

## SYNTHESIS OF NOMINAL MODELS IN CONTROL PROBLEMS

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Abstract: Statement of problem nominal models synthesis of controlled with changeable properties is considered on the basis of internal models. Method of nominal model synthesis with energy criterion is proposed. Illustrative example for interval object is included. *Copyright © 2002 IFAC*

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### 1. INTRODUCTION

Methods using internal models of both controlled plant and external disturbances are frequently used for increasing of precision and robustness of control systems for objects with varying properties (Tsytkin, 1992; Garsia, 1982). Control performance index, of such control methods depends on ranges of varying parameters and structure of chosen models. Model can reflect properties either controlled plant or closed-loop system as a whole. In the last case nominal model (NM) is considered as reference model, as reflection of system requirements (desired quality coefficients of control system). As a rule synthesis of control laws for objects with varying parameters is based on the model with fix parameters, though the properties of controlled plant are subject to variations with wide range.

The problem of synthesis of nominal model is solved in most cases intuitively, using experience of developer, though selection of nominal model has an influence on parameters of control law and quality of whole system.

In this article the problem of synthesis of nominal model using energy criterion for scheme with compensating control is solved.

### 2. REVIEW OF NOMINAL MODELS METHODS SYNTHESIS

In control systems with internal models nominal model is one of central elements in control algorithm and determines dynamic properties of whole system. Correspondence of nominal model to object, purposes and control tasks, present constraints, technical means of realization, may be carried out under certain properties. So, exact definition of many objects may be represented by the system of non-linear differential equations, but majority of engineering requirements is satisfied by the linear systems not higher than the second order. Following conclusions of theory of reference models for adaptive systems (Gromyko and Sankovskiy, 1974) the order of object model should coincide with the order of nominal model for convergence of adaptation algorithms. For this direction there are no constructive solutions of control with model for adaptive systems. For robust control methods preservation of stability of control at multiple parametric uncertainty of object (Kuntsevich, 2001). For system with signal adaptation or compensation of disturbances it is possible to control the object using nominal or simplified model with desired quality coefficients (Csaki, 1975).

Let's consider basic methods of obtaining simplified or nominal models.

The idea of *comparison principle method* (Voronov, 1985) consist in replacement of some complex model by simpler model using some coefficient so that selected coefficient for simplified model should be not less (not bigger ) than for complete model at whole time interval. This simplified model or system is called the system of comparison using selected coefficient. With reference to dynamic processes constructive solutions are obtained on the basic of Lyapunov functions, as some majorants of dynamic processes or majoring models of comparison. Parameters of this element are calculated rigorously according to singular values of positive definite matrix, the solution of Lyapunov equation. Dependence of Lyapunov equation on arbitrary matrix is used for choice of majoring according to defined degree of stability, and norm of space state vector of complete model is between majorant and minorant. Principle of comparison on the basis of Lyapunov vector functions is used for complex models, which can be decomposed into several simple ones.

*Method of singular disturbances* is based on decomposition - aggregation and belongs to methods with structural disturbances. Method of singular disturbances allows decomposing the model into component parts with slow and fast dynamics. Interest to this method has increased because of investigations of robust properties of control systems. With reference to the problem of nominal model choice constructively of this method is restricted by partition of model into elements with different time scale.

*Method of weighting functions* (Voronov, 1985). If eigen-values of complete linear model are different, then transfer function of comparison model is presented by a-periodic first order element. Time constant of this element is defined as inverse to eigen-value minimum in absolute value, and gain is defined as average geometric value of all inputs and outputs.

*Method of matrix inequalities* (Voronov, 1985). Comparison models may be obtained on the basic of estimation of equation solution of model by norm, and the difference between nominal models will be define by the difference of used matrix norm. Constructive solutions for majorants and minorants are obtained for logarithmic matrix norm.

Considered methods of obtaining comparison models of upper and lower bounds of parameters demonstrate available range of nominal model's properties, but do not answer the question - how to choose proper nominal model from obtained range.

If to synthesize nominal model from control realization conditions, for example using energy criterion, then it is necessary to formulate

requirements to nominal model and to develop synthesis methods of such model.

