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COMPARISON OF ANALYSIS METHODS FOR DISCRETIZED NON-LINEAR CONTROL LAWS WITH SMALL CONTROL DELAYS

Guido Herrmann, [1] **Sarah K. Spurgeon, Christopher Edwards**

Control Systems Research Group, University of Leicester, LE1 7RH, U.K., Tel.: +44-116-252 2531, Fax.: +44-116-252 2619, gh17@sun.engg.le.ac.uk, eon@le.ac.uk, ce14@le.ac.uk

Abstract: This paper considers two different approaches for analysis of discretized non-linear control laws subject to small computational delays. Emphasis is given to the problem of robustness analysis for the discretized control. Hsu and Sastry (1987) provided a concept for which an explicit formulae is derived here and a modification is suggested allowing robust closed loop analysis in contrast to the initial approach.

Keywords: controller discretization, robust nonlinear control, Lyapunov analysis

1. INTRODUCTION

Nonlinear sampled data control in application to continuous-time plants has been a research area of great interest during recent years (Clarke *et al.*, 1997; Nešić *et al.*, 1999) since many questions easily analyzed for linear systems still have to be verified for non-linear control. One approach is to consider controllers designed as a discrete control law for a continuous-time system (Nešić *et al.*, 1999). The opposite approach is to discretize a continuous-time control law deriving a sampled-data implementation (Hsu and Sastry, 1987; Clarke *et al.*, 1997; Teel *et al.*, 1998; Herrmann *et al.*, 2000). This approach is the 'fast-sampling'-approach. A small enough sampling time ensures that the sampled-data implementation is stable. However, many approaches so far have concentrated on the issue of existence for this sampling-time and have not considered the issue of robustness or delays in the discretized control due to computational speed. Hsu and Sastry (1987) considered rigorously the problem of computational delays for a class of nonlinear affine systems and non-linear controls. However, Hsu and Sastry (1987) only provided the concept not a constructive formulae for closed loop analysis. Furthermore, Hsu and Sastry (1987) excluded the issue of robustness as strict constraints of Lipschitz continuity for the drift of the non-linear affine system were used for analysis so that classes of uncertainty and disturbances readily described by sector or constant bounds were not easily considered. In contrast, the problem of robustness of discretized controllers has been investigated by Herrmann *et al.* (2000) and Herrmann (2000) considering sector and constant bounds for uncertainty. Thus, this paper derives the formulae for Hsu and Sastry's (1987) analysis concept and also introduces an extension to decrease conservatism. Further, the concept of robustness analysis from Herrmann (2000) is extended to Hsu and Sastry's (1987) method by suggesting a modification which allows robust closed loop analysis to be undertaken. This method and Hsu and Sastry's (1987) original approach are compared numerically for a non-linear sliding-mode based control (Herrmann *et al.*, 1998; Herrmann, 2000) showing that the set of assumptions for the discretization analysis is of importance.

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2. DISCRETIZED, NON-LINEAR CONTROL WITH VARYING COMPUTATIONAL DELAY

The discretization procedure will be introduced using a generic example allowing robust stability of the sampled-data implementation to be investigated. For simplicity, the conditions raised are global but can be easily reformulated for local results considering an invariant set, a suitable vicinity of the origin of the investigated state space.

Assumption 1: For a system with continuous and bounded input gain $\mathrm{g} : \mathbb{R}^n \to \mathbb{R}^n$, $\|\mathrm{g}\| \leq \mathcal{G}$

$$\dot{\mathbf{x}}(t)=\mathsf{f}(t,\mathbf{x})+\mathsf{g}(\mathbf{x})\mathbf{u},\ \mathbf{x}\in\mathbb{R}^n,\ \mathsf{f}:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n \quad (1)$$

a control $\mathbf{u} = \mathbf{u}^{\mathcal{C}}(\mathbf{x}(t))$ exists which stabilizes the system. Further, the usual conditions of Caratheodory (1948) are assumed for $f(t,\mathbf{x}) = \mathsf{f}(t,\mathbf{x})+\mathsf{g}(\mathbf{x})\mathbf{u}^{\mathcal{C}}(\mathbf{x}(t))$ to ensure a solution $\mathbf{x}(t)$ to (1) for $\mathbf{u} = \mathbf{u}^{\mathcal{C}}$.

Assumption 2: A Lyapunov function $\mathcal{V}(t,\mathbf{x})$ exists for the continuous-time control ($c_1,\ c_2,\ c_3,\ c_4 > 0$):

$$\left\|\frac{\delta\mathcal{V}}{\delta\mathbf{x}}\right\| \le c_4\|\mathbf{x}\|,\ c_1\|\mathbf{x}\|^2 \le \mathcal{V}(t,\mathbf{x}) \le c_2\|\mathbf{x}\|^2,$$

$$\mathcal{V}\ge\bar{c}:\ \frac{\delta\mathcal{V}}{\delta t}+\frac{\delta\mathcal{V}}{\delta\mathbf{x}}\left(\mathsf{f}(t,\mathbf{x})+\mathsf{g}(\mathbf{x})\mathbf{u}^{\mathcal{C}}(\mathbf{x})\right)\le -c_3\|\mathbf{x}\|^2 \quad (2)$$

with $\bar{c} \ge 0$ defining a set of ultimate boundedness:

$$\Pi_{\bar{c}+\varepsilon} = \{\mathbf{x}\in\mathbb{R}^n : \mathcal{V}(t,\mathbf{x}) \le \bar{c}+\varepsilon\},\ \varepsilon > 0. \quad (3)$$

Assumption 3:

$$\begin{aligned}&\left\|\mathsf{f}(t,\mathbf{x})+\mathsf{g}(\mathbf{x})\mathbf{u}^{\mathcal{C}}(\mathbf{x})\right\| \le \mathcal{K}_{\mathcal{C}}\|\mathbf{x}\| + \mathcal{K}_{\mathcal{C},c},\\ &\left\|\mathbf{u}^{\mathcal{C}}(\mathbf{x}_1)-\mathbf{u}^{\mathcal{C}}(\mathbf{x}_2)\right\| \le \mathcal{K}_u\|\mathbf{x}_1-\mathbf{x}_2\|,\\ &\qquad \mathcal{K}_u,\ \mathcal{K}_{\mathcal{C},c}\ge 0,\ \mathcal{K}_{\mathcal{C}}>0. \end{aligned} \quad (4)$$

