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BIFURCATION ANALYSIS OF THE LOTKA-VOLTERRA MODEL SUBJECT TO VARIABLE STRUCTURE CONTROL

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Abstract: In this paper the dynamics of the Lotka-Volterra predator-prey model subject to a variable structure control is studied applying bifurcation analysis techniques. This methodology allows to determine a complete characterization of the system dynamics when some selected control parameters are changed. The main contributions of the paper are: (i) an analysis of the effects of the harvesting policy for different combinations of the control parameters; (ii) the establishment of necessary and sufficient conditions in order to guarantee global stability of a desired equilibrium in predator-prey populations under harvesting action.

Keywords: Ecological Modelling, Bifurcation Analysis, Variable Structure Control, Sliding Mode, Predator-Prey.

1. INTRODUCTION

Variable structure systems appear quite frequently in engineering problems but they are not so frequent in other areas like biology or ecology.

A whole family of predator-prey models has been the object of multidisciplinar research by many authors (Bhattacharya and Begum, 1996), (Azar *et al.*, 1995), (Krivan, 1996), (Malchow, 2000), (Neubert *et al.*, 1995) but the most typical example of a dynamic model for an ecological system is the well-known Lotka-Volterra predator-prey model. This system has been extensively studied in the non-linear dynamic systems literature in its original form. However, the controlled version of this model has been less investigated and shows much more complex dynamics than the uncontrolled one. In ecological systems the control action can be associated with human interference in the natural environment, which can take the form of a harvesting policy.

Models like the Lotka-Volterra are extremely important to the concept of *Sustainable Exploitation of Natural Resources*. They allow one to evaluate the exploitation policies and their effect on the dynamic behaviour of the target ecosystem. Such analysis is useful to determine the safety levels of harvesting and to understand the limits of Nature's capacity to bear human interference.

This paper studies the dynamics of the Lotka-Volterra predator-prey model subject to a variable structure control using bifurcation analysis techniques (Kuznetsov, 1995), (Guckenheimer and Holmes, 1983). The bifurcation theory permits a complete characterization of the system dynamics when changing some selected parameters. For the system analysed in this work, the bifurcation parameters are the ones that model the human action. Some necessary conditions (not sufficient) for the global stability of the desired

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equilibrium populations of this kind of system have already been reported in a previous work (Costa *et al.*, 2000). The present work uses the bifurcation theory in order to derive necessary and sufficient conditions to guarantee global stability of this equilibrium point. This equilibrium is important because, if it is stable, the survival of both populations is guaranteed under sustainable exploitation. Furthermore, the analysis developed here allows one to determine all the different behaviour modes for the variable structure controlled Lotka-Volterra system. This analysis can also bring new insights into the problem of controlling ecological systems and into the Theory of Variable Structure Systems.

This paper is organized as follows. Some basic results on variable structure systems are presented in Section 2. Section 3 describes the controlled Lotka-Volterra model. Section 4 deals with the equilibrium analysis of the system while the bifurcation analysis is developed in Section 5.

2. SOME FACTS ABOUT VARIABLE STRUCTURE CONTROL

Consider the generic nonlinear dynamic variable structure system in the form

$$\dot{x} = f(t,x) + B(t,x)u(t,x) \tag{1}$$

with

$$u(t,x) = \begin{cases} u^+(t,x), & \text{if } \sigma(x) > 0 \\ u^-(t,x), & \text{if } \sigma(x) < 0, \end{cases} \tag{2}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}$ is a discontinuous control function, $f : D_f \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $B : D_B \subset \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous vector functions with continuous derivatives with respect to all their arguments and $\sigma : \mathbb{R}^n \to \mathbb{R}$ is a scalar function of the state vector.