Further assume, that nominal model should satisfy to following requirements:

- energy reachability (energy limitation of control signals);
- physical constraints on phase variables and parameters;
- correspondence to accepted level of adequacy;
- constructively in solving tasks of check and control algorithms synthesis;
- technical reliability in control systems.

Requirements of energy restriction of control signals is caused by either amplitude and power present constraints of actuators. On the one hand, simplification of nominal model may improve reliability of control algorithms, but on the other hand it does not satisfy energy restriction, that will result in additional expenditure on control.

Requirements to nominal models depend on schemes of control by model too. On principle, well-known schemes of forming additional control signal may be represented by two types: «differential plug» (Ivachnenko, 1966) and by scheme with internal model (Garsia, 1982). The difference between these schemes is shown in Fig.1-Fig.4, where: CM - correcting model; MO – model of object; NM - nominal model.

Different schemes of model inclusion satisfy different properties of control. Further consider problem of NM for scheme shown in Fig.3.

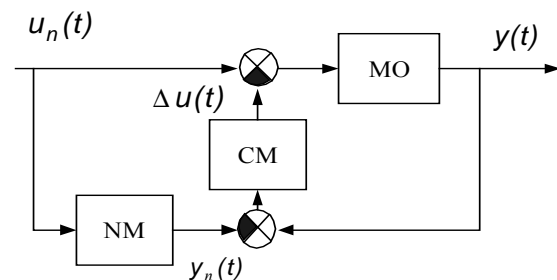


Fig. 1. Schemes of interconnection of single input NM for control problem (variant 1)

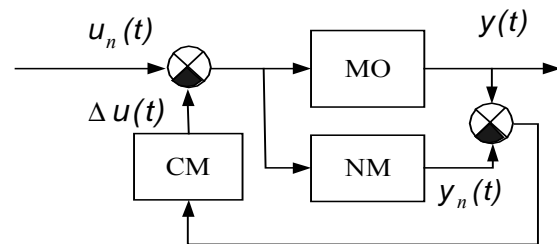


Fig. 2. Schemes of interconnection of single input NM for control problem (variant 2)

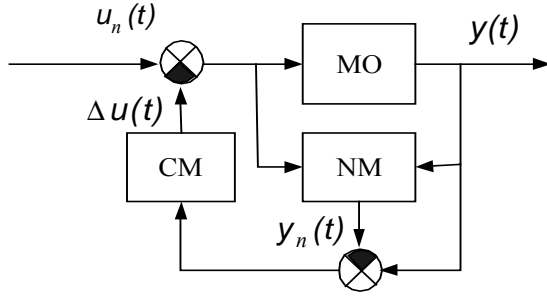


Fig. 3. Schemes of interconnection of two-input  $NM$  for control problem

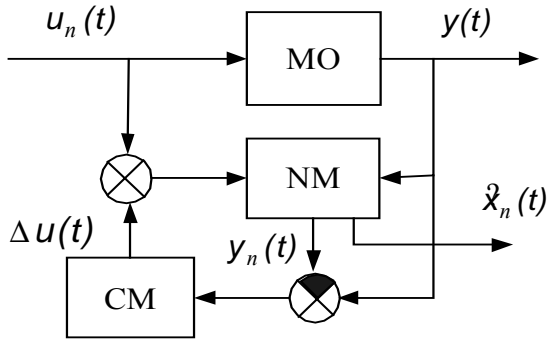


Fig. 4. Schemes of interconnection of two-input  $NM$  for observation

### 3. STATEMENT OF NOMINAL MODEL SYNTHESIS PROBLEM

Controlled object is described by system of non-linear equations

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}, \mathbf{u}, \Theta, t), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \\ \mathbf{y}(t) &= \mathbf{g}(\mathbf{x}, \mathbf{u}, \Theta, t), \end{aligned} \quad (1)$$

where  $\mathbf{x}(t) \in R^{n_x}$  - state vector,  $\mathbf{u}(t) \in R^{n_u}$  - control vector,  $\mathbf{y}(t) \in R^{n_y}$  - output vector,  $\Theta(t) \in \Omega_\Theta$  - varying vector of object parameters from some finite set  $\Omega_\Theta$ .