These assumptions, in particular Assumption 3, permit sector and constant bounded uncertainty and disturbances. They allow calculation of a sampling frequency which implies robust stability of a sampled-data implementation of $\mathbf{u}^{\mathcal{C}}$. Herrmann *et al.* (2000) used these assumptions to prove robust closed loop stability of a sampled-data implementation. The alternative approach is to assume an exponentially stabilizing control to prove the *existence* of a suitable sampling frequency of the sampled-data implementation:
Assumption 1b: $f(t,\mathbf{x})$ and $\mathsf{g}(\mathbf{x})$ are continuously differentiable, $\|\mathsf{g}\| < \mathcal{G}$
Assumption 2b: $\mathbf{u}^{\mathcal{C}}(\mathbf{x})$ is exponentially stabilizing
Assumption 3b:

$$\|\mathsf{f}(t,\mathbf{x}_1)-\mathsf{f}(t,\mathbf{x}_2)\| \le \mathcal{K}_{\mathsf{f}}\|\mathbf{x}_1-\mathbf{x}_2\|,\ \mathcal{K}_{\mathsf{f}}\ge 0$$
$$\left\|\mathbf{u}^{\mathcal{C}}(\mathbf{x}_1)-\mathbf{u}^{\mathcal{C}}(\mathbf{x}_2)\right\| \le \mathcal{K}_u\|\mathbf{x}_1-\mathbf{x}_2\|,\ \mathcal{K}_u\ge 0.$$

Assumptions 1b-3b are a subset of the Assumptions 1-3. Thus, Assumptions 1-3 can be derived from 1b-3b. $f(t,0)=0$ follows from Assumption 1b/2b which allows Assumption 3 to be derived from Assumption 2b/3b implying $\mathcal{K}_{\mathcal{C},c}=0$ in (4). Assumption 2 follows with $\bar{c}=0$ of (2) using Khalil (1992, Theorem 3.12) from Assumption 1b/2b. The set of assumptions 1b-3b is the basis for the discretization analysis by Hsu and Sastry (1987). Since the drift $\mathsf{f}(t,\mathbf{x})$ is expected in this case to be Lipschitz, sector and constant bounded uncertainty as for Assumption 3, is not easily incorporated. The permissible class of non-linearities and uncertain systems is limited. The conditions used for the sampled-data implementation of $\mathbf{u}^{\mathcal{C}}$ are:
Assumption 4: State/output measurements are available for discretization of $\mathbf{u}^{\mathcal{C}}(\mathbf{x})$ at well defined time instants t_i for constant sampling time $\tau > 0$:

$$\mathbf{x}(t_i) = \mathbf{x}(t)|_{t=t_i},\ t_i = i\tau,\ i = 0,1,2,3,\ldots.$$

Assumption 5: A discretized control can only be applied after a delay of $\Delta\tau_i$, $0 \le \Delta\tau_i < \tau$, $i \ge 0$:

$$\forall t\in(t_i+\Delta\tau_i, t_{i+1}+\Delta\tau_{i+1}]:\mathbf{u}(t)=\mathbf{u}^{\Delta}(i)=\mathbf{u}^{\mathcal{C}}(\mathbf{x}(t_i)) \quad (5)$$

and $0 \le \frac{\Delta\tau_i}{\tau} \le \chi$, for known $\chi = const.$, $1 > \chi > 0$ for all $i \ge 0$. Hence, the delays $\Delta\tau_i$ of the discretized control are not necessarily identical for each sampling interval. It is also assumed $\left\|\mathbf{u}(t=t_0)-\mathbf{u}^{\mathcal{C}}(\mathbf{x}(t_0))\right\| \le \mathcal{C}_u$ and thus, the initial control $\mathbf{u}(t=t_0)$ at time instant $t=t_0$ is not based on an actual measured value but a known bound $\mathcal{C}_u$ with respect to the desired controller value $\mathbf{u}^{\mathcal{C}}(\mathbf{x}(t_0))$ is known. It can be shown that for such a sampled-data controlled system a solution $\mathbf{x}(t)$ exists in $[0,\infty)$ for *any* $\tau > 0$ and has no finite escape time (Herrmann, 2000, Lemma 9.1).

The basis for deriving robust stability of the closed loop employing the discretized control is a Lyapunov analysis. Suppose $t\in(t_i+\Delta\tau_i, t_{i+1}+\Delta\tau_{i+1}]$, then from (1-2) the Lyapunov analysis gives for $\mathcal{V}(t,\mathbf{x}) \ge \bar{c}$:

$$\begin{aligned}\frac{d\mathcal{V}}{dt} &= \frac{\delta\mathcal{V}}{\delta t}+\frac{\delta\mathcal{V}}{\delta\mathbf{x}}\left(\mathsf{f}(t,\mathbf{x}(t))+\mathsf{g}(\mathbf{x}(t))\mathbf{u}^{\mathcal{C}}(\mathbf{x}(t))\right)\\ &\quad+\frac{\delta\mathcal{V}}{\delta\mathbf{x}}\mathsf{g}(\mathbf{x})(\underbrace{\mathbf{u}^{\Delta}(i)}_{\mathbf{u}^{\mathcal{C}}(\mathbf{x}(t_i))}-\mathbf{u}^{\mathcal{C}}(\mathbf{x}(t))). \quad (6)\\ &\le -c_3\|\mathbf{x}(t)\|^2+\left\|\frac{\delta\mathcal{V}}{\delta\mathbf{x}}\right\|\mathcal{G}\mathcal{K}_u\|\mathbf{x}(t)-\mathbf{x}(t_i)\|. \quad (7)\end{aligned}$$

Hence, a suitable upper bound for $\|\mathbf{x}(t)-\mathbf{x}(t_i)\|$ is required to prove robust stability for the discretized control. In the next Lemma, it is seen that this upper estimate exists.

Lemma 1. For a system described by Assumptions 1,3-5, there exists $\bar{\tau} > 0$ so that:

$$\begin{aligned}&\tau\in(0,\bar{\tau}],\ t\in[t_i+\Delta\tau_i, t_{i+1}+\Delta\tau_{i+1}]:\\ &\|\mathbf{x}(t)-\mathbf{x}(t_i)\| \le \Delta_1\|\mathbf{x}(t)\|+\Delta_2+\Delta_{3/i} \quad (8)\end{aligned}$$

where $\Delta_1 \ge 0$, $\Delta_2 \ge 0$, $\Delta_{3/i} \ge 0$ are continuous in $\tau\in(0,\bar{\tau}]$ and $\lim_{\tau\to 0+}(\Delta_1) = \lim_{\tau\to 0+}(\Delta_2) = \lim_{\tau\to 0+}(\Delta_{3/i})$ and $\lim_{i\to\infty}(\Delta_{3/i}) = 0$.

