15th Triennial World Congress, Barcelona, Spain

CONTROLLABILITY AND REACHABILITY OF SWITCHED LINEAR SYSTEMS: A GEOMETRIC APPROACH [1]

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Abstract: This paper investigates the controllability and reachability of switched linear control systems. It is proven that both the controllable and reachable sets are subspaces of the total space. Complete geometric characterizations for both sets are presented. The switching control design problem is also addressed.

Keywords: Switched linear systems, Controllability, Reachability

1. INTRODUCTION

During the last decade, hybrid and switched systems have attracted considerable attention (Chase, Serrano & Ramadge, 1993; Branicky, 1998; Wicks, Peleties & DeCarlo, 1998; Ye, Michel & Hou, 1998; Liberzon & Morse, 1999). Basically, a switched system consists of continuous-time/discrete-time dynamical subsystems and a rule (supervisor) that determines the switching among them.

Switched systems deserve investigation for theoretical reasons as well as for practical reasons. Switching among different system structures is an essential feature of many engineering control applications including power systems and power electronics, and switched systems have numerous applications in control of mechanical systems, air traffic control, aircrafts and satellites and many other fields (Li, Wen & Soh, 2001). Control techniques by switching among different controllers have been applied extensively in recent years. Indeed, a switched controller can provide a performance improvement over a fixed controller (Morse, 1996; Narendra & Balakrishnan, 1997; Savkin, Skafidas & Evans, 1999). The switched controller architecture is proven to be a rigorous design framework for general nonlinear systems (Kolmanovsky & McClamroch, 1996; Caines & Wei, 1998; Leonessa, Haddad & Chellaboina, 2001). A switched controller can also achieve certain control objects which cannot be accomplished by conventional methods, such as pure feedback stabilization of nonholonomic systems (Brockett, 1983).

A fundamental pre-requisite for the design of feedback control systems is full knowledge about the structural properties of the switched systems under consideration. These properties are closely related to the concepts of controllability, observability and stability which are of fundamental importance in the literature of control. For controllability and reachability, studies for low-order switched linear systems have been presented in Loparo, Aslanis & Hajek (1987) and Xu & Antsaklis (1999). Some sufficient conditions and necessary conditions for controllability were presented in Ezzine & Haddad (1989) for switched linear control systems under the assumption that the switching sequence is fixed *a priori*. The complexity of stability and controllability of hybrid

[1] An expansion of this paper has been submitted to *AUTOMATICA* for possible publication.
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systems was addressed in Blondel & Tsitsiklis (1999).

For controllability analysis of switched linear control systems, a much more difficult situation arises since both the control input and the switching rule are design variables to be determined, and thus the interaction between them must be fully understood. For a switched linear discrete-time control system, the controllable set is not a subspace but a countable union of subspaces in general case (Stanford & Conner, 1980; Ge, Sun & Lee, 2001). For a switched linear continuous-time control system, the controllable set is an uncountable union of subspaces (Sun & Zheng, 2001).

In this paper, we investigate the controllability and reachability issues for switched linear control systems in detail. We prove that, both the controllable set and the reachable set are subspaces of the total space, and the two sets always coincide with each other. Verifiable geometric characterization is presented for the controllable subspace. The switching control design problem is also addressed.

2. ELEMENTARY RESULTS

Consider a switched linear control system given by

$$\dot{x}(t) = A_\sigma x(t) + B_\sigma u_\sigma(t) \tag{1}$$

where $x \in \Re^n$ are the states, $u_k : \Re^+ \in \Re^{r_k}, k = 1, \cdots, m$ are piecewise continuous input functions, $\sigma : \Re^+ \to M = \{1, 2, \cdots, m\}$ is the switching path to be designed, and matrix pairs (A_k, B_k) for $k \in M$ are referred to as the subsystems of (1).

Given a switching path $\sigma : [0, t_f] \to M$, suppose its discontinuous (jump) time instants are $t_1 < t_2 < \cdots < t_s$, we refer to the sequence $t_0, t_1, t_2, \cdots, t_s$ as switching time sequence, and the sequence $\sigma(t_0), \sigma(t_1), \cdots, \sigma(t_s)$ as switching index sequence. It is clear that these two sequences can uniquely determine the switching path, and vice-versa.

For clarity, let $x(t; x_0, u, \sigma)$ denote the state trajectory at time t of switched system (1) starting from $x(0) = x_0$ with $u(t) = [u_1(t), \cdots, u_m(t)]^T$.