The control function is undefined when the state vector belongs to the set

$$S = \{x \in \mathbb{R}^n / \sigma(x) = 0\} \tag{3}$$

since S is a discontinuity surface between the two different structures of the system. A sliding mode may occur on this surface if the function σ is such that

$$V(x) = \frac{\sigma^2(x)}{2} > 0; \qquad \dot{V}(x) = \sigma \frac{\partial \sigma}{\partial x}\frac{dx}{dt} < 0. \tag{4}$$

Since σ is a real valued function, the subset of the state space that matches condition (4) is

$$\Psi = \left\{ x \in \mathbb{R}^n \middle/ \sigma \frac{\partial \sigma}{\partial x}\frac{dx}{dt} < 0 \right\}, \tag{5}$$

which is the domain for which $V(x)$ is a Lyapunov function. The sliding domain is given by

$$\Omega = S \cap \Psi. \tag{6}$$

If Ω is empty, sliding modes are not possible. Otherwise, the solution of (1) when $x \in S$ represents a sliding motion on the surface S and is given by the theory of variable structure systems (Utkin, 1978) in terms of the *equivalent control*

$$u_{eq} = -\left[J_\sigma(x)B(t,x)\right]^{-1} \cdot J_\sigma(x) f(t,x) \tag{7}$$

where $J_\sigma(x)$ is the jacobian matrix of the function σ with respect to the state vector x. In the case of the single input system (1) it becomes

$$J_\sigma(x) = \left[\ \frac{\partial \sigma}{\partial x_1} \quad \frac{\partial \sigma}{\partial x_2} \quad \cdots \quad \frac{\partial \sigma}{\partial x_n}\ \right]. \tag{8}$$

The next step in the analysis of this type of system is finding the equilibrium points. Two types of equilibria are possible in systems with variable structure. Those that belong to the sliding domain Ω and those that do not. In this paper the first type will be referred to as *sliding equilibrium*, whereas the second one will be called *natural equilibrium*. This last type is subdivided in two classes: *real equilibria* and *virtual equilibria* (Costa *et al.*, 2000). To define them properly it must be borne in mind that the dynamics of the system formed by (1) and (2) is governed by two different structures. Each of them corresponds to one of the control laws defined in (2) and is valid for a different region of the state space. These regions are separated from each other by the boundary S. Each structure posses its own equilibrium points, which are the solutions of the equilibrium condition $\dot{x} = 0$ imposed on the equations of the particular system structure. If any of these solutions lies in the region governed by the structure that originates it, this solution is defined as a *real equilibrium point*, whereas if it is located in another region it is said to be a *virtual equilibrium point*.

3. THE VARIABLE STRUCTURE CONTROLLED LOTKA-VOLTERRA MODEL

The system analysed in this paper corresponds to the Lotka-Volterra model subject to a switching harvesting policy with constant harvesting effort. The equations of this model are

$$\begin{cases} \dot{x}_1 = r_1 x_1 - a x_1 x_2 - u_1(x_1,x_2) \\ \dot{x}_2 = -r_2 x_2 + b x_1 x_2 - u_2(x_1,x_2), \end{cases} \tag{9}$$

where x_1 stands for the population of preys and r_1 is its specific growth rate, x_2 stands for the population of predators and r_2 is its specific death rate. The parameter a models the reduction in the population of preys as a consequence of their interaction with predators and b is the growth rate of the predator population due to interaction between prey and predators. The control function u is given by

$$u(x) = \begin{bmatrix} q(\sigma)\epsilon_1 x_1 \\ q(\sigma)\epsilon_2 x_2 \end{bmatrix} \tag{10}$$

where ϵ_1 and ϵ_2 are the harvesting efforts for the preys and predators respectively. The function q is defined as

$$q(\sigma) = \begin{cases} 0 & \text{if } \sigma < 0 \\ 1 & \text{if } \sigma > 0 \end{cases} \tag{11}$$

with

$$\sigma(x) = s_0 + s_1 x_1 + s_2 x_2 \tag{12}$$

and models the switching between the two structures of the system. Eqs. (11) and (12) can be understood as a kind of rule that allows harvesting if the weighted sum of the populations of preys and predators exceeds a certain limit represented by the value of $-s_0$. The parameters s_1 and s_2 are the weighting factors of the populations. Note that the resulting system composed by (9-12) takes the form of (1) where q in (10) plays the role of u in (1).

