\frac{1}{i_g} = (0.256, 0.35, 0.5, 0.75, 0.95)$, and $i_d = 4$. The load torque is assumed to be constant and equal to $10Nm$. The numerical results obtained for a clutch engagement at standing start with the controller proposed in (Garofalo *et al.*, 2001) are reported in Figure 3 and Figure 4.

The results obtained with the proposed LQ controller are reported in Figure 5 and Figure 6. The numerical results show the reduction in the driveline oscillations after the lock-up and the faster engagement.

The superiority of the LQ control with respect to the classical PI control becomes more evident under more

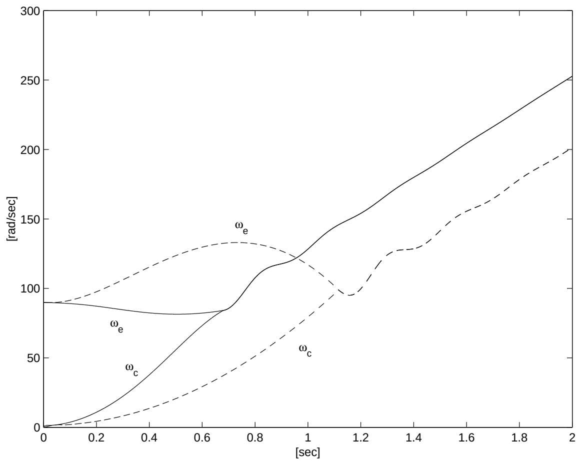

Fig. 3. Engine speed and clutch disk speed during the engagement; solid (*dashed*) lines correspond to the simulation under closed loop (*open loop*) control.

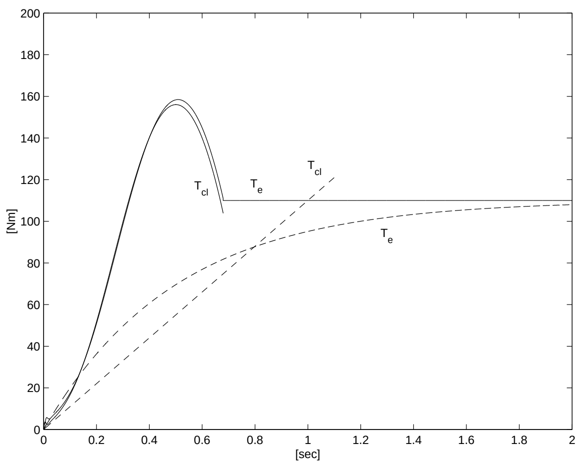

Fig. 4. Engine torque and clutch torque during the engagement; solid (*dashed*) lines correspond to the simulation under closed loop (*open loop*) control.

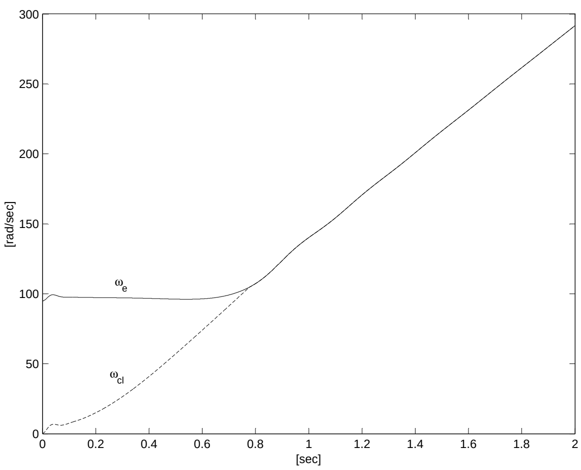

Fig. 5. Speeds under LQ control.

realistic operating conditions such as a toothed wheel sensor for the engine and clutch speed measurements or the dependence of the dynamic friction coefficient

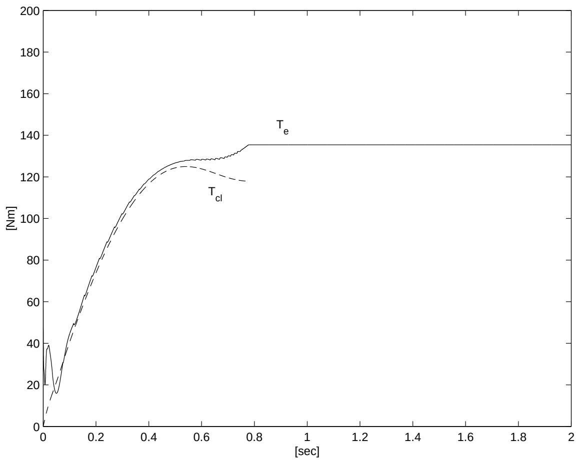

Fig. 6. Torques under LQ control.

on the slip speed. These problems have been considered and a lightly modified LQ controller has been proposed in (Garofalo *et al.*, 2002) in which it is also shown that PI control cannot guarantee the engagement.

