15th Triennial World Congress, Barcelona, Spain

# SYNCHRONIZE HIGN DIMENTIONAL CHAOTIC SYSTEMS BASED ON NONLINEAR FEEDBACK CONTROL *

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Abstract:In this paper, under some assumptions a nonlinear controller based on differential geometry theory of nonlinear control system and pole-assignment method is designed, and applied to synchronize output signals of high dimensional chaotic system. Using the method, we can synchronize output signals formed by linear or nonlinear composition of single or multiple state variables of the chaotic system. The controller is easy to be realized and can be adapted to a lot of nonlinear systems. Computer simulation results show the excellent performance of the proposed method. *Copyright © 2002 IFAC*

Keyword: chaos control, synchronization, nonlinear, pole-assignment, feedback

## 1 INTRODUCTION

In the last decade, synchronization in chaotic dynamical systems has received a great deal of interest among scientists from various fields (Carroll & Perora,1990;Chen & Dong,1998). As a particular class of nonlinear systems, chaotic systems can also be controlled well by many methods which are proved effective for numerous nonlinear systems (Bernardo, 1996;Ge *et al*., 2000; Femat *et al*., 2000; Hegazi *et al*.,2001; Fah & Tung,1995). Fah & Tung(Fah & Tung,1995) use exact linearization method to control chaotic systems, and based on it noise has been studied by Liaw & Tung (Liaw & Tung,1996). Through combining differential geometry with nonlinear dynamical system theory, a method is proposed which can synchronize arbitrarily designated scalar output signal composed of either single state variable or linear or nonlinear combination of multiple state variables of chaotic system (Gao *et al*., 2000a; Gao *et al*.,2000b). On these basis , in this paper, a controller is designed by differential geometry theory and pole-assignment method, which extends the above mentioned results to multi-input and multi-output situation.

*This research is supported by national key teachers foundation and doctorial special brance foundation of national educational commission.

## 2. PROBLEM DESCRIPTION

Consider two multi-variable nonlinear systems

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{f}}(\hat{\mathbf{x}}, t), \quad \hat{\mathbf{y}} = \hat{\mathbf{h}}(\hat{\mathbf{x}}), \hat{\mathbf{x}}(0) = \hat{\mathbf{x}}_0 \in \mathbf{R}^n, \tag{1}$$

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + \mathbf{g}(\mathbf{x})\mathbf{u}, \quad \mathbf{y} = \mathbf{h}(\mathbf{x}), \mathbf{x}(0) = \mathbf{x}_0 \in \mathbf{R}^n, \tag{2}$$

in which

$\hat{\mathbf{f}}, \mathbf{f} : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$,
$\hat{\mathbf{f}}, \mathbf{f} \in C^\infty$,
$\hat{\mathbf{y}} = col(\hat{y}_1 \dots \hat{y}_m)$,
$\hat{\mathbf{h}} = col(\hat{h}_i \dots \hat{h}_m)$,
$\mathbf{u} = col(u_1 \dots u_m)$,
$\mathbf{y} = col(y_1 \dots y_m)$,
$\mathbf{h} = col(h_1 \dots h_m)$,
$\mathbf{g}(\mathbf{x}) = [\mathbf{g}_1(\mathbf{x}) \dots \mathbf{g}_m(\mathbf{x})]$,
$\mathbf{g}_i : \mathbf{R}^n \to \mathbf{R}^n, \hat{h}_i, h_i : \mathbf{R}^n \to \mathbf{R}$,
$\hat{h}_i, h_i \in C^\infty, \hat{\mathbf{x}}(t), \mathbf{x}(t) \in \mathbf{R}^n$,
$u_i \in \mathbf{R}, i = 1, \cdots, m$,
$1 \le m \le n$.