4. EQUILIBRIUM ANALYSIS

This section deals with the equilibrium analysis of the system described in Section 3. Imposing the equilibrium condition on (9) one obtains

$$\begin{cases} r_1 x_1 - a x_1 x_2 - q\epsilon_1 x_1 = 0 \\ -r_2 x_2 + b x_1 x_2 - q\epsilon_2 x_2 = 0. \end{cases} \tag{13}$$

Any equilibrium point must satisfy (13) regardless of the function q, be it continuous or discrete. In the case of this paper, q is a value in the discrete set $\{0; 1\}$ that models the forbiddance or permission of harvesting. Eliminating q from (13) it follows that the non-trivial equilibrium points lies on the manifold

$$\Gamma = \left\{ x \in \mathbb{R}^2 \,\middle|\, x_2 = \left(\frac{\epsilon_1 r_2}{\epsilon_2 a} + \frac{r_1}{a}\right) - \frac{\epsilon_1 b}{\epsilon_2 a} x_1 \right\}. \tag{14}$$

The natural equilibria of the system can be found by substituting the two possible values of q in (13). For $q = 0$ the solution is

$$\overline{X}_0 = \begin{bmatrix} \bar{x}_{0_1} \\ \bar{x}_{0_2} \end{bmatrix} = \begin{bmatrix} \frac{r_2}{b} \\ \frac{r_1}{a} \end{bmatrix} \tag{15}$$

which is a degenerated equilibrium (a center). The second subscript index denotes the components of the state vector. For $q = 1$ the solution is

$$\overline{X}_1 = \begin{bmatrix} \bar{x}_{1_1} \\ \bar{x}_{1_2} \end{bmatrix} = \begin{bmatrix} \frac{r_2+\epsilon_2}{b} \\ \frac{r_1-\epsilon_1}{a} \end{bmatrix} \tag{16}$$

which is also a center if only the physically significant case is considered. Note that $\overline{X}_0$ and $\overline{X}_1$ belong to Γ.

The system (9) with the control given by (10), (11) and (12) can only exhibit sliding equilibria if the switching surface S intercepts the equilibrium manifold Γ. These two sets are straight lines in the state space, so there is only one intersection point between them. This point is given by

$$X_{int} = \begin{bmatrix} x_{int_1} \\ x_{int_2} \end{bmatrix} = \begin{bmatrix} \dfrac{\epsilon_1 r_2 + \epsilon_2 r_1 + \epsilon_2 a \frac{s_0}{s_2}}{\epsilon_1 b - \epsilon_2 a \frac{s_1}{s_2}} \\ -\frac{s_0}{s_2} - \frac{s_1}{s_2} \dfrac{\epsilon_1 r_2 + \epsilon_2 r_1 + \epsilon_2 a \frac{s_0}{s_2}}{\epsilon_1 b - \epsilon_2 a \frac{s_1}{s_2}} \end{bmatrix}. \tag{17}$$

X_{int} is a sliding equilibrium point if and only if it belongs to the sliding domain Ω. To compute Ω it is necessary to find the region Ψ for which $V(x) = \sigma^2/2$ is a Lyapunov function. Computing (5) for the controlled Lotka-Volterra model, the conditions obtained for Ψ are

$$\begin{cases} x_2 > \frac{s_1 r_1 x_1}{(s_1 a - s_2 b)x_1 + s_2 r_2} & \text{for } x_1 > \frac{s_2 r_2}{s_2 b - s_1 a}, \\ x_2 < \frac{s_1 r_1 x_1}{(s_1 a - s_2 b)x_1 + s_2 r_2} & \text{for } x_1 < \frac{s_2 r_2}{s_2 b - s_1 a} \end{cases} \tag{18}$$

and

$$\begin{cases} x_2 > \frac{s_1 (r_1 - \epsilon_1) x_1}{(s_1 a - s_2 b)x_1 + s_2 (r_2 + \epsilon_2)} & \text{for } x_1 < \frac{s_2 (r_2+\epsilon_2)}{s_2 b - s_1 a}, \\ x_2 < \frac{s_1 (r_1 - \epsilon_1) x_1}{(s_1 a - s_2 b)x_1 + s_2 (r_2 + \epsilon_2)} & \text{for } x_1 > \frac{s_2 (r_2+\epsilon_2)}{s_2 b - s_1 a}. \end{cases} \tag{19}$$

Therefore, the region Ψ is

$$\Psi = \left\{ x \in \mathbb{R}^2 / x \quad \text{satisfies (18) and (19)} \right\}. \tag{20}$$

So, if the point X_{int} belongs to Ψ, the sliding equilibrium point is $\overline{X}_2 = X_{int}$. If the set $\Omega = S \cap \Psi$ is not empty but does not contain X_{int}, then there is a sliding mode in Ω without sliding equilibria.