A state x is said to be controllable, if it can be transferred to the origin in a finite time by appropriate choices of input u and switching path σ.

Definition 1. State $x \in \Re^n$ is controllable, if there exist a time instant $t_f > 0$, a switching path $\sigma : [0, t_f] \to M$, and inputs $u_k : [0, t_f] \to \Re^{r_k}$, $k \in M$, such that $x(t_f; x, u, \sigma) = 0$.

Definition 2. The controllable set of system (1) is the set of states which are controllable.

Definition 3. System (1) is said to be (completely) controllable, if its controllable set is $\Re^n$.

The reachability counterparts can be defined in the same fashion and are omitted here.

Denote $\mathcal{B}_k = \mathrm{Im} B_k$, and $\mathcal{D}_k = \sum_{i=0}^{n-1} A_k^i \mathcal{B}_k$ for $k \in M$. Note that $\mathcal{D}_k$ is exactly the controllable subspace of subsystem (A_k, B_k).

By an elementary analysis (Sun & Zheng, 2001; Ge, Sun & Lee, 2001), the reachable set of system (1) is given by

$$\mathcal{R} = \cup_{k=1}^{\infty} \cup_{i_0,\cdots,i_k \in M} \cup_{h_1,\cdots,h_k > 0} (\mathcal{D}_{i_k} + e^{A_{i_k} h_k} \mathcal{D}_{i_{k-1}} + \cdots + e^{A_{i_k} h_k} \cdots e^{A_{i_1} h_1} \mathcal{D}_{i_0}) \tag{2}$$

and the controllable set of system (1) is given by

$$\mathcal{C} = \cup_{k=1}^{\infty} \cup_{i_0,\cdots,i_k \in M} \cup_{h_0,\cdots,h_k > 0} (e^{-A_{i_0} h_0} \mathcal{D}_{i_0} + \cdots + e^{-A_{i_0} h_0} \cdots e^{-A_{i_k} h_k} \mathcal{D}_{i_k}) \tag{3}$$

Given a matrix A and a subspace $\mathcal{B} \in \Re^n$, let $\Gamma_A \mathcal{B}$ denote the minimal A-invariant subspace that contains $\mathcal{B}$, i.e.,

$$\Gamma_A \mathcal{B} = \mathcal{B} + A\mathcal{B} + \cdots + A^{n-1}\mathcal{B}$$

This operation can be defined recursively as $\Gamma_{A_1}\Gamma_{A_2}\mathcal{B} = \Gamma_{A_1}(\Gamma_{A_2}\mathcal{B})$. Let us define the nested subspaces for system (1) as

$$\begin{aligned} \mathcal{V}_1 &= \mathcal{D}_1 + \cdots + \mathcal{D}_m \\ \mathcal{V}_{j+1} &= \Gamma_{A_1}\mathcal{V}_j + \cdots + \Gamma_{A_m}\mathcal{V}_j, \quad j = 1, 2, \cdots \end{aligned} \tag{4}$$

and

$$\mathcal{V} = \sum_{k=1}^{\infty} \mathcal{V}_k$$

Note that if $\dim \mathcal{V}_j = \dim \mathcal{V}_{j+1}$, then $\mathcal{V}_l = \mathcal{V}_j$ for $l > j$. This fact implies that $\mathcal{V} = \mathcal{V}_n$. It is readily seen that this subspace is the minimal subspace which is invariant under A_k, $k \in M$ and contains $\sum_{k \in M} \mathcal{B}_k$. See Ge *et al.* (2001) for the computational issues of this subspace.

Note that $e^{At}\mathrm{Im}B \subset \Gamma_A \mathrm{Im}B$ for all $A \in \Re^{n\times n}$, $B \in \Re^{n\times p}$ and $t \in \Re$. This implies that

$$\mathcal{R} \subset \mathcal{V}, \quad \mathcal{C} \subset \mathcal{V} \tag{5}$$

In the sequel, we exploit several properties of exponential matrix functions which play an important role in structural analysis for switched linear systems.