(1) is called the driving system and (2) is called the driven system. The problem considered in this paper is how to design a suitable control law $\mathbf{u}$ which makes the outputs of (1) and (2) satisfy:

$$\lim_{t\to\infty} \| \mathbf{y} - \hat{\mathbf{y}} \| = 0 \tag{3}$$

Since nonautonomous systems can always be transformed to autonomous systems by extending state space (Wiggins, 1990), therefore, we only consider the case in which (2) is autonomous.

## 3 CONTROLLER DESIGN

First, we introduce some terminologies in nonlinear control system theory (Isdori,1989; Zhang ,Cai& Bien , 2000) .

Define $j$-th Lie derivative of function $h_i(x)$ respect to $\mathbf{f}(\mathbf{x})$ as:

$$L_{\mathbf{f}}^j h_i(\mathbf{x}) = \frac{\partial (L_{\mathbf{f}}^{j-1} h_i(\mathbf{x}))}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}),$$

$$L_{\mathbf{g}} L_{\mathbf{f}}^j h_i(\mathbf{x}) = \frac{\partial (L_{\mathbf{f}}^{j} h_i(\mathbf{x}))}{\partial \mathbf{x}} \mathbf{g}(\mathbf{x}),$$

$$L_{\mathbf{f}}^0 h_i(\mathbf{x}) = h_i(\mathbf{x}),\ j = 1,2,\cdots.$$

Denote $B(\mathbf{x}^0, \delta) = \left\{ \mathbf{x} \,\middle|\, \left\| \mathbf{x} - \mathbf{x}^0 \right\| < \delta \right\}$ as the neighborhood of $\mathbf{x}^0$, in which $\delta$ >0.

**Definition** (2) has vector relative degree $\{r_1, \ldots, r_m\}$ at $\mathbf{x}^0$ if

(i) $L_{\mathbf{g}_j} L_{\mathbf{f}}^k h_i(\mathbf{x}) = 0, \forall 1 \le i, j \le m, \forall k \le r_i - 1,$

$\forall x \in B(\mathbf{x}^0, \delta),$

(ii) $m \times m$ matrix

$$\mathbf{a}(\mathbf{x}) = \begin{bmatrix} L_{\mathbf{g}_1} L_{\mathbf{f}}^{r_1-1} h_1(\mathbf{x}) \cdots L_{\mathbf{g}_m} L_{\mathbf{f}}^{r_1-1} h_1(\mathbf{x}) \\ L_{\mathbf{g}_1} L_{\mathbf{f}}^{r_2-1} h_2(\mathbf{x}) \cdots L_{\mathbf{g}_m} L_{\mathbf{f}}^{r_2-1} h_2(\mathbf{x}) \\ \cdots \quad \cdots \quad \cdots \\ L_{\mathbf{g}_1} L_{\mathbf{f}}^{r_m-1} h_m(\mathbf{x}) \cdots L_{\mathbf{g}_m} L_{\mathbf{f}}^{r_m-1} h_m(\mathbf{x}) \end{bmatrix} \tag{4}$$

is nonsingular at $\mathbf{x}^0$.

**Proposition** (Isdori,1989) Suppose (2) has a (vector) relative degree $\{r_1, \ldots, r_m\}$ at $\mathbf{x}^0$. Then $r_1 + \cdots + r_m \le n$. Set, for $1 \le i \le m$

$$\phi_1^i = h_i(\mathbf{x})$$

$$\phi_2^i = L_{\mathbf{f}} h_i(\mathbf{x})$$

$$\cdots$$

$$\phi_{r_i}^i(\mathbf{x}) = L_{\mathbf{f}}^{r_i-1} h_i(\mathbf{x}),$$

if $r = r_1 + \cdots + r_m$ is strictly less than $n$, it is always possible to find $n - r$ functions $\phi_{r+1}, \ldots, \phi_n$ such that the mapping

$$\Phi(\mathbf{x}) = col[\phi_1^1(\mathbf{x}), \ldots, \phi_{r_1}^1(\mathbf{x}), \ldots, \phi_1^m(\mathbf{x}), \ldots,$$

