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STABILIZATION OF A CLASS OF NONLINEAR PROCESSES BY LINEAR FEEDBACK

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Abstract: We discuss the stabilization of a class of nonlinear systems describing the dynamics of a stirred tank chemical reactor. The stabilization is achieved using linear dynamic feedback. Several types of controllers are defined for which closed loop global asymptotic stability of the set point can be proved. We also discuss the case where the closed loop possesses several equilibrium states and the controller is designed to ensure global convergence of the set of the equilibria. The approach relies on the direct method of Lyapunov and uses concepts from dissipativity and passivity theory.

Keywords: process control, nonlinear systems, stabilization, PID control, Lyapunov methods

1. INTRODUCTION

This paper addresses the stabilization of a class of nonlinear processes describing the mass and heat dynamics of a stirred tank chemical reactor. The processes contain two different types of nonlinear components : a nonlinear amplifier and a product type nonlinearity. There may exist several open loop equilibrium states. Various types of dynamic feedback controllers are considered : PD control, PID control as well as more complex linear controllers. Sufficient conditions are derived for the global asymptotic stabilization of the system's set point. The approach relies on well established principles for nonlinear controller design : It uses classical Lyapunov arguments and some concepts from dissipativity and passivity theory. Background information on these methods can be found e.g. in (Lozano *et al.*, 2000). Although in principle the controlled system is globally feedback linearizable, at least when both feed temperature and feed concentration are used as inputs, we wish to concentrate on the application of scalar or multivariable linear control, without trying to compensate the system's complex nonlinearities. This will improve the simplicity and robustness of the design. Section 2 describes the dynamics of a stirred tank reactor. First a single feedback loop with feed temperature as the only input and with a PD controller is investigated. Section 3 presents a Lyapunov stability analysis of the closed loop yielding sufficient conditions for global asymptotic stability of the set point. More general classes of controllers including PID control are discussed in Section 4. An interpretation of the results in the framework of passivity theory is included. Section 5 considers the design of controllers that allow the existence of several closed loop equilibrium states. Conditions are derived that ensure the convergence of all solutions to one of the equilibria. Thus the occurrence of sustained oscillations is eliminated while the availability of a global Lyapunov function may be exploited to estimate the set point's region of attraction in state space (Genesio *et al.*, 1985). The paper terminates with some conclusions and suggestions for further work.

2. PROCESS DYNAMICS

In reduced form the heat and mass balance equations of a stirred tank reactor for a single chemical reaction (Luyben, 1990) can be written as

$$\dot{x}_1 = r - (1+s)x_1 + m_2 g(x_1, x_2) + u_1 \quad (1)$$

$$\dot{x}_2 = 1 - x_2 - m_1 g(x_1, x_2) + u_2 \quad (2)$$

where $x_1 \triangleq \frac{RT}{E}10^4$, $x_2 \triangleq \frac{c}{c_0}$ and $g(x_1, x_2) \triangleq x_2 \exp\left(-\frac{10^4}{x_1}\right)$; $r \triangleq 10^4 \frac{R}{E}\left(T_0 + \frac{UT_k}{F\rho c_p}\right)$, $s \triangleq \frac{U}{F\rho c_p}$, $m_1 \triangleq \frac{Vk_0}{F}$, $m_2 \triangleq 10^4 \frac{(-\Delta H)Vk_0 c_0 R}{F\rho c_p E}$ and where time has been rescaled as $\tau \triangleq t/(\frac{V}{F})$. The state variables are proportional to reactor temperature T and reactor concentration c. The inputs $u_1 \triangleq 10^4 \frac{R}{E}\Delta T_0(t)$ and $u_2 \triangleq \frac{1}{c_0}\Delta c_0(t)$ are scaled fluctuations of feed temperature with nominal value T_0 (or eventually cooling temperature, with nominal value T_k) and possibly feed concentration with nominal value c_0. The other model parameters are the heat capacity per unit of volume c_p, the specific mass ρ, the molar reaction heat $-\Delta H$ and the reaction speed per unit of volume $k_0 c \exp\left(\frac{-E}{RT}\right)$. Corresponding to $u_1 = u_2 = 0$ there can be either one or three equilibrium states (Genesio *et al.*, 1985). One of these, say (x_{1s}, x_{2s}) is the set point. Redefining the state variables as deviations from the set values transforms (1), (2) into