Lemma 1. For any given matrix $A \in \Re^{n\times n}$ and subspace $\mathcal{B} \subset \Re^n$, the following equation holds for almost all $t_1, t_2, \cdots, t_n \in \Re$

$$e^{At_1}\mathcal{B} + e^{At_2}\mathcal{B} + \cdots + e^{At_n}\mathcal{B} = \Gamma_A\mathcal{B} \quad (6)$$

Proof. Let $\mathcal{S}$ be the smallest subspace of $\Re^n$ that contains the subspaces $e^{At}\mathcal{B}$ for all $t \in \Re$. Let B be a matrix such that $\mathcal{B}$ =ImB. By Proposition 2.1 of Drager *et al.* (1989), $\mathcal{S}$ is exactly the controllable subspace of matrix pair (A, B) :

$$\mathcal{S} = \text{span}\{e^{At}Bz : t \in \Re, z \in \Re^n\} = \Gamma_A\mathcal{B} \quad (7)$$

Suppose $e^{At_j^0}Bz_j,\ \ j = 1, \cdots, n$ spans subspace $\mathcal{S}$, i.e.,

$$\mathcal{S} = \text{span}\{e^{At_1^0}Bz_1, \cdots, e^{At_n^0}Bz_n\}$$

This implies that

$$\text{rank}[e^{At_1^0}B, \cdots, e^{At_n^0}B] = \dim(\Gamma_A\mathcal{B})$$

Denote integer $r = \dim(\Gamma_A\mathcal{B})$, and matrix function $L(t_1, \cdots, t_n) = [e^{At_1}B, \cdots, e^{At_n}B]$. Choose a nonsingular sub-matrix M_0 with maximal rank in $L(t_1^0, \cdots, t_n^0)$. Therefore, M_0 is nonsingular and rankM_0 =rank$L(t_1^0, \cdots, t_n^0)$. Denote the corresponding sub-matrix of $L(t_1, \cdots, t_n)$ as $M(t_1, \cdots, t_n)$, and its determinant as $d(t_1, \cdots, t_n)$.

Since each entry in matrix $M(t_1, \cdots, t_n)$ is an analytic function of variables $t_1, \cdots, t_n$, $d(t_1, \cdots, t_n)$ is also an analytic function of its variables. As $d(t_1^0, \cdots, t_n^0) \neq 0$, function $d(t_1, \cdots, t_n)$ is not identically zero. By Weierstrass Preparation Theorem (Kaplan, 1966, Theorem 62), its zeros forms a zero-measure set of $\Re^n$. Therefore, for almost all $t_1, \cdots, t_n$, matrix $M(t_1, \cdots, t_n)$ is nonsingular. This implies that

$$\text{rank}[e^{At_1}B, \cdots, e^{At_n}B] \geq \dim(\Gamma_A\mathcal{B})$$

for almost all $t_1, \cdots, t_n$. Together with the fact that $e^{At}\mathcal{B} \subseteq \Gamma_A\mathcal{B}$, we can conclude that

$$e^{At_1}\mathcal{B} + \cdots + e^{At_n}\mathcal{B} = \Gamma_A\mathcal{B}$$

for almost all $t_1, \cdots, t_n$. $\diamondsuit$

Lemma 2. For any given matrices $A_k \in \Re^{n\times n}$ and $B_k \in \Re^{n\times p_k}$, $k = 1, 2$, inequality

$$\text{rank}[A_1e^{A_2t}B_1, B_2] \geq \text{rank}[A_1B_1, B_2] \quad (8)$$

holds for almost all $t \in \Re$.

Proof. Denote $\Omega(t) = [A_1e^{A_2t}B_1, B_2]$. Choose a nonsingular sub-matrix G with maximal rank in $\Omega(0) = [A_1B_1, B_2]$. Denote the corresponding sub-matrix of $\Omega(t)$ as $\Delta(t)$, and its determinant as $\delta(t)$. It is standard that all elements of $\Delta(t)$ are linear combinations of the form $t^ke^{\lambda t}$, hence $\delta : \Re \to \Re$ is an analytic function on $\Re$. Because $\delta(0)$ =det$G \neq 0$, the zeros of $\delta(t)$ are isolated points (Kaplan, 1966, Theorem 43). Consequently, $\delta(t) \neq 0$ for almost all $t \in \Re$. Accordingly, for almost all t, $\Delta(t)$ is nonsingular. Therefore,

$$\text{rank}\Omega(t) \geq \text{rank}\Delta(t) = \text{rank}G = \text{rank}[A_1B_1, B_2]$$

for almost all t.

3. MAIN RESULTS

3.1 Geometric criteria

In this subsection, we shall identify the controllable set and the reachable set for switched linear systems.

Theorem 1. For switched linear system (1), the reachable set is

$$\mathcal{R} = \mathcal{V} \quad (9)$$

Proof. We are to design a switching path σ such that each point in $\mathcal{V}$ can be reached from the origin via this switching path.