$$\phi_{r_m}^m(\mathbf{x}), \phi_{r+1}(\mathbf{x}), \ldots, \phi_n(\mathbf{x})]$$

has a jacobian matrix which is nonsingular at $\mathbf{x}^0$ and therefore, qualifies as a local coordinates transformation in a neighborhood of $\mathbf{x}^0$. The value of these additional functions at $\mathbf{x}^0$ can be chosen arbitrarily. Moreover, if the distribution

$$\mathbf{G} = span\{\mathbf{g}_1, \ldots, \mathbf{g}_m\}$$

is involutive near $\mathbf{x}^0$,it is always possible to choose $\phi_{r+1}, \ldots, \phi_n$ in such a way that

$$L_{\mathbf{g}_j} \phi_i(\mathbf{x}) = 0,$$

for all $r + 1 \le i \le n,$ for all $1 \le j \le m,$ for all $\mathbf{x}$ around $\mathbf{x}^0$.

Transformed by $\Phi$ (2) becomes

$$\begin{aligned} \dot{\boldsymbol{\xi}} &= \mathbf{A}\boldsymbol{\xi} + \mathbf{B}(\mathbf{b}(\mathbf{x}) + \mathbf{a}(\mathbf{x})\mathbf{u}), \\ \dot{\boldsymbol{\eta}} &= \mathbf{q}(\boldsymbol{\xi}, \boldsymbol{\eta}), \\ \mathbf{y} &= \boldsymbol{\xi}, \\ \boldsymbol{\xi}(0) &= \boldsymbol{\xi}_0 \in \mathbf{R}^r, \\ \boldsymbol{\eta}(0) &= \boldsymbol{\eta}_0 \in \mathbf{R}^{n-r}, \end{aligned} \tag{5}$$

in which $\mathbf{A} = diag(\mathbf{A}_i),$

$$\mathbf{A}_i = \begin{bmatrix} \mathbf{0} & \mathbf{I}_{r_i-1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{6}$$

$$\mathbf{B} = diag(\mathbf{B}_i), \quad 1 \le i \le m,$$

$$\mathbf{B}_i = col(0, \cdots, 0, 1)_{r_i \times 1} \tag{7}$$

$$\boldsymbol{\xi} \in \mathbf{R}^r, \boldsymbol{\eta} \in \mathbf{R}^{n-r},$$

$$\boldsymbol{\xi} = col(\boldsymbol{\xi}^1, \ldots, \boldsymbol{\xi}^m),$$

$$\boldsymbol{\eta} = col(\phi_{r+1}, \ldots, \phi_n),$$

$$\boldsymbol{\xi}^i = col(\xi_1^i, \ldots, \xi_{r_i}^i) = col(\phi_1^i, \ldots, \phi_{r_i}^i).$$

$\mathbf{a}(\mathbf{x})$ is defined as in (4),

$$\mathbf{b}(\mathbf{x}) = \begin{bmatrix} L_{\mathbf{f}}^{r_1} h_1(\mathbf{x}) \\ L_{\mathbf{f}}^{r_2} h_2(\mathbf{x}) \\ \cdots \\ L_{\mathbf{f}}^{r_m} h_m(\mathbf{x}) \end{bmatrix}. \tag{8}$$

when $\boldsymbol{\xi} = \mathbf{0}$, $\dot{\boldsymbol{\eta}} = \mathbf{q}(\mathbf{0}, \boldsymbol{\eta})$ is called zero dynamics of system (5).

Next we discuss the controller design.