The real or virtual characters of the equilibrium points $\overline{X}_0$ and $\overline{X}_1$ also depend on X_{int} because this belongs to the surface that separates the two different structures of the state space dynamics. Therefore the point X_{int}, besides determining the position of the sliding equilibrium, also determines if the natural equilibria are real or virtual.

5. BIFURCATION ANALYSIS

This section analyses the bifurcation phenomena that the Lotka-Volterra model can display due to changes in the parameters of the variable structure control defined by Eqs. (10-12).

For the sake of simplicity, the harvesting efforts ϵ_1 and ϵ_2 will be kept constant while the surface parameters will be varied keeping s_2 positive.

With σ given by (12), the switching surface S corresponds to the solutions of $\sigma(x) = 0$ that can be written as

$$x_2 = K + \alpha x_1 \tag{21}$$

where

$$K = -\frac{s_0}{s_2} \qquad \text{and} \qquad \alpha = -\frac{s_1}{s_2}$$

will be treated as bifurcation parameters. Because the state variables represent the size of the populations, only the first quadrant of the state space has a physical meaning. Consequently, if K and α are both negative, the switching surface lies outside the physically meaningful region of the parameter space and does not represent a real switching. Note that the position of the switching surface S and the point X_{int} can be put in terms of these new parameters.

As stated in Section 4, $\overline{X}_0$ and $\overline{X}_1$ are real or virtual depending on the position of X_{int}. If X_{int} lies between $\overline{X}_0$ and $\overline{X}_1$ both of them are real or virtual. Otherwise one of them is real and the other virtual. To distinguish between these two cases it is sufficient to consider the condition

$$\bar{x}_{0_1} < \bar{x}_{2_1} < \bar{x}_{1_1} \tag{22}$$

since $\overline{X}_0$, $\overline{X}_1$ and X_{int} lie on the same straight line. It is straightforward to show that (22) is true if K and α satisfy

$$\alpha > -\frac{\epsilon_1}{\epsilon_2}\frac{b}{a} \quad \Rightarrow \quad \begin{cases} \alpha < \frac{b}{a}\frac{r_1}{r_2} - \frac{b}{r_2}K \\ \alpha > \frac{b}{a}\frac{(r_1-\epsilon_1)}{(r_2+\epsilon_2)} - \frac{b}{(r_2+\epsilon_2)}K \end{cases} \tag{23}$$

$$\alpha < -\frac{\epsilon_1}{\epsilon_2}\frac{b}{a} \quad \Rightarrow \quad \begin{cases} \alpha > \frac{b}{a}\frac{r_1}{r_2} - \frac{b}{r_2}K \\ \alpha < \frac{b}{a}\frac{(r_1-\epsilon_1)}{(r_2+\epsilon_2)} - \frac{b}{(r_2+\epsilon_2)}K. \end{cases} \tag{24}$$

The condition (23) bounds the region of the parameter space $(K;\alpha)$ for which the natural equilibrium points are both virtual and (24) delimits the region where $\overline{X}_0$ and $\overline{X}_1$ are both real equilibrium points. Fig. 1 shows a graphic representation of the parameter space and its regions considering $a = b = r_1 = r_2 = s_2 = 1$, $\epsilon_1 = 0.5$ and $\epsilon_2 = 0.4$.