Let $t_0, \cdots, t_l$ and $i_0 = \sigma(t_0), \cdots, \sigma(t_l)$ denote the switching time/index sequences, respectively. Assume that the switching index sequence is periodic. i.e.,

$$\begin{aligned} i_0 = 1, \quad i_1 = 2, \quad \cdots, \quad i_{m-1} = m, \\ i_m = 1, i_{m+1} = 2, \cdots, i_{2m-1} = m, \cdots \end{aligned} \quad (10)$$

The switching time sequence $t_0, \cdots, t_l$ and the number l are to be designed later.

Let $t_f > t_l$. Let $\mathcal{R}_f$ denote the set of states which are reachable at t_f from the origin at $t_0 = 0$. It can be computed that

$$\begin{aligned} \mathcal{R}_f = e^{A_{i_l}h_l} \cdots e^{A_{i_1}h_1}\mathcal{D}_{i_0} + e^{A_{i_l}h_l} \cdots e^{A_{i_2}h_2}\mathcal{D}_{i_1} \\ + \cdots + e^{A_{i_l}h_l}\mathcal{D}_{i_{l-1}} + \mathcal{D}_{i_l} \end{aligned} \quad (11)$$

where $h_j = t_{j+1} - t_j,\ \ j = 0, 1, \cdots, l-1$ and $h_l = t_f - t_l$.

Since

$$\begin{aligned} & e^{A_{i_l}h_l} \cdots e^{A_2h_1}\mathcal{D}_1 + \cdots + e^{A_{i_l}h_l}\mathcal{D}_{i_{l-1}} + \mathcal{D}_{i_l} = \\ & e^{A_{i_l}h_l}(e^{A_{i_{l-1}}h_{l-1}} \cdots e^{A_2h_1}\mathcal{D}_1 + \cdots + \mathcal{D}_{i_{l-1}}) + \mathcal{D}_{i_l} \end{aligned}$$

it follows from Lemma 2 that

$$\begin{aligned} & \dim(e^{A_{i_l}h_l} \cdots e^{A_2h_1}\mathcal{D}_1 + \cdots + e^{A_{i_l}h_l}\mathcal{D}_{i_{l-1}} + \mathcal{D}_{i_l}) \\ & \quad \geq \dim(e^{A_{i_{l-1}}h_{l-1}} \cdots e^{A_2h_1}\mathcal{D}_1 + \cdots \\ & \quad + e^{A_{i_{l-1}}h_{l-1}}\mathcal{D}_{i_{l-2}} + \mathcal{D}_{i_{l-1}} + \mathcal{D}_{i_l}) \end{aligned} \quad (12)$$

for almost all h_l.

By repeatedly applying Lemma 2, for almost all $h_l, \cdots, h_{l-m+1}$, we have

$$\begin{aligned}
&\dim(e^{A_{i_l}h_l}\cdots e^{A_2h_1}\mathcal{D}_1+\cdots+e^{A_{i_l}h_l}\mathcal{D}_{i_{l-1}}+\mathcal{D}_{i_l})\\
&\geq\dim(e^{A_{i_{\tau_1}}h_{\tau_1}}\cdots e^{A_2h_1}\mathcal{D}_1+\cdots+e^{A_{i_{\tau_1}}h_{\tau_1}}\mathcal{D}_{i_{\tau_1-1}}\\
&\quad+\mathcal{D}_{i_{\tau_1}}+\mathcal{D}_{i_{\tau_1+1}}+\cdots+\mathcal{D}_{i_l})\\
&=\dim(e^{A_{i_{\tau_1}}h_{\tau_1}}\cdots e^{A_2h_1}\mathcal{D}_1+\cdots+\mathcal{D}_{i_{\tau_1}}+\mathcal{V}_1)
\end{aligned}$$

where $\tau_1=l-m$.