**Hypothesis 1** Suppose the reference signal

$\hat{y}_i \in C^{r_i}$, we denote the $j$-th derivative of $\hat{y}_i$ as $\hat{y}_i^{(j)}, 1 \le j \le r_i$, and

$$|\hat{y}_i^{(j)}(t)| < M_1, \forall t \in [0,\infty), M_1 \in \mathbf{R}^+,$$

**Hypothesis 2** To system (2), no matter whether the case is in equilibrium, period, quasi-period, chaos or hyperchaos, the following conditions always hold:

$$\|\mathbf{x}(t)\| < M_2, \forall t \in [0,\infty), M_2 \in \mathbf{R}^+,$$

$$\|\mathbf{a}(\mathbf{x})\| < M_3, \forall t \in [0,\infty), M_3 \in \mathbf{R}^+,$$

$$\|\mathbf{b}(\mathbf{x})\| < M_4, \forall t \in [0,\infty), M_4 \in \mathbf{R}^+,$$

$$\left\|\mathbf{b}(\mathbf{x}) - \hat{\mathbf{y}}^{(r)}\right\| < W, \forall t \in [0,\infty), W \in \mathbf{R}^+$$

in which

$$\hat{\mathbf{y}}^{(r)} = col(\hat{y}_1^{(r_1)}, \ldots, \hat{y}_m^{(r_m)}) = col(\hat{h}_1^{(r_1)}, \ldots, \hat{h}_m^{(r_m)}).$$

**Remark 1:** In this paper, we suppose all outputs of (1) and (2) are available.

**Remark 2:** Although hypothesis2 looked quite strict**,** the hypothesis is easy to be satisfied since the move- ment of chaotic system is in a bounded region.

Denote partial-variable-errors of synchronization as

$$\mathbf{e} = \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \vdots \\ \mathbf{e}_m \end{bmatrix} = \begin{bmatrix} \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \\ \vdots \\ \boldsymbol{\xi}_m \end{bmatrix} - \begin{bmatrix} \hat{\mathbf{y}}_1 \\ \hat{\mathbf{y}}_2 \\ \vdots \\ \hat{\mathbf{y}}_m \end{bmatrix}$$

in which $\mathbf{e}_i = \boldsymbol{\xi}_i - \hat{\mathbf{y}}_i = \begin{bmatrix} \xi_i^1 \\ \xi_i^2 \\ \vdots \\ \xi_i^{r_i} \end{bmatrix} - \begin{bmatrix} \hat{y}_i^{(0)} \\ \hat{y}_i^{(1)} \\ \vdots \\ \hat{y}_i^{(r_i-1)} \end{bmatrix}$,

$\hat{y}_i^{(j)}$ is the $j$-th derivative of $\hat{y}_i$, $1 \le j \le r_i - 1$, $1 \le i \le m$, therefore

$$\dot{\mathbf{e}} = \begin{bmatrix} \dot{\boldsymbol{\xi}}_1 \\ \dot{\boldsymbol{\xi}}_2 \\ \vdots \\ \dot{\boldsymbol{\xi}}_m \end{bmatrix} - \begin{bmatrix} \dot{\hat{\mathbf{y}}}_1 \\ \dot{\hat{\mathbf{y}}}_2 \\ \vdots \\ \dot{\hat{\mathbf{y}}}_m \end{bmatrix} \quad (9)$$

$$= \mathbf{Ae} + \mathbf{B}(\mathbf{b}(\mathbf{x}) + \mathbf{a}(\mathbf{x})\mathbf{u}) - \mathbf{B}\hat{\mathbf{y}}^{(r)}$$

**Theorem** If multi-input and multi-output system(2) has vector relative degree $r$ respect to some **g(x)** and satisfies hypothesis1 and hypothesis2, suppose control law **u** is selected as

$$\mathbf{u} = -(\mathbf{a}^{-1}(\mathbf{x})(\mathbf{b}(\mathbf{x}) + \mathbf{v} - \hat{\mathbf{y}}^{(r)})) \quad (10)$$