4. CONCLUSIONS

The control of the dry clutch engagement process for automotive systems has been considered. The presence of the clutch and, more specifically, its different operating conditions during the automotive cycles (slipping or engaged) makes reasonable the use of a piecewise linear time-invariant model for the description of the driveline dynamics. A sixth order model has been considered in order to detect the driveline oscillations after the lock-up, which drastically influence the driver comfort. A simplified second order model, which still captures the main driveline dynamics during the engagement, has been presented and used to design the proposed LQ controller. Some numerical results have shown the improvement achievable by means of the proposed controller, also through comparison with a classical engagement control strategy.

It seems important to say that more realistic operating conditions (about the measurement of the speeds and about the friction behaviour) are considered in (Garofalo *et al.*, 2002) and other problems such as thermal effects, damper spring nonlinearity or more realistic constraints on the engine torque are object of current investigation by the authors.

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Appendix A. ACCELERATION DISCONTINUITY

In this appendix we show how it is obtained (10). Since after the lock-up $\omega_e = \omega_c$, from (5) one can write:

$$\dot{\omega}_c(\bar{t}^+) - \dot{\omega}_c(\bar{t}^-) = \frac{1}{J_e + J_c}\left[T_e(\bar{t}^+) - \beta_e\omega_c(\bar{t}^+)\right.$$
$$\left. - \underbrace{(k_{cm}(\theta_c - \theta_m) + \beta_{cm}(\omega_c - \omega_m))}_{T_2}\right]$$
$$- \dot{\omega}_c(\bar{t}^-), \qquad \text{(A.1)}$$

where for notational simplicity the dependence on i_g and i_d has been omitted. Since the engine speed is continuous at lock-up one can substitute $\omega_c(\bar{t}^+) = \omega_e(\bar{t}^+)$ in (A.1) with $\omega_e(\bar{t}^-)$ and compute $\omega_e(\bar{t}^-)$ from (1), thus obtaining

$$\dot{\omega}_c(\bar{t}^+) - \dot{\omega}_c(\bar{t}^-) = \frac{1}{J_e + J_c}\left[T_e(\bar{t}^+) + J_e\dot{\omega}_e(\bar{t}^-)\right.$$
$$\left. + T_{cl}(\bar{t}^-) - T_e(\bar{t}^-) - T_2(\bar{t}^+)\right]$$
$$- \dot{\omega}_c(\bar{t}^-) - \frac{J_e}{J_e + J_c}\dot{\omega}_c(\bar{t}^-)$$
$$+ \frac{J_e}{J_e + J_c}\dot{\omega}_c(\bar{t}^-) \qquad \text{(A.2)}$$

Now, by assuming $T_e(\bar{t}^+) = T_e(\bar{t}^-)$, one can write:

$$\dot{\omega}_c(\bar{t}^+) - \dot{\omega}_c(\bar{t}^-) = \frac{J_e}{J_e + J_c}\dot{\omega}_{sl}(\bar{t}^-)$$
$$- \frac{1}{J_e + J_c}\left[J_c\dot{\omega}_c(\bar{t}^-) - T_{cl}(\bar{t}^-)\right.$$
$$\left. + T_2(\bar{t}^+)\right] \qquad \text{(A.3)}$$

By considering (2) and by assuming $T_{cl}(\bar{t}^-) = T_{cl}(\bar{t}^+)$ and $T_2(\bar{t}^+) = T_2(\bar{t}^-)$ one obtains (10).