A partial proof of this Lemma is given in Section 2.1 while a full proof can be found in Herrmann (2000) and Herrmann *et al.* (2000). Employing Lemma 1, it follows for the Lyapunov analysis (7) for $\tau\in(0,\bar{\tau}]$, $\mathcal{V}(t,\mathbf{x}) \ge \bar{c}$ and a κ, $0<\kappa<1$, for $t \ge t_0+\Delta\tau_0$:

$$\begin{aligned}\frac{d\mathcal{V}}{dt} &\le -c_3\|\mathbf{x}\|^2\\ &\quad+c_4\mathcal{G}\mathcal{K}_u\left(\Delta_1\|\mathbf{x}(t)\|^2+\Delta_2\|\mathbf{x}(t)\|+\Delta_{3/i}\|\mathbf{x}(t)\|\right)\\ &\le -(1-\kappa)\frac{c_3}{c_2}\mathcal{V}(t,\mathbf{x})+c_4\mathcal{G}\mathcal{K}_u\frac{\Delta_{3/i}}{\sqrt{c_1}}\sqrt{\mathcal{V}(t,\mathbf{x})}\\ &\quad-\left(\kappa c_3-c_4\mathcal{G}\mathcal{K}_u\left(\Delta_1+\frac{\Delta_2\sqrt{c_2}}{\sqrt{\bar{c}}}\right)\right)\|\mathbf{x}\|^2 \quad (9)\end{aligned}$$

The parameter κ can be used for a compromise between performance and stability of the discretized controller. From Lemma 1 for any arbitrary $\varepsilon > 0$, it is possible to choose an $i > 0$ large enough so that

$$\epsilon_i = \Delta_{3/i}^2\frac{{c_2}^2c_4^2\mathcal{G}^2\mathcal{K}_u^2}{(1-\kappa)^2c_3^2c_1} < \bar{c}+\varepsilon.$$

Furthermore, there must be a $\hat{\tau} > 0$, so that for any $\tau\in(0,\hat{\tau}]$:

$$\left(\kappa c_3-c_4\mathcal{G}\mathcal{K}_u\left(\Delta_1+\frac{\Delta_2\sqrt{c_2}}{\sqrt{\bar{c}}}\right)\right) \ge 0. \quad (10)$$

This implies robust ultimate bounded stability of the discretized control with delayed control action for small enough sampling time τ which is summarized in the following theorem:

Theorem 1. Suppose Assumptions 1-3 are satisfied and a controller discretization scheme as given with Assumptions 4-5 is implemented. Then there is a $\hat{\tau} > 0$ so that for $\tau > 0$, $\tau \leq \hat{\tau}$ the discretized control with a varying controller delay $\Delta\tau_i$ is ultimately bounded with respect to $\Pi_{\bar{c}+\varepsilon}$ (3).

The next section derives an exact formula for Δ_1 based on ideas adopted from Hsu and Sastry (1987).

2.1 *The approach of Hsu & Sastry*

This section gives an explicit formula for (8) derived from the approach of Hsu and Sastry (1987). The non-linear affine system and the control satisfying Assumptions 1b-3b are used which also ensure a $\mathcal{V}(t, \mathbf{x})$ (6) with $c_5 = 0$. Hsu and Sastry (1987) additionally had to assume that

$$\mathsf{f}(\cdot, 0) = 0,\ \mathbf{u}^{\mathcal{C}}(0) = 0 \quad (11)$$

$$t \in [t_0, t_0 + \Delta\tau_0] : \mathbf{u} = 0. \quad (12)$$

The main idea is to show that for the discretized control there is for small enough τ a scalar $\Delta_{HS} > 0$:

$$\left\|\mathbf{x}(t_{(i+1)}) - \mathbf{x}(t_i)\right\| \leq \Delta_{HS} \left\|\mathbf{x}(t_{(i+1)})\right\|,\ i \geq 0 \quad (13)$$

$$\lim_{\tau\to 0+} \Delta_{HS} = 0. \quad (14)$$

This result is then used to show the claim of Lemma 1. The proof may be structured into three steps.

Step I: Employing the principle of complete induction, it can be proved that there is $\Delta_{HS,i}$ for (13-14), where Δ_{HS} is replaced by $\Delta_{HS,i}$. Later it can be shown that there is $\Delta_{HS,i}$ independent of $i \geq 0$:

• *Step 0*: Consider for (1) the interval $t \in [t_0 + \Delta\tau_0, t_1]$ using (12) and Assumption 5:

$$\|\mathbf{x}(t) - \mathbf{x}(t_0)\| = \left\| \int_{t_0}^{t_0+\Delta\tau_0} \mathsf{f}(s, \mathbf{x}(s)) ds + \int_{t_0+\Delta\tau_0}^{t} \left(\mathsf{f}(s, \mathbf{x}(s)) + \mathsf{g}(\mathbf{x}(s))\mathbf{u}^{\mathcal{C}}(\mathbf{x}(t_0))\right) ds \right\|$$

$$\leq \int_{t_0}^{t} \|\mathsf{f}(s, \mathbf{x}(s)) - \mathsf{f}(s, \mathbf{x}(t_0))\| \, ds + \int_{t_0}^{t} \|\mathsf{f}(s, \mathbf{x}(t_0))\| \, ds + \int_{t_0+\Delta\tau_0}^{t} \left\|\mathsf{g}(\mathbf{x}(s))\mathbf{u}^{\mathcal{C}}(\mathbf{x}(t_0))\right\| ds$$

Using Assumption 1, Assumption 3b, (11) and the Gronwall-Bellman Inequality (Khalil, 1992):

$$\|\mathbf{x}(t) - \mathbf{x}(t_0)\| \leq \int_{t_0}^{t} \mathcal{K}_{\mathsf{f}} \|\mathbf{x}(s) - \mathbf{x}(t_0)\| \, ds + \int_{t_0}^{t} \mathcal{K}_{\mathsf{f}} \|\mathbf{x}(t_0)\| \, ds + \int_{t_0+\Delta\tau_0}^{t} \mathcal{G}\mathcal{K}_u \|\mathbf{x}(t_0)\| \, ds \quad (15)$$

$$\leq \tau e^{\tau\mathcal{K}_{\mathsf{f}}} (\mathcal{K}_{\mathsf{f}} + \mathcal{G}\mathcal{K}_u) \|\mathbf{x}(t_0)\|$$

$$\leq \tau e^{\tau\mathcal{K}_{\mathsf{f}}} (\mathcal{K}_{\mathsf{f}} + \mathcal{G}\mathcal{K}_u) (\|\mathbf{x}(t) - \mathbf{x}(t_0)\| + \|\mathbf{x}(t)\|) \quad (16)$$

For small enough τ follows $1 - \tau e^{\tau\mathcal{K}_{\mathsf{f}}} (\mathcal{K}_{\mathsf{f}} + \mathcal{G}\mathcal{K}_u) > 0$ and

$$\|\mathbf{x}(t) - \mathbf{x}(t_0)\| \leq \frac{\tau e^{\tau\mathcal{K}_{\mathsf{f}}} (\mathcal{K}_{\mathsf{f}} + \mathcal{G}\mathcal{K}_u) \|\mathbf{x}(t)\|}{1 - \tau e^{\tau\mathcal{K}_{\mathsf{f}}} (\mathcal{K}_{\mathsf{f}} + \mathcal{G}\mathcal{K}_u)} \quad (17)$$

With the choice of $t = t_1$, the existence of $\Delta_{HS,0}$ is shown.