It follows from Lemma 2 that

$$\begin{aligned}
&\dim(e^{A_{i_{\tau_1}}h_{\tau_1}}\cdots e^{A_2h_1}\mathcal{D}_1+\cdots+\mathcal{D}_{i_{\tau_1}}+\mathcal{V}_1)\\
&=\dim(e^{A_{i_{\tau_1}}h_{\tau_1}}e^{A_{i_{\tau_1-1}}h_{\tau_1-1}}(e^{A_{i_{\tau_1-2}}h_{\tau_1-2}}\cdots\\
&\times e^{A_2h_1}\mathcal{D}_1+\cdots+e^{A_{i_{\tau_1-2}}h_{\tau_1-2}}\mathcal{D}_{i_{\tau_1-3}}\\
&\quad+\mathcal{D}_{i_{\tau_1-2}})+e^{A_{i_{\tau_1}}h_{\tau_1}}\mathcal{D}_{i_{\tau_1-1}}+\mathcal{D}_{i_{\tau_1}}+\mathcal{V}_1)\\
&\geq\dim(e^{A_{i_{\tau_1}}h_{\tau_1}}(e^{A_{i_{\tau_1-2}}h_{\tau_1-2}}\cdots e^{A_2h_1}\mathcal{D}_1+\cdots\\
&\quad+e^{A_{i_{\tau_1-2}}h_{\tau_1-2}}\mathcal{D}_{i_{\tau_1-3}}+\mathcal{D}_{i_{\tau_1-2}})\\
&\quad+e^{A_{i_{\tau_1}}h_{\tau_1}}\mathcal{D}_{i_{\tau_1-1}}+\mathcal{D}_{i_{\tau_1}}+\mathcal{V}_1)\\
&=\dim(e^{A_{i_{\tau_1}}h_{\tau_1}}(e^{A_{i_{\tau_1-2}}h_{\tau_1-2}}\cdots e^{A_2h_1}\mathcal{D}_1+\cdots\\
&\quad+e^{A_{i_{\tau_1-2}}h_{\tau_1-2}}\mathcal{D}_{i_{\tau_1-3}}+\mathcal{D}_{i_{\tau_1-2}}+\mathcal{D}_{i_{\tau_1-1}})\\
&\quad+\mathcal{D}_{i_{\tau_1}}+\mathcal{V}_1)
\end{aligned}$$

for almost all h_{τ_1-1}.

Proceed by the same reasonings, we have

$$\begin{aligned}
&\dim(e^{A_{i_{\tau_1}}h_{\tau_1}}\cdots e^{A_2h_1}\mathcal{D}_1+\cdots+\mathcal{D}_{i_{\tau_1}}+\mathcal{V}_1)\\
&\geq\dim(e^{A_{i_{\tau_1}}h_{\tau_1}}(e^{A_{i_{\tau_1-m}}h_{\tau_1-m}}\cdots e^{A_2h_1}\mathcal{D}_1\\
&\quad+\cdots+e^{A_{i_{\tau_1-m}}h_{\tau_1-m}}\mathcal{D}_{i_{\tau_1-m-1}}+\mathcal{D}_{i_{\tau_1-m}}\\
&\quad+\cdots+\mathcal{D}_{i_{\tau_1-1}})+\mathcal{D}_{i_{\tau_1}}+\mathcal{V}_1)\\
&=\dim(e^{A_{i_{\tau_1}}h_{\tau_1}}e^{A_{i_{\tau_1-m}}h_{\tau_1-m}}\\
&\quad(e^{A_{i_{\tau_1-m-1}}h_{\tau_1-m-1}}\cdots e^{A_2h_1}\mathcal{D}_1+\cdots\\
&\quad+\mathcal{D}_{i_{\tau_1-m-1}})+e^{A_{i_{\tau_1}}h_{\tau_1}}\mathcal{V}_1+\mathcal{D}_{i_{\tau_1}}+\mathcal{V}_1)
\end{aligned}$$

for almost all $h_j, j=\tau_1-1,\cdots,\tau_1-m+1$.