To analyse the existence of sliding modes implies finding the condition under which the set $\Omega = S \cap \Psi$ is not empty. It is straightforward to show that if $\alpha < 0$ then Ω is not empty. For $\alpha > 0$ there is a lower boundary on the region Ψ represented by the point

$$X_c = \begin{bmatrix} x_{c_1} \\ x_{c_2} \end{bmatrix} = \begin{bmatrix} \frac{\epsilon_1 r_2 + \epsilon_2 r_1}{\epsilon_1 b + \epsilon_1 a\alpha} \\ \alpha\frac{\epsilon_1 r_2 + \epsilon_2 r_1}{\epsilon_2 b + \epsilon_2 a\alpha} \end{bmatrix} \tag{25}$$

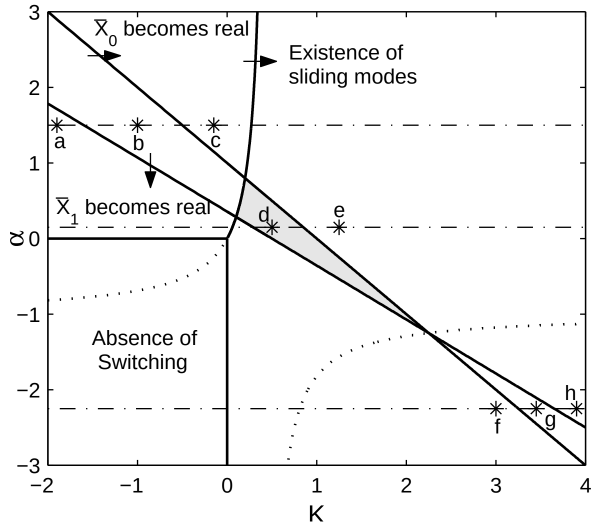

Fig. 1. Bifurcation set for the Lotka-Volterra model subject to a variable structure control.

which is a common point between Γ and the two boundaries of Ψ. A sliding mode will exist if and only if S intercepts Γ above the point X_c. The condition that guarantees this is

$$x_{int_1} < x_{c_1}, \tag{26}$$

which after some algebra becomes

$$\begin{cases} \alpha < \dfrac{bK}{\left(\frac{\epsilon_2}{\epsilon_1} - 1\right) r_1 - \left(\frac{\epsilon_1}{\epsilon_2} - 1\right) r_2 + aK} \\ \qquad \text{for } \alpha < \left(\frac{\epsilon_1}{\epsilon_2} - 1\right)\frac{r_2}{a} - \left(\frac{\epsilon_2}{\epsilon_1} - 1\right)\frac{r_1}{a}, \\ \alpha > \dfrac{bK}{\left(\frac{\epsilon_2}{\epsilon_1} - 1\right) r_1 - \left(\frac{\epsilon_1}{\epsilon_2} - 1\right) r_2 + aK} \\ \qquad \text{for } \alpha > \left(\frac{\epsilon_1}{\epsilon_2} - 1\right)\frac{r_2}{a} - \left(\frac{\epsilon_2}{\epsilon_1} - 1\right)\frac{r_1}{a}. \end{cases} \tag{27}$$

The condition (27) describes a region in the space $(K;\alpha)$ bounded by a hyperbola that is valid for $\alpha > 0$. In Fig. 1, the hyperbola is a continuous line in the region where it is meaningful and a dashed line where it is not. When the parameters are inside this region sliding modes can appear in the state space.

The set formed by the boundaries of these different regions in the parameter space is referred to as bifurcation set (Fig. 1) and it describes all possible combinations of the bifurcation parameters that lead to different topologies of the flow in the state space.

To show the richness of the possible dynamics that the system can exhibit, several combinations of the parameters K and α were chosen to build phase diagrams. Three different levels of α are considered (see Fig. 1). For each one, K varies in order to sample one point for each region. These