• *Assumption i*: $\lim_{\tau\to 0+} \Delta_{HS,i} = 0$, $\left\|\mathbf{x}(t_{(i+1)}) - \mathbf{x}(t_i)\right\| \leq \Delta_{HS,i} \left\|\mathbf{x}(t_{(i+1)})\right\|,\ i \geq 0$

• *Step i*: For $t \in [t_{i+1} + \Delta\tau_{i+1}, t_{i+2}]$, the state $\mathbf{x}(t)$ satisfies:

$$\|\mathbf{x}(t) - \mathbf{x}(t_{i+1})\| \leq \int_{t_{i+1}}^{t} \left\|\mathsf{g}(\mathbf{x}(s))\mathbf{u}^{\mathcal{C}}(\mathbf{x}(t_{i+1}))\right\| ds + \int_{t_{i+1}}^{t} \|\mathsf{f}(s, \mathbf{x}(t_{i+1}))\| ds + \int_{t_{i+1}}^{t} \|\mathsf{f}(s, \mathbf{x}(t)) - \mathsf{f}(s, \mathbf{x}(t_{i+1}))\| ds + \int_{t_{i+1}}^{t_{i+1}+\Delta\tau_{i+1}} \left\|\mathsf{g}(\mathbf{x}(s))(\mathbf{u}^{\mathcal{C}}(\mathbf{x}(t_i)) - \mathbf{u}^{\mathcal{C}}(\mathbf{x}(t_{i+1})))\right\| ds \quad (18)$$

With $t \in [t_{i+1} + \Delta\tau_{i+1}, t_{i+2}]$ and the respective Lipschitz constants it follows

$$\|\mathbf{x}(t) - \mathbf{x}(t_{i+1})\| \leq \tau e^{\tau\mathcal{K}_{\mathsf{f}}} (\mathcal{G}\mathcal{K}_u + \mathcal{K}_{\mathsf{f}}) \|\mathbf{x}(t_{i+1})\| + \tau e^{\tau\mathcal{K}_{\mathsf{f}}} \chi\mathcal{G}\mathcal{K}_u \|\mathbf{x}(t_i) - \mathbf{x}(t_{i+1})\|$$

Using *Assumption i*, for small enough τ:

$$\|\mathbf{x}(t) - \mathbf{x}(t_{i+1})\| \leq \tau \frac{e^{\tau\mathcal{K}_{\mathsf{f}}} (\mathcal{G}\mathcal{K}_u + \mathcal{K}_{\mathsf{f}} + \chi\mathcal{G}\mathcal{K}_u\Delta_{HS,i}) \|\mathbf{x}(t)\|}{1 - \tau e^{\tau\mathcal{K}_{\mathsf{f}}} (\mathcal{G}\mathcal{K}_u + \mathcal{K}_{\mathsf{f}} + \chi\mathcal{G}\mathcal{K}_u\Delta_{HS,i})} \quad (19)$$

Thus, $\Delta_{HS,i+1}$ exists and $\lim_{\tau\to 0+} \Delta_{HS,i+1} = 0$ for small enough τ.

• *Conclusion*: Using the principle of complete induction, $\Delta_{HS,i}$ exists for $i \geq 0$ and small enough τ.

Step II: Now the existence of Δ_{HS} is shown. Note that Hsu and Sastry (1987) did not provide this partial result. It is readily seen that $\Delta_{HS,0} \leq \Delta_{HS,i}$ for all $i \geq 0$. Thus, it is sufficient to prove that there is $\Delta_{HS,i} \leq \Delta_{HS}$ satisfying (13-14) or

$$0 < \frac{\tau e^{\tau\mathcal{K}_{\mathsf{f}}} (\mathcal{G}\mathcal{K}_u + \mathcal{K}_{\mathsf{f}} + \chi\mathcal{G}\mathcal{K}_u\Delta_{HS})}{1 - \tau e^{\tau\mathcal{K}_{\mathsf{f}}} (\mathcal{G}\mathcal{K}_u + \mathcal{K}_{\mathsf{f}} + \chi\mathcal{G}\mathcal{K}_u\Delta_{HS})} \leq \Delta_{HS}, \quad (20)$$

$$\lim_{\tau\to 0+} \Delta_{HS} = 0 \quad (21)$$

for small enough $\tau > 0$. If $\mathcal{K}_{\Delta/1} \stackrel{def}{=} e^{\tau\mathcal{K}_{\mathsf{f}}} (\mathcal{G}\mathcal{K}_u + \mathcal{K}_{\mathsf{f}})$, $\mathcal{K}_{\Delta/2} \stackrel{def}{=} e^{\tau\mathcal{K}_{\mathsf{f}}} \chi\mathcal{G}\mathcal{K}_u$, (20) is satisfied if

$$\tau\mathcal{K}_{\Delta/1} + \left(\tau(\mathcal{K}_{\Delta/1} + \mathcal{K}_{\Delta/2}) - 1\right)\Delta_{HS} + \tau\mathcal{K}_{\Delta/2}\Delta_{HS}^2 \leq 0, \quad (22)$$

$$0 \leq 1 - \tau(\mathcal{K}_{\Delta/1} + \mathcal{K}_{\Delta/2}),$$

$$0 \leq \left(\tau(\mathcal{K}_{\Delta/1} + \mathcal{K}_{\Delta/2}) - 1\right)^2 - 4\tau^2\mathcal{K}_{\Delta/1}\mathcal{K}_{\Delta/2}. \quad (23)$$

The fast-sampling constraints of (23) ensure that $\Delta_{HS} > 0$ for (22) exists, where the smallest Δ_{HS} satisfying (22) and (21) is for $\mathcal{K}_{\Delta/2} > 0$

$$\Delta_{HS} = \frac{1-\tau(\mathcal{K}_{\Delta/1}+\mathcal{K}_{\Delta/2})}{2\tau\mathcal{K}_{\Delta/2}} - \sqrt{\frac{(1-\tau(\mathcal{K}_{\Delta/1}+\mathcal{K}_{\Delta/2}))^2}{4\tau^2\mathcal{K}_{\Delta/2}^2} - \frac{\mathcal{K}_{\Delta/1}}{\mathcal{K}_{\Delta/2}}}. \quad (24)$$

Step III: Now Δ_1, Δ_2 and $\Delta_{3/i}$ (8) are derived. It follows from (15) a relation similar to (17) for $t \in [t_0, t_1 + \Delta\tau_1]$:

$$\|\mathbf{x}(t)-\mathbf{x}(t_0)\| \leq \frac{\tau(1+\chi)e^{(1+\chi)\tau\mathcal{K}_\mathsf{f}}(\mathcal{K}_\mathsf{f}+\mathcal{G}\mathcal{K}_u)\|\mathbf{x}(t)\|}{1-\tau(1+\chi)e^{(1+\chi)\tau\mathcal{K}_\mathsf{f}}(\mathcal{K}_\mathsf{f}+\mathcal{G}\mathcal{K}_u)} \quad (25)$$