$$\begin{aligned}\dot{x}_1 = &-(1+s)x_1 \\ &+m_2\left[g(x_{1s}+x_1, x_{2s}) - g(x_{1s}, x_{2s})\right] \\ &+m_2 g(x_{1s}+x_1, x_2) + u_1 \quad (3)\\ \dot{x}_2 = &-x_2 - m_1\left[g(x_{1s}+x_1, x_{2s}) - g(x_{1s}, x_{2s})\right] \\ &-m_1 g(x_{1s}+x_1, x_2) + u_2 \quad (4)\end{aligned}$$

(3), (4) imply that

$$\begin{aligned}m_1\dot{x}_1 + m_2\dot{x}_2 = &-m_1(1+s)x_1 \\ &-m_2 x_2 + u_0 \quad (5)\end{aligned}$$

where $u_0 \triangleq m_1 u_1 + m_2 u_2$. Choose u_0 such that

$$m_2 x_2 = \alpha x_1 + \beta \dot{x}_1 \quad (6)$$

where α and β are design parameters. Then eliminating x_2 from (5), (6) results in the control law

$$\begin{aligned}u_0 = &\left[m_1(1+s) + \alpha\right]x_1 \\ &+(m_1+\beta)\dot{x}_1 + m_2\dot{x}_2 \quad (7)\end{aligned}$$

Now the state equations (6), (4) take the form

$$M\dot{x} = Ax - b\left[f_1(c'x) + f_2(c'x)c'\dot{x} - u_2\right] \quad (8)$$

where $M = \begin{bmatrix}\beta & 0\\ 0 & 1\end{bmatrix}$, $A = \begin{bmatrix}-\alpha & m_2\\ 0 & -1\end{bmatrix}$, $b = \begin{bmatrix}0\\ 1\end{bmatrix}$, $c = \begin{bmatrix}1\\ 0\end{bmatrix}$, $f_2(y) = \frac{m_1}{m_2}\beta\exp\left(-\frac{10^4}{x_{1s}+y}\right)$, $f_1(y) = m_1 x_{2s}\left[\exp\left(-\frac{10^4}{x_{1s}+y}\right) - \exp\left(-\frac{10^4}{x_{1s}}\right)\right] + \frac{m_1}{m_2}\alpha y \exp\left(-\frac{10^4}{x_{1s}+y}\right)$.

First we consider the case of a a PD controlled process with $u_1 = u_0/m_1$ and u_0 defined by (7) as the input, while

$$u_2(t) \equiv 0 \quad (9)$$

Generally speaking, (1), (2) is a special case of the class of nonlinear systems

$$\dot{x} = Ax + k - bf(g'x)c'x + Du\,;\ x \in R^n \quad (10)$$

which, besides chemical reactor processes, have been used to model various other nonlinear technical systems. Some examples can be found in (Chiang *et al.*, 1988).

Below we analyze the global asymptotic stability of the PD-controlled process (8), (9).

3. PD CONTROL

For simplicity rewrite (8), (9) as

$$\dot{x} = A_0 x - b_0 f_0(x, \dot{x}) \quad (11)$$

with $A_0 \triangleq M^{-1}A$, $b_0 \triangleq M^{-1}b$, $f_0(x,\dot{x}) \triangleq f_1(c'x)$ $+f_2(c'x)c'\dot{x}$ and where we assume that M is nonsingular. Also assume that in the transfer function

$$H(s) = c'(Ms - A)^{-1}b \quad (12)$$

of the system's linear part there are no pole-zero cancellations, or equivalently that (A_0, b_0, c) is a controllable and observable triple. Furthermore let $H(s)$ be asymptotically stable. Let

$$\begin{aligned}V(x) \triangleq\ & x'Px + \alpha_0\int_0^{c'x} f_1(\theta)d\theta \\ &+\int_0^{c'x} f_2(\theta)\theta d\theta \quad (13)\end{aligned}$$