For system (1) assume existence of linear model, sufficiently close to plant according to norm

$$\|\mathbf{y}(t) - \mathbf{y}_n(t)\|_{R^{n_y}} < \Delta_y$$

with comparatively small value  $\Delta_y$ , represented the system

$$\begin{aligned} \dot{\mathbf{x}}_n(t) &= \mathbf{A}^n \mathbf{x}_n(t) + \mathbf{B}_u^n \mathbf{u}_n(t), \\ \mathbf{y}_n(t) &= \mathbf{C}^n \mathbf{x}_n(t) + \mathbf{D}_u^n \mathbf{u}_n(t). \end{aligned} \quad (2)$$

Define linear model of plant using nominal model

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}^n \mathbf{x}(t) + \mathbf{B}_u^n (\mathbf{u}_n(t) + \Delta \mathbf{u}(t)) + \mathbf{B}_v^n \mathbf{v}(t), \\ \mathbf{y}(t) &= \mathbf{C}^n \mathbf{x}(t) + \mathbf{D}_u^n (\mathbf{u}_n(t) + \Delta \mathbf{u}(t)) + \mathbf{D}_v^n \mathbf{v}(t), \end{aligned} \quad (3)$$

where  $\mathbf{v}(t) \in R^{n_v}$  - vector of equivalent disturbances, taking into account internal and external disturbances,  $\Delta \mathbf{u}(t)$  - additional control according to compensation scheme.

For more precise definition of solution of  $NM$  synthesis problem give the following definitions.

*Definition 1.* Linear nominal model (2) is called output reachable for different uncertainties of plant, if for any finite vector of equivalent disturbances  $\mathbf{v}(t) \in \Omega_v$ , some nominal vector of parameters  $\Theta_n \in \Omega_\Theta$  bounded vector of control  $\mathbf{u}_n(t) + \Delta \mathbf{u}(t) \in L_2$  exists such that for models (2) and (3)

$$\|\mathbf{y}(t) - \mathbf{y}_n(t)\|_{R^{n_y}} \rightarrow 0$$

is valid when  $t \rightarrow \infty$ .

*Definition 2.* Linear nominal model (2) is called partially achievable for all forms of plant uncertainty, if for any bounded vector of equivalent disturbances  $\mathbf{v}(t) \in \Omega_v$ , some nominal vector of parameters  $\Theta_n \in \Omega_\Theta$  and for given bounded level of coordination  $\Delta_y$  of models (2) and (3) finite control vector  $\mathbf{u}_n(t) + \Delta \mathbf{u}(t) \in \Omega_u$  exists such that inequality

$$\|\mathbf{y}(t) - \mathbf{y}_n(t)\|_{R^{n_y}} < \Delta_y$$

is valid for all interval of system work.

Represent the dependence between input and output as linear model (3) in operational form

$$\mathbf{y}(t) = \mathbf{W}_0^u(p) \mathbf{u}(t) + \mathbf{W}_0^v(p) \mathbf{v}(t) + \mathbf{y}_0(t, \mathbf{x}_0), \quad (4)$$

where  $\mathbf{W}_0^u(p), \mathbf{W}_0^v(p)$  - dynamic operator of plant model with respect to,  $p$ - differentiation operator,  $\mathbf{y}_0(t, \mathbf{x}_0) = \mathbf{W}_0^{x_0}(p) \mathbf{x}_0$  - output component of intrinsic motion, defined by initial conditions vector  $\mathbf{x}_0$ . Write down two-input model in state estimator form

$$\hat{\mathbf{y}}_e(t) = \mathbf{W}_e^u(p) \mathbf{u}(t) + \mathbf{W}_e^y(t) \mathbf{y}(t), \quad (5)$$

where  $\mathbf{W}_e^u(p), \mathbf{W}_e^y(p)$  - dynamic operators of  $NM$  with respect to control and observation,  $\hat{\mathbf{y}}_0(t, \mathbf{x}_0) = \mathbf{W}_e^{x_0}(p) \hat{\mathbf{x}}_0$  - output component of

intrinsic motion of  $NM$ . At parameterization of  $NM$  assume, that  $\|\hat{\mathbf{y}}_0(t, \mathbf{x}_0)\|_{R^{n_y}} \rightarrow 0$  when  $t \rightarrow \infty$ .