in which

$$\mathbf{v} = \mathbf{Ke}$$
$$\mathbf{K} = diag(\mathbf{K}_i), \quad (11)$$

$\mathbf{K}_i = (c_i^0, c_i^1, \ldots, c_i^{r_i-1})$, $1 \le i \le m, 1 \le j \le r_i - 1$, $c_i^0, c_i^1, \ldots, c_i^{r_i-1}$ are coefficients of Hurwitz polynomial

$$P(s) = s^{r_i} + c_i^{r_i-1}s^{r_i-1} + c_i^{r_i-2}s^{r_i-2} + \cdots + c_i^0$$

then for all initial condition

$$\mathbf{x(0)} \in \mathbf{R}^n, \mathbf{e(0)} \in \mathbf{R}^r$$

we always have

$$\lim_{t\to\infty} \mathbf{e}(t) = \mathbf{0}.$$

**proof:** Since (2) satisfies hypothesis 1 and hypothesis 2, from (10) we can know that the control is bounded. Substitute (10), (11) into (9), we get

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{BK})\mathbf{e}, \quad (12)$$

from (6),(7) we have,

$$\det(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{BK}))$$
$$= \prod_{i=1}^{m} \det(\lambda\mathbf{I}_{r_i} - (\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_i))$$

According to the selection method of **K**, we know all eigenvalues of $\mathbf{A}_i - \mathbf{B}_i\mathbf{K}_i$ are negative , so (9) is asymptotic stable, and moreover, in above procedure we don't use initial value, so (9) is global asymptotic stable. □

**Remark 3:** Since the choice of **g(x)** is quite flexible, usually we can find appropriate **g(x)** which makes (2) have vector relative degree.

## 4 SIMULATION

Consider following nonlinear system

$$\dot{\hat{\mathbf{x}}} = \mathbf{f}(\hat{\mathbf{x}}),$$
$$\hat{y}_1 = \hat{x}_1, \quad (13)$$
$$\hat{y}_2 = \hat{x}_5$$

it is the model of heart-blood coupling movement, in which

$$\mathbf{f}(\hat{\mathbf{x}}) = \begin{bmatrix} E\hat{x}_2 - E\hat{x}_1 + B\hat{x}_4, \\ A\hat{x}_1 - \hat{x}_2 - \hat{x}_1\hat{x}_3, \\ \hat{x}_1\hat{x}_2 - G\hat{x}_3, \\ P_1\hat{x}_5 + P_2\hat{x}_4 - P_3\hat{x}_4^3 + M\hat{x}_7, \\ -P_1\hat{x}_4 + D\hat{x}_1, \\ P_0\hat{x}_7 + 5.0\hat{x}_6 - 6.0\times10^4\hat{x}_6^3 + H\hat{x}_1, \\ -P_0\hat{x}_6 - M\hat{x}_4. \end{bmatrix}$$

where parameters $E = 16.0$, $A = 45.92$, $B = 5.0$ are constants. When $G = 4.0$, $P_0 = 10.0$, $P_1 = 3.4$, $P_2 = 60.0$, $P_3 = 12.0\times10^5$, $M = 1.5$, $D = 0.012$, $H = 0.012$, the maximal Lyapunov exponent of the system is 2.162(He *et al.*, 2000) therefore, the system is chaotic, which is shown in figure1. Suppose the dynamics and output of the controlled system equal those of (13), but the initial condition is not the case, so the controlled system can be written as:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}, \\ y_1 &= x_1, \\ y_2 &= x_5, \end{aligned} \qquad (14)$$