For the interval $t \in [t_i + \Delta\tau_i, t_{i+1} + \Delta\tau_{i+1}]$ follows with (18) similar to (19) $\|\mathbf{x}(t) - \mathbf{x}(t_i)\| \leq \Delta_1 \|\mathbf{x}(t)\|$, where $\Delta_2 = \Delta_{3/i} = 0$,

$$\Delta_1 = \frac{\tau e^{\tau(1+\chi)\mathcal{K}_\mathsf{f}}((1+\chi)(\mathcal{G}\mathcal{K}_u+\mathcal{K}_\mathsf{f})+\chi\mathcal{G}\mathcal{K}_u\Delta_{HS})}{1-\tau e^{\tau(1+\chi)\mathcal{K}_\mathsf{f}}((1+\chi)(\mathcal{G}\mathcal{K}_u+\mathcal{K}_\mathsf{f})+\chi\mathcal{G}\mathcal{K}_u\Delta_{HS})} \quad (26)$$

and the fast-sampling constraint for (25) is necessary

$$1-\tau e^{\tau(1+\chi)\mathcal{K}_\mathsf{f}}((1+\chi)(\mathcal{G}\mathcal{K}_u+\mathcal{K}_\mathsf{f})+\chi\mathcal{G}\mathcal{K}_u\Delta_{HS}) > 0 \quad (27)$$

It is seen in (15) and (18) that the Lipschitz continuity of the drift f is explicitly exploited.

Remark 1. The approach of Hsu and Sastry (1987) can be adapted to derive the result from Lemma 1 so that *only* Assumptions 1-3 are used *instead* of Assumptions 1b-3b and (11-12). Assumptions 1-3 are sufficient to derive the result from Lemma 1. This can be readily seen from

$$t \in [t_i + \Delta\tau_i, t_{i+1} + \Delta\tau_{i+1}] : \|\mathbf{x}(t) - \mathbf{x}(t_i)\| =$$
$$\left\| \int_{t_i}^{t} \left(\mathsf{f}(s,\mathbf{x}(s)) + \mathsf{g}(\mathbf{x}(s))\mathbf{u}^\mathcal{C}(\mathbf{x}(s))\right) ds + \int_{t_i}^{t} \mathsf{g}(\mathbf{x}(s)) \left(\mathbf{u}^\mathcal{C}(\mathbf{x}(t_i)) - \mathbf{u}^\mathcal{C}(\mathbf{x}(s))\right) ds + \int_{t_i}^{t_i+\Delta\tau_i} \mathsf{g}(\mathbf{x}(s)) \left(\mathbf{u}^\mathcal{C}(\mathbf{x}(t_{i-1})) - \mathbf{u}^\mathcal{C}(\mathbf{x}(t_i))\right) ds \right\|$$
$$\leq \int_{t_i}^{t} (\mathcal{K}_\mathcal{C}\|\mathbf{x}(s)\| + \mathcal{K}_{\mathcal{C},c}) ds + \int_{t_i}^{t} \mathcal{G}\mathcal{K}_u \|\mathbf{x}(s)-\mathbf{x}(t_i)\| ds + \Delta\tau_i\mathcal{G}\mathcal{K}_u\|\mathbf{x}(t_i)-\mathbf{x}(t_{i-1})\|$$
$$\leq \int_{t_i}^{t} (\mathcal{K}_\mathcal{C}+\mathcal{G}\mathcal{K}_u)\|\mathbf{x}(s)-\mathbf{x}(t_i)\| ds + (t-t_i)(\mathcal{K}_\mathcal{C}\|\mathbf{x}(t_i)\|+\mathcal{K}_{\mathcal{C},c}) + \Delta\tau_i\mathcal{G}\mathcal{K}_u\|\mathbf{x}(t_i)-\mathbf{x}(t_{i-1})\| \quad (28)$$

This inequality allows $\Delta_{HS,i}$ and Δ_{HS} to be derived as in *Step I-II* of this section. The full derivation of the expressions for Δ_1, Δ_2 and $\Delta_{3/i}$ is omitted. However, the formula for Δ_1 is

$$\Delta_1 = \frac{\tau e^{\tau(1+\chi)(\mathcal{K}_\mathcal{C}+\mathcal{G}\mathcal{K}_u)}((1+\chi)\mathcal{K}_\mathcal{C}+\chi\mathcal{G}\mathcal{K}_u\Delta_{HS})}{1-\tau e^{\tau(1+\chi)(\mathcal{K}_\mathcal{C}+\mathcal{G}\mathcal{K}_u)}((1+\chi)\mathcal{K}_\mathcal{C}+\chi\mathcal{G}\mathcal{K}_u\Delta_{HS})} \quad (29)$$
$$0 < 1-\tau e^{\tau(1+\chi)(\mathcal{K}_\mathcal{C}+\mathcal{G}\mathcal{K}_u)}((1+\chi)\mathcal{K}_\mathcal{C}+\chi\mathcal{G}\mathcal{K}_u\Delta_{HS})$$

where Δ_{HS} is defined in this case by $\mathcal{K}_{\Delta/1} = e^{\tau(\mathcal{K}_\mathcal{C}+\mathcal{G}\mathcal{K}_u)}\mathcal{K}_\mathcal{C}$, $\mathcal{K}_{\Delta/2} = e^{\tau(\mathcal{K}_\mathcal{C}+\mathcal{G}\mathcal{K}_u)}\chi\mathcal{G}\mathcal{K}_u$ and the relations of (22-24). This is in particular useful for the case when $\bar{c} = \mathcal{K}_{\mathcal{C},c} = 0$ for Assumptions 1-3, which also implies $\Delta_2 = 0$ (Herrmann, 2000).

The next section introduces a non-linear control law (Herrmann *et al.*, 1998) which is used to compare the analysis approaches.