For plant model operator additive assume separation of  $NM$

$$\mathbf{W}_o^u(p) = \mathbf{W}_n^u(p) + \Delta\mathbf{W}_o^u(p), \quad (6)$$

where  $\Delta\mathbf{W}_o^u(p)$  - additive non-parametric uncertainties,  $\mathbf{W}_n^u(p)$  - nominal part of model.

According to scheme in Fig.3 write down additional control  $\Delta\mathbf{u}(t) = -\mathbf{W}_c(p)(\mathbf{y}(t) - \hat{\mathbf{y}}_e(t))$ , where  $\mathbf{W}_c(p)$  - correcting model operator.

The problem is structural and parametric synthesis with respect to known operator of plant with multiple values of plant parameters  $\Theta \in \Omega_\Theta$ .

#### 4. CRITERIONS OF NOMINAL MODEL SYNTHESIS

The choice of  $NM$  should be defined by of additional control signal constraints, then corresponding close-loop system with respect to reachability property is robust. Preservation of robust control using model, as a necessary condition of reachability of  $NM$ , is possible under conditions to be satisfied, which are given by the following theorem.

*Theorem 1 (Q - criterion of reachability with respect to output of NM).*

Nominal linear model, defined by vector of parameters  $\Theta_n \in \Omega_\Theta$ , will be completely reachable with respect to output for all types of plant uncertainties and all possible values of parameters  $\Theta \in \Omega_\Theta$ , if:

- vector of equivalent disturbance is bounded  $\mathbf{v}(t) \in \Omega_v$ ;
- nominal model is stable;
- $n_u = n_y$  and vector of additional control is

defined by operator of CM  $\mathbf{W}_c(p) = (\mathbf{W}_e^u(p))^{-1}$ , and

$\mathbf{W}_e^u(p)$  - inverse operator;

- the following condition

$$\left\| (\mathbf{W}_c^u(p))^{-1} \Delta\mathbf{W}_o^u(p, \Theta) \right\|_{H_\infty} = q < 1$$

is fulfilled for all set of values of parameters  $\Theta \in \Omega_\Theta$ .

Remarks to criterion of reachability.

*Remark 1.* The difference between plant model and  $NM$  should be comparatively not large and such, that condition of convergence in norm of Neyman series for inverse operator (6).

*Remark 2.* The value of additional control signal depends on value of external disturbances and degree of discrepancy between plant model and nominal model.

*Remark 3.* Q-criterion is used for «square models», in which the number of inputs is equal to the number of outputs. For non-«square models», partial reachability is possible. The following theorem formulates the condition of partial reachability.

*Theorem 2 (NQ - criterion of partial reachability with respect to output).*

Nominal model, defined by vector of parameters  $\Theta_n \in \Omega_\Theta$ , will be reachable for all types of plant uncertainties and all possible values of parameters  $\Theta \in \Omega_\Theta$  and given level of co-ordination  $\Delta_y$ , if:

- vector of equivalent disturbance is bounded  $\mathbf{v}(t) \in \Omega_v$ ;
- nominal model is stable;
- $\dim(\mathbf{W}_e^u(p)) = n_y \times n_u, n_y > n_u$ ,

$\text{rank}(\mathbf{W}_e^u(p)) = n_y$ , vector of additional control is

defined by operator CM  $\mathbf{W}_k(p) = (\mathbf{W}_e^u(p))^+$ , where

$( )^+$  - denotes of pseudo- inversion operator;

- the following condition

$$\left\| (\mathbf{W}_n^u(p))^+ \Delta\mathbf{W}_o^u(p, \Theta) \right\|_{H_\infty} = q < 1$$

is fulfilled for all set of values of parameters  $\Theta \in \Omega_\Theta$ .