Appendix B. OPTIMAL CONTROL TRACKING

The system (13)-(15) can be rewritten into the following compact form

$$\dot{x}(t) = Ax(t) + Bu(t) + \Gamma \qquad \text{(B.1)}$$

where we assumed the load torque to be constant (in the vector Γ). Using the Hamiltonian approach and the so called adjoint equation it is simple to show that solving the minimization problem

$$u^*(t) = \arg\min V \qquad \text{(B.2)}$$

with V given by (16), subject to (B.1) with $x(0) = x_0$ and $x_2(\bar{t}) = 0$, corresponds to solving the following set of differential equations (Bryson and Ho, 1975)

$$\begin{cases} \dot{x}(t) = Ax(t) - BR^{-1}B^T\lambda(t) + \Gamma \\ \dot{\lambda}(t) = -Qx(t) - A^T\lambda(t) + Q\tilde{x}(t) \end{cases} \qquad \text{(B.3)}$$

with the set of conditions

$$x(0) = x_0, \quad x_2(\bar{t}) = 0, \quad \lambda_1(\bar{t}) = 0, \quad \lambda_3(\bar{t}) = 0. \qquad \text{(B.4)}$$

Some of the conditions (B.4) are given at the initial time instant and some at the final one $\bar{t}$. Moreover $\lambda_2(\bar{t})$, $x_1(\bar{t})$ and $x_3(\bar{t})$ are not given and the controller must also take into account the presence of the vector Γ in (B.1). Both these problems can be solved by imposing

$$\lambda(t) = P(t)x(t) + m(t)\nu + h(t), \qquad \text{(B.5)}$$
$$\psi = g(t)x(t) + f(t)\nu + k(t) \qquad \text{(B.6)}$$

where $\psi = x_2(\bar{t})$, $\nu = \lambda_2(\bar{t})$, $P(t)$ is the matrix that solves the Riccati equation, $h(t)$ has been introduced in order to compensate for the presence of the disturbance vector Γ and to solve the tracking problem, $m(t)$ is used to solve the problem of the unknown final condition $\lambda_2(\bar{t})$. By computing (B.5)-(B.6) at $\bar{t}$ one obtains that $P(\bar{t})$, $h(\bar{t})$, $f(\bar{t})$ and $k(\bar{t})$ must all be zero and

$$m(\bar{t}) = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T, \quad g(\bar{t}) = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}, \qquad \text{(B.7)}$$

By substituting (B.5)-(B.6) in (B.3), after simple algebraic manipulations one obtains the following three matrix differential equations:

$$-\dot{P}(t) - A^TP(t) - P(t)A + P(t)BR^{-1}B^TP(t) - Q = 0, \qquad \text{(B.8)}$$
$$P(t)BR^{-1}B^Tm(t) - \dot{m}(t) - A^Tm(t) = 0, \qquad \text{(B.9)}$$
$$P(t)BR^{-1}B^Th(t) - P(t)\Gamma - \dot{h}(t) - A^Th(t) + Q\tilde{x} = 0, \qquad \text{(B.10)}$$

which can be solved backward in time from the above terminal conditions. Now, by differentiating (B.6) we have

$$\dot{g}(t)x(t) + g(t)\dot{x}(t) + \dot{f}(t)\nu + \dot{k} = 0, \qquad \text{(B.11)}$$

and, by using (B.3)

$$\left[\dot{g}(t) + g(t)A - g(t)BR^{-1}B^TP(t)\right]x(t) +$$
$$+ \left[\dot{f}(t) - g(t)BR^{-1}B^Tm(t)\right]\nu + \dot{k}(t)$$
$$- g(t)BR^{-1}B^Th(t) + g(t)\Gamma = 0 \qquad \text{(B.12)}$$

that holds for any $x(t)$ and ν. Therefore

$$\dot{g}(t) + g(t)A - g(t)BR^{-1}B^TP(t) = 0, \qquad \text{(B.13)}$$
$$\dot{f}(t) - g(t)BR^{-1}B^Tm(t) = 0, \qquad \text{(B.14)}$$
$$\dot{k}(t) - g(t)BR^{-1}B^Th(t) + g(t)\Gamma = 0, \qquad \text{(B.15)}$$

which allow to determine $g(t)$, $f(t)$ and $k(t)$. From (B.13), (B.9) and the final conditions on $g(t)$ and $m(t)$, one can conclude that $g(t) = m^T(t)$. From (B.6), since ν is a constant one obtains

$$\nu = f^{-1}(0)\left[\psi - g(0)x(0) - k(0)\right]. \qquad \text{(B.16)}$$

Finally the control variable $u^*(t)$ has the following expression,

$$u^*(t) = -R^{-1}B^T\lambda(t)$$
$$= -R^{-1}B^TP(t)x(t) - R^{-1}B^T[m(t)\nu + h(t)], \qquad \text{(B.17)}$$

where $P(t)$, $m(t)$, ν and $h(t)$ can be obtained as described above.