3. A SLIDING-MODE LIKE CONTROL LAW

Linear, uncertain systems are considered

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} + \mathbf{F}(t,\mathbf{x}),\ \mathbf{x} \in \mathbb{R}^n,\ \mathbf{u} \in \mathbb{R}^m \quad (30)$$

where the known matrix pair (A, B) is controllable with B of full rank and $\mathbf{F}(.,.) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ models parametric uncertainties and non-linearities lying in the range space of B. Thus, there is a linear transformation $\tilde{\mathbf{z}} = \tilde{T}\mathbf{z}$ such that (30) becomes:

$$\dot{\mathbf{z}}_1 = \Sigma\mathbf{z}_1 + A_{12}\phi,\ \mathbf{z}_1 \in \mathbb{R}^{n-m} \quad (31)$$
$$\dot{\phi} = (\Theta+\Delta\Theta)\,\mathbf{z}_1 + (\Omega+\Delta\Omega)\,\phi + B_2\mathbf{u},\ \phi \in \mathbb{R}^m \quad (32)$$

where Σ is a stable design matrix and $[\Delta\Theta(t,\mathbf{x})\mathbf{z}_1\ \Delta\Omega(t,\mathbf{x})\phi] = \tilde{T}\mathbf{F}(.,.)$ has known finite bounds $\|\Delta\Theta\| \leq K_\Theta$, and $\|\Delta\Omega\| \leq K_\Omega$. The control has two parts $\mathbf{u}(t) = \mathbf{u}_L^\mathcal{C}(\cdot) + \mathbf{u}_{NL}^\mathcal{C}(\cdot)$. The linear control component $\mathbf{u}_L^\mathcal{C}(\cdot)$ is:

$$\mathbf{u}_L^\mathcal{C}(\mathbf{z}_1(t),\phi(t)) \stackrel{def}{=} -B_2^{-1}\left(\Theta\mathbf{z}_1(t)+(\Omega-\Omega^*)\phi(t)\right). \quad (33)$$

Ω^* is a stable design matrix and P_2 satisfies $P_2\Omega^* + \Omega^{*T}P_2 = -I_m$, where I_m is the $(m \times m)$ identity matrix. Lyapunov functions $V_1(t)$ appropriate for (31) and $V_2(t)$ suitable for (32) are used:

$$V_2(t) \stackrel{def}{=} \frac{1}{2}\phi^T(t)P_2\phi(t),\ V_1(t) \stackrel{def}{=} \frac{1}{2}\mathbf{z}_1{}^T(t)P_1\mathbf{z}_1(t) \quad (34)$$

with $P_1\Sigma + \Sigma^T P_1 = -I_{(n-m)}$. The non-linear control component

$$\mathbf{u}_{NL}^\mathcal{C} \stackrel{def}{=} \begin{cases} \dfrac{-\rho(\mathbf{z}_1(t),\phi(t))B_2^{-1}P_2\phi(t)}{\|P_2\phi(t)\|+\delta(\|\mathbf{z}_1(t)\|+\|\phi(t)\|)} & if\,[\mathbf{z}_1^T\,\phi^T] \neq 0 \\ 0 & if\,[\mathbf{z}_1^T\,\phi^T] = 0 \end{cases} \quad (35)$$

$$\rho(\mathbf{z}_1(t),\phi(t)) \stackrel{def}{=} \gamma_1(\eta_1\|P_2\phi(t)\|+\eta_3\|\mathbf{z}_1(t)\|),\ \gamma_1 > 1 \quad (36)$$

$$\eta_1 \stackrel{def}{=} \max\left(\sup_{\Delta\Omega}\left(\frac{1}{2}\lambda_{max}\{P_2^{-1}\Delta\Omega^T+\Delta\Omega P_2^{-1}\}\right), 0\right) \quad (37)$$

$$\eta_3 \stackrel{def}{=} \omega^2\|P_1A_{12}P_2^{-1}\| + \frac{\omega}{2}\|P_1^{-\frac{1}{2}}\|\|P_2^{-\frac{1}{2}}\| + \sup_{\Delta\Theta}\|\Delta\Theta\| \quad (38)$$

$$\omega \stackrel{def}{=} \frac{\lambda_{max}(P_2^{\frac{1}{2}})\delta/\lambda_{min}(P_1^{\frac{1}{2}})}{(\lambda_{min}(P_2)(\gamma_1-1)-\delta)},\ \delta,\omega \in \mathbb{R}^+ \quad (39)$$

achieves robustness by counteracting the matched uncertainties. For $\omega \in \mathbb{R}^+$ (39) to be positive the constraint $\lambda_{min}(P_2) > \frac{\delta}{\gamma_1 - 1}$ is imposed. The expression $\delta(\|\mathbf{z}_1(t)\| + \|\phi(t)\|)$ in (35) results in a cone shaped layer around $\phi = 0$, which is defined by the relation $V_2 \leq \omega^2 V_1$. It is easily verified from (30), (33-35) and (36) that Assumptions 1 and 3 are satisfied justifying the use of Lemma 1. A proof of stability (Herrmann, 2000) makes it necessary to show that the Lyapunov function $V_1(t)$ in (34) is decreasing as soon as the states have entered the cone shaped layer $V_2 \leq \omega^2 V_1$ which implies that the choice of ω has to be limited by an upper bound (Herrmann, 2000).

3.1 *Discretization of the sliding-mode based control*

The closed loop system using the discretized control $\mathbf{u}_{NL}^{\Delta}$ and $\mathbf{u}_{L}^{\Delta}$ of $\mathbf{u}_{L}^{\mathcal{C}}$ (33) and $\mathbf{u}_{NL}^{\mathcal{C}}$ from (35) for (31-32) can be written for $t \in (t_i + \Delta\tau_i, t_{i+1} + \Delta\tau_{i+1}]$:

$$\begin{aligned}\dot{\mathbf{z}}_1 &= \Sigma \mathbf{z}_1 + A_{12}\phi \\ \dot{\phi} &= \Delta\Theta \mathbf{z}_1 + (\Omega^* + \Delta\Omega)\,\phi + B_2 \mathbf{u}_{NL}^{\mathcal{C}}(\mathbf{z}_1(t), \phi(t)) \\ &\quad + B_2(\mathbf{u}_{NL}^{\Delta}(i) - \mathbf{u}_{NL}^{\mathcal{C}} + \mathbf{u}_{L}^{\Delta}(i) - \mathbf{u}_{L}^{\mathcal{C}}).\end{aligned}$$

A global non-smooth Lyapunov function follows from non-smooth analysis theory (Herrmann, 2000):

$$V(\mathbf{z}_1(t), \phi(t)) = \begin{cases} V_1(\mathbf{z}_1(t)) + kV_2(\phi(t)) & if\ V_2 \geq \omega^2 V_1 \\ (k\omega^2 + 1)V_1(\mathbf{z}_1(t)) & if\ V_2 \leq \omega^2 V_1 \end{cases} \quad (40)$$

where $k > 0$ has to be determined. This non-smooth function is differentiable for almost all t and $\int^t \dot{V}(\mathbf{z}_1(s), \phi(s))ds = (V(\mathbf{z}_1(t), \phi(t)) + const.)$ applies. This allows the non-smooth Lyapunov function to be used as $\mathcal{V}$ within the proof for Theorem 1. Furthermore, the relation $\dot{V} \leq -\tilde{\vartheta} V$, $(\tilde{\vartheta} > 0)$, holds for almost all t. This implies exponential stability of the continuous-time controlled system. For the discretization analysis, $V_1(\mathbf{z}_1(t)) + kV_2(\phi(t))$ may be investigated first. There is $\breve{\vartheta} > 0$ for large enough $k > 0$ and suitable $0 < \kappa < 1$:

$$\breve{\vartheta} = -(1-\kappa)\lambda_{max}\left(\begin{bmatrix} P_1^{-\frac{1}{2}}\Sigma^T P_1^{\frac{1}{2}} + P_1^{\frac{1}{2}}\Sigma P_1^{-\frac{1}{2}} & \dfrac{P_1^{\frac{1}{2}} A_{12} P_2^{-\frac{1}{2}}}{\sqrt{k}} \\ \dfrac{P_2^{-\frac{1}{2}} A_{12}^T P_1^{\frac{1}{2}}}{\sqrt{k}} & -P_2^{-1} \end{bmatrix}\right) \quad (41)$$