Continuing the above process gives

$$\begin{aligned}
&\dim(e^{A_{i_{\tau_1}}h_{\tau_1}}\cdots e^{A_2h_1}\mathcal{D}_1+\cdots+\mathcal{D}_{i_{\tau_1}}+\mathcal{V}_1)\\
&\geq\dim(e^{A_{i_{\tau_1}}h_{\tau_1}}e^{A_{i_{\tau_1-m}}h_{\tau_1-m}}\cdots e^{A_{i_{\tau_1-nm}}h_{\tau_1-nm}}\\
&\quad(e^{A_{i_{\tau_1-nm-1}}h_{\tau_1-nm-1}}\cdots e^{A_2h_1}\mathcal{D}_1+\cdots\\
&\quad+\mathcal{D}_{i_{\tau_1-nm-1}})\\
&\quad+e^{A_{i_{\tau_1}}h_{\tau_1}}\cdots e^{A_{i_{\tau_1-nm+m}}h_{\tau_1-nm+m}}\mathcal{V}_1\\
&\quad+\cdots+e^{A_{i_{\tau_1}}h_{\tau_1}}\mathcal{V}_1+\mathcal{D}_{i_{\tau_1}}+\mathcal{V}_1)\\
&=\dim(e^{A_{i_{\tau_1}}h_{\tau_1}}e^{A_{i_{\tau_1-m}}h_{\tau_1-m}}\cdots e^{A_{i_{\tau_1-nm}}h_{\tau_1-nm}}\\
&\quad(e^{A_{i_{\tau_1-nm-1}}h_{\tau_1-nm-1}}\cdots e^{A_2h_1}\mathcal{D}_1+\cdots\\
&\quad+\mathcal{D}_{i_{\tau_1-nm-1}})+e^{A_{i_{\tau_1}}(h_{\tau_1}+\cdots+h_{\tau_1-nm+m})}\mathcal{V}_1\\
&\quad+\cdots+e^{A_{i_{\tau_1}}h_{\tau_1}}\mathcal{V}_1+\mathcal{D}_{i_{\tau_1}}+\mathcal{V}_1)
\end{aligned}\tag{13}$$

for almost all $h_j, \tau_1-mn+1\leq j\leq\tau_1-1, (\tau_1-j)$ mod $m\neq 0$. The relationships $i_j=i_{j+m}, j=1,2,\cdots$ have been used in the last equation.

From Lemma 1, we have

$$\begin{aligned}
&e^{A_{i_{\tau_1}}(\sum_{k=0}^{m-1}h_{\tau_1-km})}\mathcal{V}_1+\cdots+e^{A_{i_{\tau_1}}h_{\tau_1}}\mathcal{V}_1\\
&\qquad=\Gamma_{A_{i_{\tau_1}}}\mathcal{V}_1
\end{aligned}\tag{14}$$

for almost all $h_j, j=\tau_1,\tau_1-m,\cdots,\tau_1-mn$. Accordingly, we can rewrite (13) as

$$\begin{aligned}
&\dim(e^{A_{i_{\tau_1}}h_{\tau_1}}\cdots e^{A_2h_1}\mathcal{D}_1+\cdots+\mathcal{D}_{i_{\tau_1}}+\mathcal{V}_1)\\
&\geq\dim(e^{A_{i_{\tau_1}}h_{\tau_1}}\cdots e^{A_{i_{\tau_1-nm}}h_{\tau_1-nm}}\\
&\quad(e^{A_{i_{\tau_1-nm-1}}h_{\tau_1-nm-1}}\cdots e^{A_2h_1}\mathcal{D}_1+\cdots\\
&\quad+\mathcal{D}_{i_{\tau_1-nm-1}})+\Gamma_{A_{i_{\tau_1}}}\mathcal{V}_1+\mathcal{D}_{i_{\tau_1}})
\end{aligned}\tag{15}$$

By applied Lemma 2 repeatedly, for almost all $h_j, j=\tau_1,\tau_1-m,\cdots,\tau_1-mn$, we have

$$\begin{aligned}
&\dim(e^{A_{i_{\tau_1}}h_{\tau_1}}\cdots e^{A_{i_{\tau_1-nm}}h_{\tau_1-nm}}\\
&\quad(e^{A_{i_{\tau_1-nm-1}}h_{\tau_1-nm-1}}\cdots e^{A_2h_1}\mathcal{D}_1\\
&\quad+\cdots+\mathcal{D}_{i_{\tau_1-nm-1}})+\Gamma_{A_{i_{\tau_1}}}\mathcal{V}_1+\mathcal{D}_{i_{\tau_1}})\\
&\geq\dim(e^{A_{i_{\tau_1-nm}}h_{\tau_1-nm}}(e^{A_{i_{\tau_1-nm-1}}h_{\tau_1-nm-1}}\cdots\\
&\times e^{A_2h_1}\mathcal{D}_1+\cdots+\mathcal{D}_{i_{\tau_1-nm-1}})+\Gamma_{A_{i_{\tau_1}}}\mathcal{V}_1+\mathcal{D}_{i_{\tau_1}})\\
&=\dim(e^{A_{i_{\tau_1-nm}}h_{\tau_1-nm}}e^{A_{i_{\tau_1-nm-1}}h_{\tau_1-nm-1}}\cdots\\
&\times e^{A_2h_1}\mathcal{D}_1+\cdots+e^{A_{i_{\tau_1-nm}}h_{\tau_1-nm}}\mathcal{D}_{i_{\tau_1-nm-1}}\\
&\quad+\mathcal{D}_{i_{\tau_1-nm}}+\Gamma_{A_{i_{\tau_1}}}\mathcal{V}_1)
\end{aligned}\tag{16}$$

where the relationship $\mathcal{D}_{i_{\tau_1}}=\mathcal{D}_{i_{\tau_1-mn}}$ has been used.