The aim of simulation is to synchronize the output of (13) and (14) by devising controller **u**. Choose

$$\mathbf{g}(\mathbf{x}) = \begin{bmatrix} \mathbf{g}_1(\mathbf{x}) & \mathbf{g}_2(\mathbf{x}) \end{bmatrix},$$

in which

$$\mathbf{g}_1(\mathbf{x}) = col(0, \frac{-1}{EMP}, 0, \frac{1}{BMP}, 0, 0, \frac{3BP_3x_4^2 - BP_2 + 1}{B^2 + M^2P_1}),$$

$$\mathbf{g}_2(\mathbf{x}) = col(0,0,0,0,0,1,1).$$

By simple calculating we know (14) has vector relative degree (3,2) and from (4), (8) we get

$$\mathbf{a}(\mathbf{x}) = \begin{bmatrix} \frac{B-1}{MP_1B} & BM \\ \frac{-1}{BM} & 0 \end{bmatrix},$$

$$\det(\mathbf{a}(\mathbf{x})) = 1 \neq 0,$$

$$\mathbf{b}(\mathbf{x}) = \begin{bmatrix} b_1(\mathbf{x}) \\ b_2(\mathbf{x}) \end{bmatrix},$$

in which

$$\begin{aligned} b_1(\mathbf{x}) &= (E^2 + E(A - x_3))(Ex_2 - Ex_1 + \\ &Bx_4) - (E^2 - E)(Ax_1 - x_2 - x_1x_3) - \\ &Ex_1(x_1x_2 - Gx_3) + (-EB + B(P_2 - \\ &3P_3x_4^2))(P_1x_5 + P_2x_4 - P_3x_4^3 + Mx_7) + \\ &BP_1(-P_1x_4 + Dx_1) + BM(-P_0x_6 - Mx_4), \\ b_2(\mathbf{x}) &= D(Ex_2 - Ex_1 + Bx_4) - \\ &P_1(P_1x_5 + P_2x_4 - P_3x_4^3 + Mx_7), \end{aligned}$$

$$\hat{y}_1^{(3)}(\hat{\mathbf{x}}) = b_1(\hat{\mathbf{x}}), \quad \hat{y}_2^{(2)}(\hat{\mathbf{x}}) = b_2(\hat{\mathbf{x}}),$$

the controller is devised according to (13). Assign all poles of (15) at -2, and therefore, $\mathbf{K}_1$=(8, 16, 6), $\mathbf{K}_2$=(4, 4). Figure2 shows that the outputs of (16) and (17) are synchronized, which proves that the controller is effective.

## 5 CONCLUSION

In this paper, based on multi-input and multi-output differential geometry theory and pole-assignment method, a controller is designed. The designing method is easy to be realized and computer simulation has proved its effectiveness. The research results in this paper can be adapted to quite a wide class of nonlinear systems such as can be applied to secret communication especially in the situation without enough channel resources.

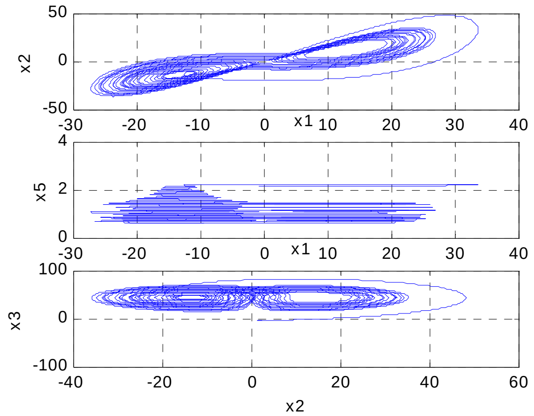


Fig1. phase diagram of system (13)

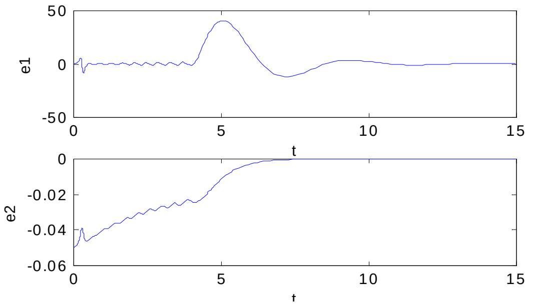


Fig2**.** errors of synchronization, the controller is taken on at t=4. $e_1 = x_1 - \hat{x}_1, e_2 = x_5 - \hat{x}_5$

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