It can be shown that for almost all $t \in (t_i + \Delta\tau_i, t_{i+1} + \Delta\tau_{i+1}]$:

$$\begin{aligned}&\dot{V}_1(\mathbf{z}_1(t)) + k\dot{V}_2(\phi(t)) \leq -(V_1(\mathbf{z}_1(t)) + kV_2(\phi(t)))\breve{\vartheta} \\ &+ \underbrace{k\phi^T P_2\left(B_2 \mathbf{u}_{NL}^{\mathcal{C}}(\mathbf{z}_1(t), \phi(t)) + \frac{\rho(\mathbf{z}_1(t), \phi(t))}{\gamma_1}\right)}_{\leq 0\ \ for\ \ V_2 \geq \omega^2 V_1} \\ &+ \kappa \begin{bmatrix} \mathbf{z}_1(t) \\ \phi(t) \end{bmatrix}^T \begin{bmatrix} \dfrac{(\Sigma^T P_1 + P_1 \Sigma)}{2} & \dfrac{P_1 A_{12}}{2} \\ \dfrac{A_{12}^T P_1}{2} & -k\dfrac{I}{2} \end{bmatrix} \begin{bmatrix} \mathbf{z}_1(t) \\ \phi(t) \end{bmatrix} \\ &+ k\phi^T(t) P_2 B_2(\mathbf{u}_{NL}^{\Delta}(i) - \mathbf{u}_{NL}^{\mathcal{C}}(\mathbf{z}_1, \phi) + \mathbf{u}_{L}^{\Delta}(i) - \mathbf{u}_{L}^{\mathcal{C}}(\mathbf{z}_1, \phi)).\end{aligned}$$

The uncertainty with upper bound $\rho(\cdot)$ is compensated by $\mathbf{u}_{NL}^{\mathcal{C}}$ for $V_2 \geq \omega^2 V_1$. From Section 3, Lemma 1 applies and the expression (8) can be represented for $\tilde{\mathbf{z}} = \left[\mathbf{z}_1^T\ \phi^T\right]^T$ implying $\Delta_2 = 0$. Using the Lipschitz constant $\mathcal{K}_u$ of the control $\mathbf{u}_{NL}^{\mathcal{C}} + \mathbf{u}_{L}^{\mathcal{C}}$, it follows for $V_2 \geq \omega^2 V_1$

$$\begin{aligned}&\dot{V}_1(\mathbf{z}_1(t)) + k\dot{V}_2(\phi(t)) \leq -\,(V_1(\mathbf{z}_1(t)) + kV_2(\phi(t)))\,\breve{\vartheta} \\ &+ \sqrt{V_1(\mathbf{z}_1(t)) + kV_2(\phi(t))}\sqrt{2k}\|P_2^{\frac{1}{2}} B_2\|\mathcal{K}_u \Delta_{3/i} \quad (42)\end{aligned}$$

provided the fast-sampling constraint

$$\underbrace{\kappa\lambda_{max}\left(\begin{bmatrix} \dfrac{(\Sigma^T P_1 + P_1\Sigma)}{2} & \dfrac{P_1 A_{12}}{2} \\ \dfrac{A_{12}^T P_1}{2} & -k\dfrac{I}{2} \end{bmatrix}\right)}_{-c_3} + \underbrace{\|kP_2 B_2\|}_{c_4\mathcal{G}}\mathcal{K}_u \Delta_1 \leq 0 \quad (43)$$

can be satisfied for small enough $\tau \leq \hat{\tau}$. This parallels the relation of (10) for $\Delta_2 = 0$. It is possible to find ω and $\bar{\vartheta} > 0$ small enough so that the second part of the Lyapunov analysis (40) for $V_2(\phi(t)) \leq \omega^2 V_1(\mathbf{z}_1(t))$ follows $(k\omega^2 + 1)\dot{V}_1 \leq -(k\omega^2 + 1)\bar{\vartheta} V_1$. The final part of the Lyapunov analysis is implied from non-smooth analysis (Herrmann, 2000):

$$\begin{aligned}\dot{V}(\mathbf{z}_1(t), \phi(t)) &\leq -\min(\breve{\vartheta}, \bar{\vartheta}) V(\mathbf{z}_1(t), \phi(t)) \\ &+ \sqrt{V(\mathbf{z}_1(t), \phi(t))}\sqrt{2k}\|P_2^{\frac{1}{2}} B_2\|\mathcal{K}_u \Delta_{3/i}.\end{aligned}$$

This implies using $\lim_{i\to\infty}\Delta_{3/i} = 0$ that the sampled-data controlled system is asymptotically stable for small enough sampling time $\tau > 0$. An example for the sliding-mode based control will be given next showing characteristics of the discretization approaches.

3.2 *Example of discretized sliding-mode based controls*

The two techniques are now applied to a simple second order system. Consider the nonlinear model of an inverted pendulum, which can be expressed in the form of (30) with

$$A = \begin{bmatrix} 0 & 1 \\ 0.5 & -0.1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{F} = \begin{bmatrix} 0 & 0 \\ 0.5(\mathrm{sinc}(x_1) - 1) & 0 \end{bmatrix}\mathbf{x},$$

$$\mathrm{sinc}(t) = \begin{cases} \dfrac{\sin(t)}{t} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases} \quad (44)$$

where $\mathbf{x}^T = [x_1\ x_2]$. The non-linear controller satisfies $\gamma_1 = 1.6$ and $\eta_1 = 0$ (37) $(\Delta\Omega = 0)$. The poles $\Sigma = \Omega^* = p_1,\ p_1 \in \mathbb{R}^-$ have been chosen to be the same. The respective Lyapunov matrices P_1 and P_2 (34) follow from Σ and Ω^*, which allows δ and ω (39) to be determined using the constraint:

$$\underbrace{-\lambda_{max}\left(P_1\Sigma + \Sigma^T P_1\right)}_{=1} - 2|P_1 A_{12} P_2^{-\frac{1}{2}}|\omega|P_1^{\frac{1}{2}}| = 0.85$$