The proofs of theorems are based on matrix identities of linear systems operators. Constructive approach to synthesis of  $NM$  may be obtained for optimum reachable of  $NM$ .

*Definition 3.* Nominal model, defined by vectors of parameters  $\Theta_n^* \in \Omega_\Theta$ , is called optimum reachable for all types of plant uncertainties, if for bounded vector of equivalent disturbance  $\mathbf{v}(t) \in \Omega_v$  bounded control vector  $\mathbf{u}_n(t) + \Delta\mathbf{u}(t) \in \Omega_u$  exists, condition of reachability with respect to output is satisfied and parameters of nominal model are optimal with respect to criterion of energy cost of additional control vector  $\Delta\mathbf{u}(t)$

$$\Theta_n^* = \arg \min_{\Theta_n \in \Omega_{\Theta}} \|\Delta \mathbf{u}(t, \Theta, \Theta_n)\|_{L_2}^2 d\Theta \quad (7)$$

for the set of possible values of nominal parameters.

Optimal reachability of nominal model is defined by technique of estimation of equivalent disturbance, which imposes particularities to provide reachability. Usually it is important to provide reachability only for some operating frequency range, so it is necessary to modify criterions of reachability as well.

## 5. PARAMETRICAL SYNTHESIS OF NM

As a criterion of nominal model we take an integral characteristic on the set of possible values of nominal model from norm of additional control vector

$$J(\Theta_n) = \int_{\Omega_{\Theta}} \|\Delta \mathbf{u}(t, \Theta, \Theta_n)\|_{L_2}^2 d\Theta. \quad (8)$$

According to scheme of connection of NM, as it is shown in Fig.3, write down the vector  $\Delta \mathbf{u}(t, \Theta, \Theta_n)$  in the following

$$\Delta \mathbf{u}(t, \Theta, \Theta_n) = -(\mathbf{W}_0^u(p, \Theta))^{-1} \times \left( (\mathbf{W}_0^u(p, \Theta) - \mathbf{W}_n^u(p, \Theta_n)) \mathbf{u}_n(t) - \mathbf{W}_0^v(p, \Theta) \mathbf{v}(t) \right). \quad (9)$$

The expression (9) comprises difference of outputs of complete and nominal models, connected with the same input signal  $\mathbf{u}_n(t)$ . We may neglect of the second addend in (9), because optimal values of nominal parameters do not depend on disturbances and control signals. So, in general case the criterion of parametric synthesis is written as

$$J = \int_{\Omega_{\Theta}} \left\| (\mathbf{W}_0^u(p, \Theta))^{-1} (\mathbf{y}(t, \Theta) - \mathbf{y}_n(t, \Theta_n)) \right\|_{L_2}^2 d\Theta. \quad (10)$$

If compare output signals of complete and nominal models at considered random input signal, we should suppose the energy power of difference between outputs, which were transformed by operator  $(\mathbf{W}_0^u(p))^{-1}$  as a comparison criterion.

Criterion (10) may be modified on the basis of available statistical information about distribution density of parameters, then write down criterion of average risk

$$J(\Theta_n) = \int_{\Omega_{\Theta}} \|\Delta \mathbf{u}(t, \Theta, \Theta_n)\|_{L_2}^2 f_{\Theta}(\Theta) d\Theta. \quad (11)$$

where  $f_{\Theta}(\Theta)$  - distribution density function of parameters.

Parametrical synthesis of nominal model with given structure is the choice of optimal with respect to criterion (11) parameters taking into account present constraints

$$\Theta_n^* = \arg \min_{\Theta_n \in \Omega_{\Theta}} J(\Theta_n), J_i(\Theta_n) \leq J_i^*, i = \overline{1, n_J}. \quad (12)$$

It is difficult to solve the optimization problem in general, but in some cases we can obtain analytical solution for specific models and present constraints.