be a candidate Lyapunov function for the system (11). Along the solutions of (11) :

$$\begin{aligned}\dot{V}(x) = &\dot{x}'Px + x'P\dot{x} + \alpha_0 f_1(c'x)c'\dot{x} \\ &+f_2(c'x)c'xc'\dot{x}\end{aligned}$$

$$= x'[PA_0 + A_0'P]x - x'[2Pb_0 - A_0'c\alpha_0].$$
$$f_0(x,\dot{x}) - \alpha_0 c' b_0 f_0^2(x,\dot{x})$$
$$-\alpha_0 f_2(c'x)(c'\dot{x})^2 + f_2(c'x)c'xc'\dot{x}$$

Writing this in the form

$$\dot{V}(x) = -[q'x + tf_0(x,\dot{x})]^2 - c'xf_1(c'x)$$
$$-\alpha_0 f_2(c'x)(c'\dot{x})^2 \qquad (14)$$

and expanding the first term in the right hand side produces the equations

$$PA_0 + A_0'P = -qq' \qquad (15)$$
$$2Pb_0 - 2qt = A_0'c\alpha_0 + c \qquad (16)$$
$$t^2 = \alpha_0 c' b_0 \qquad (17)$$

Under the assumptions above and by an application of the Kalman-Yacubovich-Popov (KYP) lemma (Lozano *et al.*, 2000) the system (15), (16), (17) has a real solution $P = P' > 0$ (positive definite), q, t if and only if

$$\alpha_0 c' b_0 + \mathrm{Re}\,(c' + \alpha_0 c' A_0)\,(j\omega I - A_0)^{-1}\,b_0 \geq 0\,;$$
$$\forall \omega \in R \qquad (18)$$

Using the identity

$$sc'(sI - A_0)^{-1}b_0 \equiv c'b_0 + c'A_0(sI - A_0)^{-1}b_0$$

(18) can be transformed to

$$\mathrm{Re}(1 + \alpha_0 j\omega)c'(j\omega I - A_0)^{-1}b_0 \geq 0\,;$$
$$\forall \omega \in R \qquad (19)$$

Choose $\alpha_0 \geq 0$. Then $V(x)$ is positive definite and radially unbounded if

$$yf_1(y) > 0\,; \forall y \in R, y \neq 0 \qquad (20)$$
$$f_2(y) > 0\,; \forall y \in R \qquad (21)$$

By (14), $\dot{V}(x) \leq 0$ for all $x \in R^n$. Furthermore because of the observability of (A_0, c) the largest invariant subset of R^n where $\dot{V}(x) \equiv 0$ consists of the set point $x = 0$. Summarizing the above the set point is globally asymptotically stable if (20), (21) hold and if for some $\alpha_0 \geq 0$:

$$(1 + \alpha_0 s)H(s) \text{ is positive real} \qquad (22)$$

For the PD controlled stirred tank reactor the conditions (20), (21) are fulfilled. From (8), (12) we obtain

$$(1 + \alpha_0 s)H(s) = m_2 \frac{1 + \alpha_0 s}{(\beta s + \alpha)(s + 1)} \qquad (23)$$

which satisfies (22) if $\alpha > 0$, $\beta > 0$ and α_0 is selected such that $\alpha_0 \geq \beta/(\alpha + \beta)$. Figure 1 displays a block diagram of the system (8), (9), which is equivalent to the diagram of Figure 2. The obtained stability conditions express that in the latter diagram the component in the forward path is passive while the feedback path is a parallel connection of two passive components. Indeed for every input applied to these components at $t = 0$,