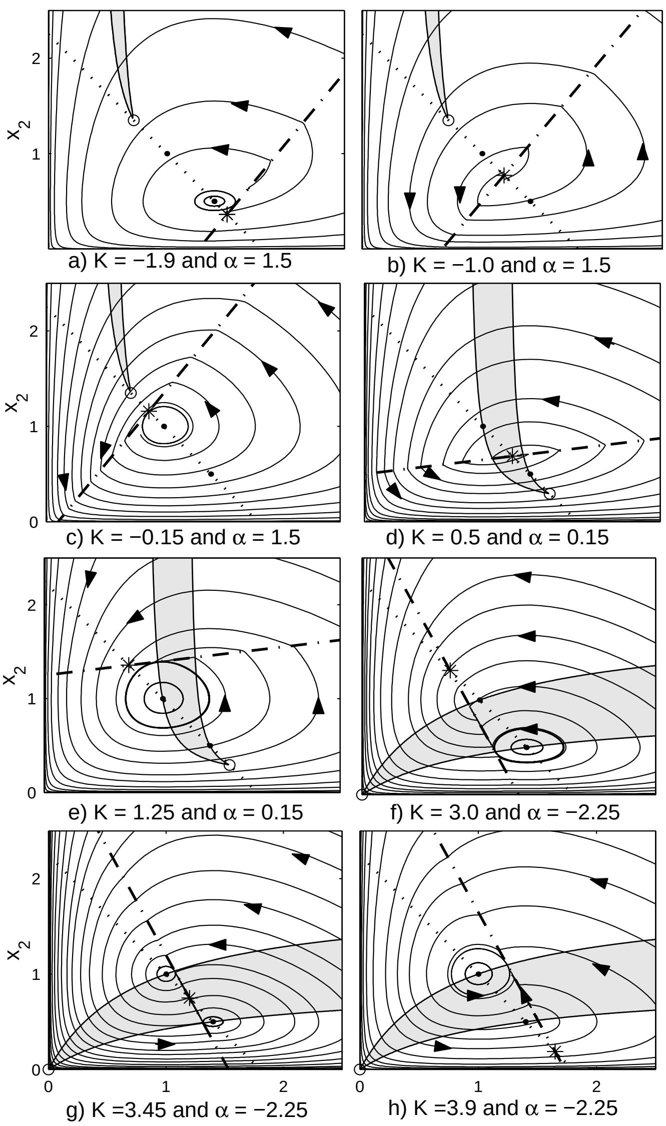

Fig. 2. Phase diagrams for different regions of the parameter space as indicated by the points a-h in Fig. 1. The points $\overline{X}_0$ and $\overline{X}_1$ are indicated by dots, X_{int} is marked with a star and X_c with a circle. The region Ψ is filled in grey, Γ is a dashed line and Ω is dashed/doted.

points are identified in the figure by the letters from a to h, which are also used in Fig. 2 to specify the phase diagrams of these possible cases.

The flow shown in Fig. 2a has a *semi-unstable* limit cycle. The trajectories that start from initial conditions inside the cycle remain in a closed orbit containing the initial point. This behavior shows the degenerated character of the equilibrium inside the limit cycle. Trajectories starting outside this cycle diverge from it in a spiral orbit.

The second case, shown in Fig. 2b, is simpler since there is no limit cycle. The intersection point between S and Γ is a source of the flow. All trajectories diverge from it in a spiral orbit. Note that there is no sliding motion.

Increasing the value of K even more, the flow assumes a behavior very close to the first case. The only difference is that now the degenerated equilibrium point is $\overline{X}_0$ instead of $\overline{X}_1$.

Fig. 2d shows the case in which both $\overline{X}_0$ and $\overline{X}_1$ are virtual. For this combination of parameters, the surface S intercepts the region Ψ (shown in grey) giving rise to the sliding domain Ω. Besides, there is a sliding equilibrium point on the intersection between S and Γ. When the trajectories hit Ω the state vector starts to move along the switching surface asymptotically converging to the point $\overline{X}_2$. This is the most important case since the equilibrium point is globally asymptotically stable. This case has already been identified in Costa *et al.* (2000), in which a necessary stability condition was demonstrated.

Moving up the switching surface a little further, the point of intersection of S and Γ leaves the region Ψ, being no longer an equilibrium point. In spite of that, the sliding mode still exists. $\overline{X}_0$ becomes a real degenerated equilibrium. A *semi-stable* limit cycle appears around it. Inside the limit cycle any initial condition gives rise to a closed orbit that returns to its initial point, the same way it does in the case of Fig. 2a and c. The difference in this case is that all the trajectories that start outside this limit cycle eventually hit the sliding domain and slide along Ω until leaving it. At that instant, the sliding motion ceases and the trajectory follows the closed orbit of the limit cycle. This scenario can be seen in Fig. 2e.

Fig. 2f shows a similar phenomenon, but in this case the real equilibrium point is $\overline{X}_1$.