## 6. SYNTHESIS OF NOMINAL MODEL FOR INTERVAL OBJECT

*Example 1.* As an illustrative example consider the problem of nominal model synthesis for the plant, described by equation

$$T \dot{x}(t) = x(t) + K(u(t) + v(t)), \quad (13)$$

with the following coefficients, defined by inequalities

$$\underline{T} < T < \bar{T}, \quad \underline{K} < K < \bar{K}.$$

Energy criterion (10) for definition of nominal model parameters is written as

$$J(K_n, T_n) = \frac{1}{(\bar{T} - \underline{T})(\bar{K} - \underline{K})} \times \int_{\underline{T}}^{\bar{T}} \int_{\underline{K}}^{\bar{K}} \frac{(T - T_n)^2 K_n + (K_n - K)^2 T_n}{2T_n^2 (K_n + T_n/T_u) K_n/T_u} dT dK, \quad (14)$$

where  $T_u$  - time constant of forming filter.

Sections of cost functions are represented in Fig.5.

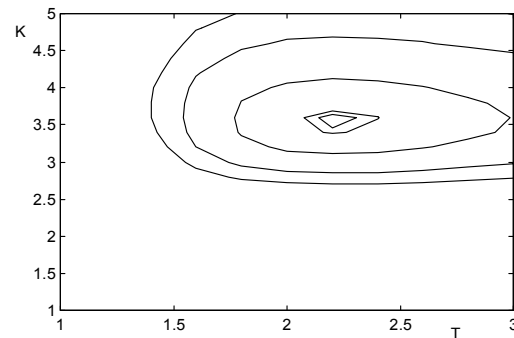


Fig. 5. Level curves of cost function in the plane of nominal parameters of the model

Interconnections of optimal with respect to criterion (14) parameters are defined by approximated expressions

$$\begin{aligned} \frac{T_n^*}{T} &= 0.96 + 0.63 \frac{(\bar{T} - T)}{T} + 0.016 \frac{(\bar{K} - K)}{K}, \\ \frac{K_n^*}{K} &= 0.60 + 0.64 \frac{(\bar{K} - K)}{K} + 0.10 \frac{(\bar{T} - T)}{T}. \end{aligned} \quad (15)$$

From dependency (15) conclude, that optimal parameters depend on value of ranges and are linear.

General recommendations of parametrical synthesis of first order nominal model consist of the following:

- gain of nominal model  $K_n^*$  lies in the region upper than average value of the range and depends on time constant  $T_n^*$  and its range;
- time constant of nominal model  $T_n^*$  is in the region of greater values of the range and defined according to approximate dependence – minimum value + 60% of possible value range.

For concerned example parameters of nominal model are equal to  $K_n^* = 3.36$ ,  $T_n^* = 2.26$ .

*Example 2.* Calculate the nominal model for the plant, described by equation (3) with matrices

$$\mathbf{A} = \begin{bmatrix} -1.0 & -5.0 & 1.0 \\ 1.0 & -2.0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0.3 & 1.8 & -1.0 \\ 0.0 & 1.0 & 0 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 1.0e-4 & 0 \\ 0 & 1.0e-4 \end{bmatrix}.$$

energy criterion for definition of the nominal model parameters shown in Fig.6. For given example the parameters of the nominal model are obtained the following  $a_{11}^* = -3.67$ ,  $b_{11}^* = 1.3$ .

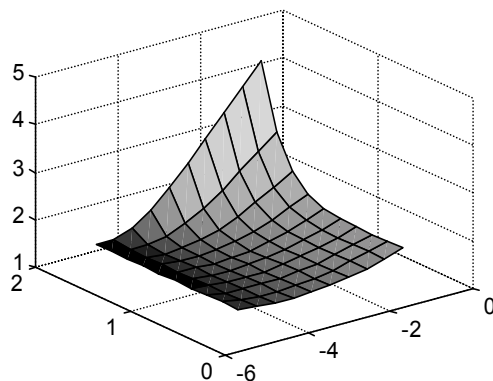


Fig.6. Cost function of nominal parameters for example 2

## 7. CONCLUSIONS

Control systems with internal model allow obtaining desired control performance indexes, but used internal models are chosen intuitively without rigorous substantiation. The nominal model synthesis with respect to energy integral criterion from additional control signal is suggested in this article. Synthesized *NM* with respect to proposed criterion reduces additional control expenses costs, connecting with the difference between plant and nominal model.

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