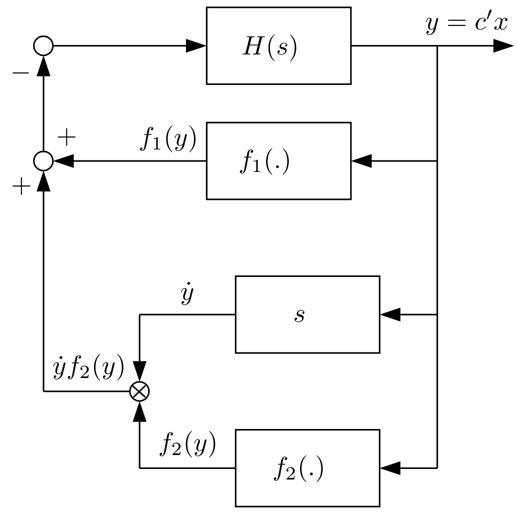

Fig. 1. Block diagram of the system (8), (9)

$$\int_0^{+\infty} f_1(z_1)(z_1 + \alpha_0 \dot{z}_1)dt \quad \geq \quad 0$$
$$\int_0^{+\infty} f_2(z_2)\dot{z}_2(z_2 + \alpha_0 \dot{z}_2)dt \quad \geq \quad 0$$

Invoking the invariance theory for autonomous systems we have shown that these passivity properties are sufficient for the set point's global asymptotic stability.

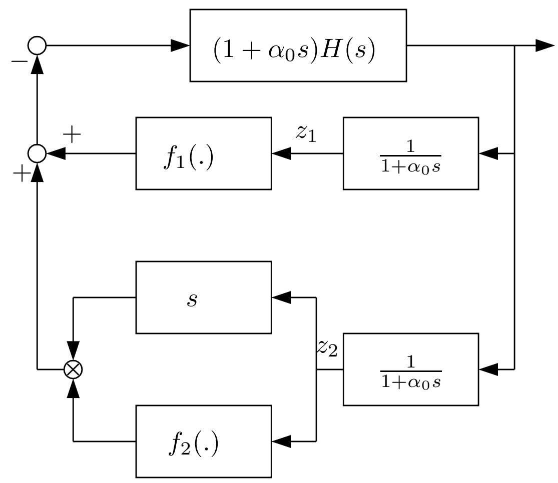

Fig. 2. Transformation of the diagram of Fig. 1

4. HIGHER ORDER CONTROLLERS

Here we design a controller for the system (8) with input $u_2 \in R$, which we rewrite as

$$\dot{x} = A_0 x - b_0 f_0(x, \dot{x}) + b_o u_2 \tag{24}$$

Along the solutions of (24) and assuming (15), (16), (17) are satisfied

$$\begin{aligned}\dot{V}(x, w) = &-[q'x + t f_0(x, \dot{x})]^2 - c'x f_1(c'x)\\ &-\alpha_0 f_2(c'x)(c'\dot{x})^2 + u_2 w\end{aligned} \tag{25}$$

where

$$w \triangleq 2x'Pb_0 + \alpha_0 c' b_0 f_0(x, \dot{x}) \tag{26}$$

Hence under the conditions (20), (21), (22), the system (24) is dissipative with storage function $V(x)$ and supply rate $u_2 w$. If

$$c'b_0 = 0 \tag{27}$$

then (26) simplifies to

$$\begin{aligned}w = 2x'Pb_0 &= c'x + \alpha_0 c' A_0 x\\ &= c'x + \alpha_0 c'\dot{x}\end{aligned}$$

(27) means that in the expression of $H(s) \triangleq q(s)/p(s)$,

$$\text{degree } q(s) \le n - 2 \tag{28}$$

Indeed $\lim_{s\to\infty} sH(s) = c'b_0 = 0$ if and only if (28) holds. (28) ensures that the system (8) will not degenerate (decrease of dynamical order) in certain points of its state space. Assume (27) holds and let

$$u_2(s) = -G(s)w(s) \tag{29}$$

be a feedback law from w to u_2 resulting in the block diagram of Figure 3. Let

$$\begin{aligned}\dot{z} &= \hat{A}z + \hat{b}w \quad ; \quad z \in R^{n_0}\\ -u_2 &= \hat{c}'z\end{aligned} \tag{30}$$

be a controllable and observable state representation of (29) and let $\hat{A}$ be Hurwitz. Let