The most interesting case is represented in Fig. 2g. The choice of K and α makes $\overline{X}_0$ and $\overline{X}_1$ both be real equilibrium points and also places X_{int} in the sliding domain. This means that $\overline{X}_2 = X_{int}$ is an equilibrium point. But now it is unstable, repelling the sliding trajectories, while some non-sliding orbits seem to be attracted to it. $\overline{X}_2$ thus behaves like a saddle point. And, like all saddle points, it divides the state space into two basins of attraction. Their respective attractors are the *semi-stable* limit cycles around $\overline{X}_0$ and $\overline{X}_1$ as can be seen in Fig. 3.

The flow of Fig. 2g is a transition between the flow of Fig. 2f and Fig. 2h, for the latter is simply the same case of the former, only inverting the qualities of real and virtual between $\overline{X}_0$ and $\overline{X}_1$.

The existence of a set of control parameters that globally stabilize the system was demonstrated in a previous work (Costa *et al.*, 2000). That was done under the implicit assumption that the intersection of the equilibrium manifold and the switching line lies on the sliding domain. As can be seen in Fig. 1, this is not always the case. X_{int}

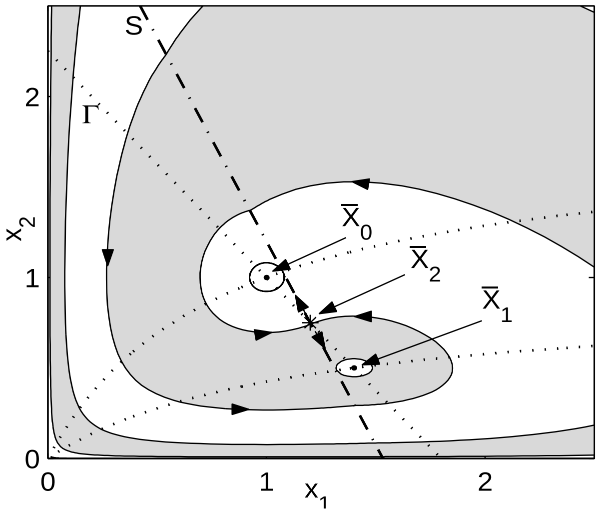

Fig. 3. Regions of attraction of the limit cycles for the case shown in Fig.2*g*.

is only a sliding equilibrium point if the values of the bifurcation parameters lie inside the region of the space $(K; \alpha)$ that satisfies (23), (24) and (27).

In this paper, it is contended that necessary and sufficient conditions to guarantee global stability of the desired equilibrium populations can be formulated through the following proposition:

Proposition: Under the hypothesis $s_2 > 0$, the system (9 - 12) has a globally stable equilibrium point given by (17) if and only if the control parameters K and α satisfy (23) and (27).

This proposition establishes the boundaries of the region $\boldsymbol{d}$ in the parameter space (K, α) shown in Fig. 1. The corresponding phase portrait is shown in Fig. 2*d*.

The proof of the stated proposition involves demonstrating that there is only one stable equilibrium point and that there is no limit cycle, since autonomous second order systems can only exhibit these two types of attractors (Guckenheimer and Holmes, 1983). In this work these two conditions are verified by the bifurcation analysis.

6. FINAL REMARKS

The analysis performed in this work established all the possible behaviours that the Lotka-Volterra system can exhibit when subject to variable structure control, as can be seen in Fig. 1.

In order to guarantee that the harvesting imposed to the predator and prey populations will not lead them to extinction, the harvesting policy must be chosen properly through the selection of control parameters inside the region $\boldsymbol{d}$. It is to be remarked that $\boldsymbol{d}$ is the smallest of all regions in the parameter space, imposing constraints upon the selection of the values of control parameters.

The control strategy applied in this paper allows for the coexistence of the two species and provides a safe way to exploit natural resources on a regular basis. Further research will be conducted in order to extend the results obtained here to more complex ecosystem models that take into account the interactions with other environment elements.

The analysis performed in Section 5 varying the parameters of the control law detected some bifurcation-like phenomena related to the sliding modes. As far as the authors are aware, this kind of result has not been reported yet. It possibly indicates the existence of a family of *sliding mode bifurcations* that can be of great significance to the better understanding of nonlinear phenomena in systems exhibiting sliding motions. Further research will be conducted in order to characterize these phenomena in a rigorous mathematical fashion.

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