Because each of (14) and (16) holds for almost all $h_j, j=\tau_1,\tau_1-m,\cdots,\tau_1-mn$, almost all choice of $h_j, j=\tau_1,\tau_1-m,\cdots,\tau_1-mn$ satisfies (14) and (16) simultaneously.

Continuing this process, we can prove that, for almost all $h_j, j=\tau_1-mn,\cdots,\tau_1-m^2n+1$, we have

$$\begin{aligned}
&\dim(e^{A_{i_{\tau_1}}h_{\tau_1}}\cdots e^{A_2h_1}\mathcal{D}_1+\cdots+\mathcal{D}_{i_{\tau_1}}+\mathcal{V}_1)\\
&\geq\dim(e^{A_{i_{\tau_1-mn}}h_{\tau_1-mn}}\cdots e^{A_2h_1}\mathcal{D}_1+\cdots\\
&\quad+e^{A_{i_{\tau_1-mn}}h_{\tau_1-mn}}\mathcal{D}_{i_{\tau_1-mn-1}}\\
&\quad+\mathcal{D}_{i_{\tau_1-mn}}+\Gamma_{A_{i_{\tau_1}}}\mathcal{V}_1)\\
&\vdots\\
&\geq\dim(e^{A_{i_{\tau_2}}h_{\tau_2}}\cdots e^{A_2h_1}\mathcal{D}_1+\cdots+\mathcal{D}_{i_{\tau_2}}\\
&\quad+\Gamma_{A_{i_{\tau_1}}}\mathcal{V}_1+\cdots+\Gamma_{A_{i_{\tau_1-m+1}}}\mathcal{V}_1)\\
&=\dim(e^{A_{i_{\tau_2}}h_{\tau_2}}\cdots e^{A_2h_1}\mathcal{D}_1+\cdots+\mathcal{D}_{i_{\tau_2}}+\mathcal{V}_2)
\end{aligned}$$

where $\tau_2=\tau_1-m^2n$.

Proceed the above reasonings, we finally have

$$\begin{aligned}
&\dim(e^{A_{i_l}h_l}\cdots e^{A_2h_1}\mathcal{D}_1+\cdots+e^{A_{i_l}h_l}\mathcal{D}_{i_{l-1}}+\mathcal{D}_{i_l})\\
&\geq\dim(e^{A_{i_{\tau_n}}h_{\tau_n}}\cdots e^{A_2h_1}\mathcal{D}_1+\cdots+\mathcal{D}_{i_{\tau_n}}+\mathcal{V})\\
&\geq\dim\mathcal{V}
\end{aligned}\tag{17}$$

where $\tau_n = l - \sum_{k=0}^{n-1} m(mn)^k$.

Let $l \geq \sum_{k=0}^{n-1} m(mn)^k - 1$, then from (5) and (17) it follows that

$$\mathcal{R}_f = \mathcal{V} \tag{18}$$

which implies (9). $\diamondsuit$

By Theorem 1, the controllable set is subspace $\mathcal{V}$. We thus refer to $\mathcal{V}$ as controllable subspace of system (1).

Theorem 2. For switched linear system (1), the controllable set is

$$\mathcal{C} = \mathcal{V} \tag{19}$$

The proof is completely parallel to that of Theorem 1 and hence is omitted.

Corollary 3. For switched linear system (1), the following statements are equivalent

(i) The system is completely controllable;
(ii) The system is completely reachable; and
(iii) $\mathcal{V} = \Re^n$.

Remark 1. For a non-switched linear system (A, B), Corollary 3 degenerate to the well known geometric characterization for controllable subspace (Wonham, 1979)

$$\mathcal{C} = \mathcal{B} + A\mathcal{B} + \cdots + A^{n-1}\mathcal{B}$$

3.2 Switching control design

By Theorems 1 and 2, any states in subspace $\mathcal{V}$ can transfer to each other in finite time. In this subsection, we study the following switching control design problem for switched system (1).