$$W(z) \triangleq z'Qz \tag{31}$$

Then

$$\begin{aligned}\dot{W}(z, w) &= \dot{z}'Qz + z'Q\dot{z}\\ &= -(\hat{q}'z)^2 - u_2 w\end{aligned} \tag{32}$$

if

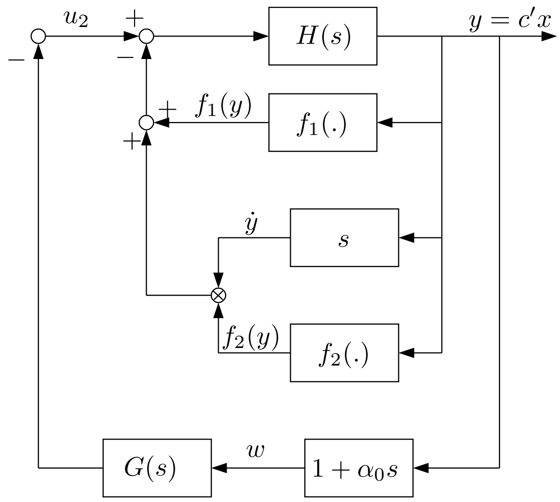

Fig. 3. Block diagram of the system (24), (29)

$$Q\hat{A} + \hat{A}'Q = -\hat{q}\hat{q}' \tag{33}$$

$$2Q\hat{b} = \hat{c} \tag{34}$$

(33), (34) has a real solution $Q = Q' > 0$, $\hat{q}$ if and only if

$$G(s) = \hat{c}'(sI - \hat{A})^{-1}\hat{b} \text{ is positive real} \tag{35}$$

Define

$$V_c(x_c) \triangleq V(x) + W(z) \tag{36}$$

Under the assumptions above $V_c(x_c)$ is positive definite and radially unbounded in $x_c \triangleq \begin{bmatrix} x \\ z \end{bmatrix}$ while

$$\begin{aligned}\dot{V}_c(x_c) = &-[q'x + t f_0(x, \dot{x})]^2 - c'x f_1(c'x)\\ &-\alpha_0 f_2(c'x)(c'\dot{x})^2 - (\hat{q}'z)^2\\ \le &\, 0 \quad ; \quad \forall x_c \in R^{n+n_0}\end{aligned}$$

Furthermore since (A_0, c) is observable and by the KYP-lemma $(\hat{A}, \hat{q})$ is also observable, the largest invariant subset of R^{n+n_0} where $\dot{V}_c(x_c) \equiv 0$ consists of the closed loop set point $x_c = 0$. The block diagram of Figure 3 is equivalent to the diagram of Figure 4. The conditions for global asymptotic stability (20), (21), (22), (35) express that in the loop of Figure 4 the forward path is passive and the feedback path is a parallel connection of three passive components. The special case

$$G(s) = \frac{\beta_0}{s} \quad ; \quad \beta_0 > 0 \tag{37}$$

yields

$$u_2(s) = -\beta_0 \left(\frac{1}{s} + \alpha_0\right) y(s) \tag{38}$$

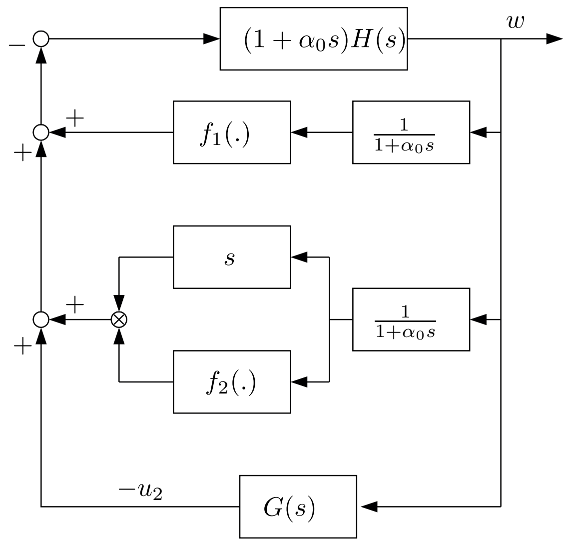

Fig. 4. Transformation of the diagram of Fig. 3

where $y \stackrel{\triangle}{=} c'x$. For the stirred tank reactor with feed temperature and feed concentration as inputs, $H(s) = m_2/(\beta s + \alpha)(s+1)$ such that (28) holds and (7), (29) represent a globally asymptotically stabilizing PID controller.