For the single input, second order systems considered, this constraint is necessary and sufficient so that $\dot{V}_1 \leq -0.85\,|z_1|^2$ $(\tilde{\mathbf{z}}^T = [z_1\ \phi])$ for any state satisfying $V_2 \leq \omega^2 V_1$. This ensures comparability of the control and implies stable performance. The value of κ, $0 < \kappa < 1$, is set as a trade-off to 0.5. Note that no uncertainty or non-linearity with non-Lipschitz character has been introduced for (44) so that the control and also the uncertainty $[\Delta\Theta\ \ \Delta\Omega]$ are globally Lipschitz and Assumptions 2b-3b are satisfied; Hsu and Sastry's (1987) technique is applicable. It has been found simpler to express $\mathcal{K}_u$, c_3, $c_4\mathcal{G}$, $\mathcal{G}$, $\mathcal{K}_{\mathrm{f}}$ and $\mathcal{K}_{\mathcal{C}}$ from Assumptions 1-3 and (43) in terms of the original states $[x_1\ x_2]^T$ instead of $\tilde{\mathbf{z}}$:

$$\begin{aligned}\mathcal{K}_u = \Bigg(&\left(|2p_1 + 0.1| + \left|\frac{\gamma_1\eta_3}{2p_1(-\frac{1}{2p_1} + \delta)}\right| + \left|\frac{\gamma_1\eta_3}{2p_1\delta}\right|\right)^2 \\ &+ \left(\left|-0.5 - p_1^2\right| + \left|\frac{\gamma_1\eta_3}{2(\frac{1}{2} + \delta)}\right| + \left|\frac{\gamma_1\eta_3(-1/2 + \delta(p_1 - 1))}{2p_1(-\frac{1}{2p_1} + \delta)\delta}\right|\right)^2\Bigg)^{\frac{1}{2}},\end{aligned}$$

$$\mathcal{K}_{\mathcal{C}} = \left\|\begin{bmatrix} 0 & 1 \\ -p_1^2 - \dfrac{\gamma_1\eta_3}{2\delta} - \dfrac{0.2172}{2} & 2p_1 + \dfrac{\gamma_1\eta_3}{2p_1\delta} \end{bmatrix}\right\|,$$

$$\mathcal{K}_\mathrm{f} = \frac{\sqrt{226}+\sqrt{26}}{20},\; c_4\mathcal{G} = \frac{k}{2p_1}\sqrt{p_1^2+1},\; \mathcal{G}=1,$$

$$c_3 = \frac{1}{2}\left(\frac{k(p_1^2+1)}{2} - \sqrt{\frac{k^2(p_1^2+1)^2}{4} - k + \frac{1}{p_1^2 4}}\right).$$

These values allow the calculation of the sampling time bounds $\hat{\tau}$ for different values of χ and p_1, so that for any $\tau \leq \hat{\tau}$ the closed loop system is stable. The value of k from the closed loop max-Lyapunov function has been adjusted so that the sampling time is maximal for given χ and p_1. The formulas from (26) for Hsu and Sastry's (1987) original technique and of (29) for the modification of Hsu and Sastry's (1987) approach are used for numerical evaluation.

The robust approach (29) delivers values for $\hat{\tau}$ up to 13% larger (better) than those for the non-robust method of Hsu and Sastry (1987) (Figure 1 and 2).

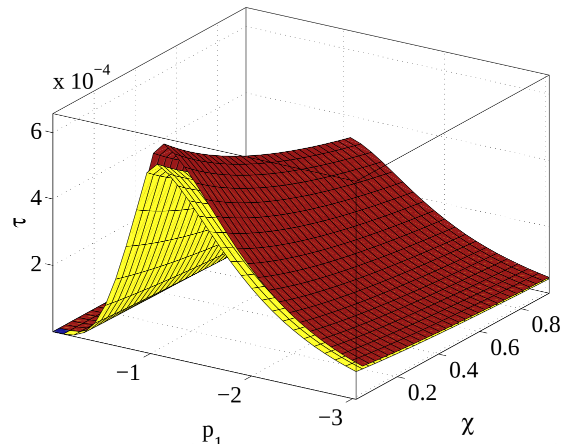

Fig. 1. Bound $\hat{\tau}$ for two approaches, robust modif. meth. of Hsu & Sastry (29) (upper surf.), orig. meth. of Hsu & Sastry (26) (lower surf.)

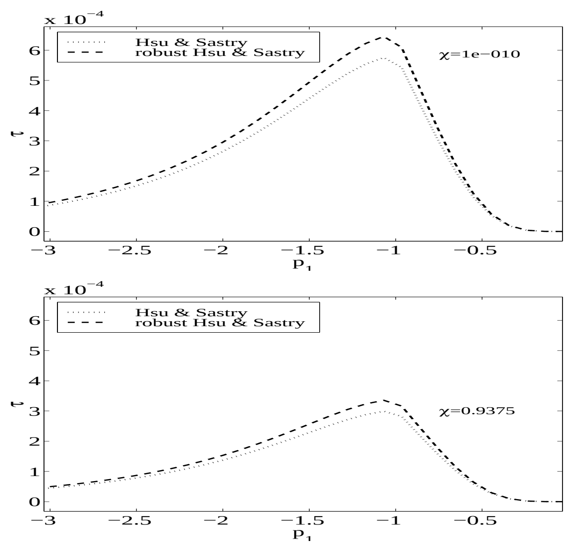

Fig. 2. Sampling time bound $\hat{\tau}$ for two approaches

Note that the non-linear control for the inverted pendulum has been designed to counteract the non-linearity and to achieve sliding-mode based behaviour. Since the method for analysis of robust control laws has taken particular account of parametric uncertainty or non-linearity, the advantage of the extension of the original method of Hsu and Sastry (1987) to the analysis of robust control problems is recognized in the example. The value of $\hat{\tau}$ as a function of χ is decreasing with χ which is also expected from the analysis (Figure 2). Within Herrmann (2000), it has been pointed out that it is important to compensate for the large value of the Lipschitz constant; this can be beneficial for decreasing the bounds for the sampling frequency for the presented case study and is of interest for future research.

4. CONCLUSIONS

Approaches for the analysis of discretized control laws subject to small computational delays have been compared with respect to the issue of robust closed loop stability. For Hsu and Sastry's (1987) original approach, an explicit formulae for closed loop analysis is provided. It has been seen that, with a more general set of assumptions, closed loop analysis of discretized *robust* controllers can be readily carried out complementing ideas of Hsu and Sastry (1987). It has been verified for a numerical example that Hsu and Sastry's (1987) *modified* approach can improve results for the discretization of a robust control which are calculated with Hsu and Sastry's (1987) original set of assumptions.

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