Switching Control Design Problem *Given any two states x_0 and x_f in the controllable subspace $\mathcal{V}$, find a switching path σ and control input u to steer the system from x_0 to x_f in finite time.*

Combining the proof of Theorem 1 and the geometric approach of linear systems (Wonham, 1979), we can formulate a procedure to solve this problem.

From the proof of Theorem 1, we can find a natural number l, positive real numbers $h_1, \cdots, h_l$, and an index sequence $i_0, \cdots, i_l$, such that equation (18) holds. That is

$$e^{A_{i_l} h_l} \cdots e^{A_2 h_1} \mathcal{D}_1 + \cdots + \mathcal{D}_{i_l} = \mathcal{V} \tag{20}$$

Fix a positive real number h_0. Define the switching time sequence as

$$t_0 = 0, t_k = t_{k-1} + h_{k-1}, k = 1, \cdots, l+1 \tag{21}$$

From the proof of Theorem 1.1 in Wonham (1979), for any $k \in M$ and $t > 0$, we have

$$\mathcal{D}_k = \mathrm{Im}\ W_t^k \tag{22}$$

where

$$W_t^k = \int_0^t e^{A_k(t-\tau)} B_k B_k^T e^{A_k^T(t-\tau)} d\tau$$

Combining (20) with (22) leads to

$$e^{A_{i_l} h_l} \cdots e^{A_2 h_1} \mathrm{Im} W_{h_0}^1 + \cdots + \mathrm{Im} W_{h_l}^{i_l} = \mathcal{V} \tag{23}$$

If we can formulate a control input u satisfying the equation

$$\begin{aligned} x_f = x(t_{l+1}) &= e^{A_l h_l} \cdots e^{A_1 h_0} x_0 \\ &+ e^{A_l h_l} \cdots e^{A_2 h_1} \times \int_{t_0}^{t_1} e^{A_1(t_1-\tau)} B_1 u_1(\tau) d\tau \\ &+ \cdots + \int_{t_l}^{t_{l+1}} e^{A_{i_l}(t_{l+1}-\tau)} B_{i_l} u_{i_l}(\tau) d\tau \end{aligned} \tag{24}$$

then the switching control problem will be solved. To this end, consider the piecewise continuous control strategy

$$\begin{aligned} u_{i_k}(t) = B_{i_k}^T e^{A_{i_k}^T(t_{k+1}-t)} a_{k+1},\ t_k \leq t < t_{k+1} \\ k = 0, 1, \cdots, l \end{aligned} \tag{25}$$

where $a_k \in \Re^n$, $k = 1, \cdots, l+1$ are vector variables to be determined.

Combining (24) with (25) gives

$$\begin{aligned} x_f - e^{A_l h_l} \cdots e^{A_1 h_0} x_0 &= e^{A_l h_l} \cdots e^{A_2 h_1} \\ \times \int_{t_0}^{t_1} & e^{A_1(t_1-\tau)} B_1 B_1^T e^{A_1^T(t_1-t)} d\tau a_1 + \cdots \\ + \int_{t_l}^{t_{l+1}} & e^{A_{i_l}(t_{l+1}-\tau)} B_{i_l} B_{i_l}^T e^{A_{i_l}^T(t_{l+1}-t)} d\tau a_{l+1} \end{aligned} \tag{26}$$

This is equivalent to

$$\begin{aligned} x_f - e^{A_l h_l} \cdots e^{A_1 h_0} x_0 = \\ [e^{A_{i_l} h_l} \cdots e^{A_2 h_1} W_{h_0}^1, \cdots, W_{h_l}^{i_l}] a \end{aligned} \tag{27}$$

where $a = [a_1^T, \cdots, a_{l+1}^T]^T$.

As $x_f - e^{A_l h_l} \cdots e^{A_1 h_0} x_0 \in \mathcal{V}$, it follows from (23) that linear equation (27) with unknown a has at least has one solution. Solutions of linear equations (27) can be computed by symbolic or numerical softwares.

4. CONCLUSION

In this paper, detailed controllability and reachability analysis has been carried out for switched linear control systems. It has been proven that, the controllable (reachable) set is exactly the minimal A_k- invariant subspace for $k \in M$ which contains $\sum_{k \in M} \mathcal{B}_k$. The switching control design problem has been also addressed and solved in Wonham's geometric approach. These results set up an elementary and solid framework towards a comprehensive geometric theory for switched linear systems.

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