5. SYSTEMS WITH SEVERAL CLOSED LOOP EQUILIBRIUM POINTS

Below we consider controllers that allow the existence of several closed loop equilibrium states. Our approach is to find conditions that ensure the convergence of every solution to one of the equilibria, thus eliminating the occurrence of sustained oscillations. In general the conditions for global convergence to the set of the equilibria are less restrictive than those for the global asymptotic stability of a single equilibrium point. Specifically the obtained control laws are less likely to generate inputs beyond the system's physical saturation levels. Hence they constitute a basis for controller design in such cases where a bounded (but sufficiently large) region of attraction in state space for the set point is acceptable.

Our approach is to find a global Lyapunov function for the system whose derivative does not vanish along any nonconstant solution. Standard invariance theory then implies that every solution that remains bounded for increasing time will converge to an equilibrium point. This property is called the system's nonoscillating behaviour. For nonoscillating systems, if all solutions remain bounded for $t \geq 0$ the set of the equilibria is globally convergent. The boundedness of solutions for $t \geq 0$ can often easily be verified, for example using a suitable Lyapunov function. Furthermore the availability of a global Lyapunov function allows to evaluate the set point's region of attraction in state space using well established techniques (Genesio *et al.*, 1985). Below we present a few criteria for nonoscillating behaviour.

Returning to the reactor equations (1), (2) choose

$$u_0 = [m_1(1+s) + \alpha]x_1 + (m_1 + \beta)\dot{x}_1 + m_2\dot{x}_2 + \gamma - m_1 r - m_2 \quad (39)$$

such that

$$m_2 x_2 = \alpha x_1 + \beta \dot{x}_1 + \gamma \quad (40)$$

The equations (40), (2) take the form

$$M\dot{x} = Ax + k - b[f_1(c'x) + f_2(c'x)c'\dot{x}] + bu_2 \quad (41)$$

where M, A, b and c are as before, $k = \begin{bmatrix} -\gamma \\ 1 \end{bmatrix}$ and

$$f_1(y) = \frac{m_1}{m_2}(\alpha y + \gamma)\exp\left(-\frac{10^4}{y}\right)$$
$$f_2(y) = \frac{m_1}{m_2}\beta \exp\left(-\frac{10^4}{y}\right)$$

Again assume that the linear part of (41) is controllable, observable and asymptotically stable. First consider the case of a PD controller :

$$u_2(t) \equiv 0 \quad (42)$$

while $u_1 = u_0/m_1$ is given by (39). For a system of the form (41), (42) let

$$V(x) \stackrel{\triangle}{=} x'Px + \alpha_0 \int_0^{c'x} f_1(\theta)d\theta + p'x \quad (43)$$

Then

$$\dot{V}(x) = \dot{x}'Px + x'P\dot{x} + \alpha_0 f_1(c'x)c'\dot{x} + p'\dot{x}$$

Substituting x by $x = A^{-1}[M\dot{x} - k + bf_1(c'x) + bf_2(c'x)c'\dot{x}]$ and choosing

$$p \stackrel{\triangle}{=} 2PA^{-1}k \quad (44)$$
$$2PA^{-1}b + \alpha_0 c = 0 \quad (45)$$
$$PA^{-1}M + M'A^{-1\prime}P = -qq' \quad (46)$$

yields

$$\dot{V}(x) = \dot{x}'[PA^{-1}M + M'A^{-1\prime}P]\dot{x} + 2\dot{x}'PA^{-1}bf_2(c'x)c'\dot{x} = -(q'\dot{x})^2 - \alpha_0 f_2(c'x)(c'\dot{x})^2 \quad (47)$$

The system (45), (46) has a real solution $P = P'$, q if $\forall\omega \in R$:

$$-\alpha_0 \text{Re } c'(j\omega I - A^{-1}M)^{-1}A^{-1}b \geq 0$$

which can be rewritten as

$$\frac{\alpha_0}{\omega}\text{Im } c'(j\omega M - A)^{-1}b = \\ \frac{\alpha_0}{\omega}\text{Im } H(j\omega) \leq 0 \quad ; \quad \forall\omega \in R \qquad (48)$$

Choose $\alpha_0 > 0$ and assume $f_2(.)$ satisfies (21). Then using (47) and the observability of (A_0, c) it is easy to show that nonoscillating closed loop behaviour is ensured. In the case of the stirred tank reactor the imposed conditions, in particular (48) are satisfied for $\alpha > 0$ and $\beta > 0$.

The extension to the case $u_2(t) \not\equiv 0$ is straightforward. We now have

$$\dot{V}(x) = -(q'\dot{x})^2 - \alpha_0 f_2(c'x)(c'\dot{x})^2 + \alpha_0 u_2 \dot{w}$$

with $w \triangleq c'x$. Again let $u_2(s) = -G(s)w(s)$ with state representation (30). However write $\dot{W}(z, w)$ as

$$\dot{W}(z, w) = \dot{z}'Q\hat{A}^{-1}(\dot{z} - \hat{b}w) + (\dot{z} - \hat{b}w)'\hat{A}^{-1\prime}Q\dot{z} \\ = -(\hat{q}'\dot{z})^2 + \alpha_1 \dot{u}_2 w$$

where we let

$$Q\hat{A}^{-1} + \hat{A}^{-1\prime}Q = -\hat{q}\hat{q}' \qquad (49)$$

$$2Q\hat{A}^{-1}\hat{b} = -\alpha_1\hat{c} \qquad (50)$$

(49), (50) requires that $\forall\omega \in R$:

$$-\text{Re } \alpha_1\hat{c}'(j\omega I - \hat{A}^{-1})^{-1}\hat{A}^{-1}\hat{b} \geq 0$$

or equivalently

$$\frac{\alpha_1}{\omega}\text{Im } \hat{c}'(j\omega I - \hat{A})^{-1}\hat{b} = \\ \frac{\alpha_1}{\omega}\text{Im } G(j\omega) \leq 0 \quad ; \quad \forall\omega \in R \qquad (51)$$

Defining

$$V_c(x_c) \&pdef V(x) - \alpha_0 u_2 w + \frac{\alpha_0}{\alpha_1} W(z, w)$$

yields

$$\dot{V}_c(x_c) = -(q'\dot{x})^2 - \alpha_0 f_2(c'x)(c'\dot{x})^2 - \frac{\alpha_0}{\alpha_1}(\hat{q}'\dot{z})^2$$

which for $\alpha_1 > 0$ ensures nonoscillating closed loop behaviour. So the conditions on the controlled system and on the controller are (48) and (51) with $\alpha_0 > 0$ and $\alpha_1 > 0$. The chemical reactor model satisfies (48).

6. DISCUSSION AND CONCLUSIONS

We have discussed the stabilization of a class of nonlinear stirred tank reactor models possessing two different types of nonlinearities. Stabilization is achieved using linear dynamic output feedback. The system's global asymptotic stabilization by means of PD, PID and more complex controllers has been studied. However the practical applicability of the proposed techniques must be further analysed. E.g. bounds on the admissible values of state and control variables must be taken into account. The local dynamics around the set point and the system's time domain response must be examined. Specific algorithms for the determination of control parameters must be developed, simulation results will be needed and the method must be compared with existing design techniques.

We have also considered the case where the control law allows the existence of several closed loop equilibrium states. We have presented sufficient conditions for the controller structure to ensure nonoscillating closed loop dynamics. When using this approach the design must include an analysis of the influence of the control parameters on the set point's region of attraction in state space. The availability of a global Lyapunov function will facilitate the solution of this problem. Also we must investigate what can be gained in terms of the characteristics of the control strategies and the system's response when discarding global asymptotic stability as a design requirement. These are subjects for further research whose outcome will be reported later.

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