15th Triennial World Congress, Barcelona, Spain

A DISCRIMINATION METHOD OF ROOTS AREA FOR POLYTOPIC POLYNOMIALS

Yoshifumi OKUYAMA, Fumiaki TAKEMORI *

* *Faculty of Engineering, Tottori University*
4-101, Koyama Minami, Tottori, 680-8552 Japan
Phone: +81-857-31-5211, Fax: +81-857-31-0879
E-mail: oku@ike.tottori-u.ac.jp

Abstract: The physical parameters of controlled systems (plants) are uncertain and are accompanied by nonlinearity. The state space equation and the characteristic polynomial of the control system should, therefore, be expressed by an interval set of parameters. This paper examines the robust performance evaluation of that type of control system based on the existing area of characteristic roots (i.e., eigenvalues). In particular, in this paper, a sufficient condition for the roots area which is enclosed by a specified circle on a complex variable plane is given by applying the classic Sturm's theorem (division algorithm) to the four corners of a segment polynomial. The result that is obtained by finite calculations in regard to the coefficients of the segment polynomial, can be extended to general interval polynomials with multiple uncertain parameters. *Copyright ©2002 IFAC*

Keywords: Robust performance; uncertain polynomials; characteristic roots; Sturm's theorem; edge theorem

1. INTRODUCTION

The physical parameters of controlled systems (plants) are uncertain and accompanied by nonlinearity. The state space equation and the characteristic polynomial of the control system should, therefore, be expressed by an interval matrix and an interval (polytopic) polynomial, respectively (Kharitonov, 1979; Ackermann, 1993; Barmish, 1994; Bhattacharyya, 1994). The robust stability and the poles location for that type of control system has been elucidated by many researchers (Bartlett *et al.*, 1988; Soh *et al.*, 1997). Nonetheless, in our previous paper, we tried to apply the classic Sturm's theorem to interval polynomials in regard to the existing area of characteristic roots in a specified contour on a complex variable plane, and derived theorems which correspond to weak-Kharitonov's theorem based on an assumption for the uncertain systems(Okuyama *et al.*, 1999).

In this paper, by applying Sturm's theorem (a division algorithm) to the four corners of a segment polynomial, a sufficient condition will be given for the characteristic roots area which is enclosed by a specified circle on the complex plane. The concept of finite calculations based on the division algorithm in regard to the coefficients of the segment polynomial will be extended to general interval polynomials with multiple uncertain parameters. The robust performance of that type of control system will be evaluated by using the discrimination method.

2. INTERVAL SYSTEMS

The state space equation of controlled systems with uncertainty (and nonlinearity) can be expressed by the interval set parameters as follows:

$$\frac{dx}{dt} = (A + \tilde{\Delta})x, \quad x \in \mathrm{R}^n, \quad A, \tilde{\Delta} \in \mathrm{R}^{n\times n}, \quad (1)$$
$$\tilde{\Delta} = \sum_{i=1}^{m} \tilde{q}_i A_i, \quad \tilde{q}_i \in [q_i^-,\ q_i^+].$$

If the entries of matrix A_i exist in only the k-th row (or column) (i.e., $\mathrm{Row}_{j=k}A_i \neq 0$, $\mathrm{Row}_{j\neq k}A_i = 0$, $\forall j \in \{1, 2, \cdots, n\}$, and $k \in \{1, 2, \cdots, n\}$), the characteristic polynomial can be written as the following form:

$$\tilde{F}(s) = \sum_{i=1}^{m} \tilde{q}_i \left(\sum_{\ell=0}^{n} c_{i,\ell}\ s^{n-\ell} \right), \qquad (2)$$

where $c_{i,\ell}$ is a real constant coefficient and s is a complex variable which corresponds to the eigenvalue of matrix $A + \tilde{\Delta}$. Of course, if $\tilde{q}_i$ is an element of interval set parameters, the eigenvalue (i.e., the characteristic root) must be an element of bounded sets of complex numbers. In other words, if $\tilde{q}_i$ is represented by a set of interval parameters, the eigevalues must be represented as a bounded area of complex numbers.

Since interval polynomial (2) is not the generalized form with respect to the state equation (1), the characteristic polynomial of Eq. (1) can be reduced to the above form. When the entries of matrix A_i exist in two or more rows (columns), interval polynomial (2) can be obtained by applying the interval arithmetic for multiplication, i.e.,

$$\begin{aligned} [q_i^-,\ q_i^+] &= [q_j^-,\ q_j^+] \cdot [q_k^-,\ q_k^+], \\ q_i^- &= \min\{q_j^- q_k^-,\ q_j^- q_k^+,\ q_j^+ q_k^+,\ q_j^+ q_k^-\}, \\ q_i^+ &= \max\{q_j^- q_k^-,\ q_j^- q_k^+,\ q_j^+ q_k^+,\ q_j^+ q_k^-\}. \end{aligned}$$

At any rate, the dynamics of the system will be dependent on the form of interval polynomial (2).

In this paper, the relationship between the mathematical model (2) (i.e.,(1)) and the location of characteristic roots (in the strict sense, the area of the roots) is examined. The discrimination of the roots area based on Eq. (2), however, becomes a considerably complicated problem. Thus, the problem based on the concept of a set of segment polynomials will be analyzed at first.

3. SEGMENT POLYNOMIAL

Let us consider the following segment polynomial (i.e., a polynomial with only one interval set coefficient):

$$\tilde{F}(s) = \sum_{i=1}^{m} \tilde{q}_i \left(\sum_{\ell=0}^{n} c_{i,\ell}\ s^{n-\ell} \right), \qquad (3)$$
$$\tilde{q}_h \in [q_h^-, q_h^+], \quad \tilde{q}_i = q_i, \quad i \neq h, \quad h, i = 1, 2, \cdots, m.$$

Here, a_ℓ and q_i (without a mark) indicate fixed coefficients. The segment polynomial can be rewritten as:

$$\tilde{F}(s) = F(s, \mu) = \mu F^+(s) + (1-\mu) F^-(s), (4)$$

for an arbitrary parameter $\mu \in [0,\ 1]$. The extreme polynomials at the both ends of Eq. (3) are expressed as:

$$F^+(s) = \sum_{i=1}^{m} \tilde{q}_i \left(\sum_{\ell=0}^{n} c_{i,\ell}\ s^{n-\ell} \right), \qquad (5)$$
$$\tilde{q}_h = q_h^+, \quad \tilde{q}_i = q_i, \quad i \neq h,$$

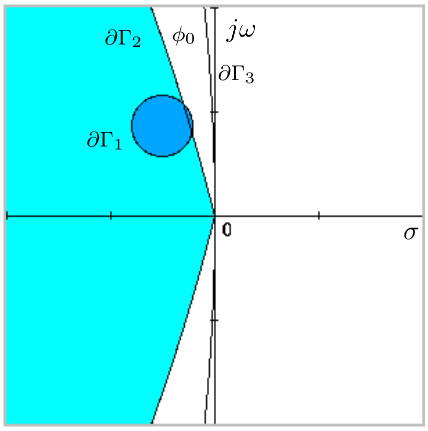

Fig. 1. Sectorial and circular contours and areas.

$$F^-(s) = \sum_{i=1}^{m} \tilde{q}_i \left(\sum_{\ell=0}^{n} c_{i,\ell}\ s^{n-\ell} \right), \qquad (6)$$
$$\tilde{q}_h = q_h^-, \quad \tilde{q}_i = q_i, \quad i \neq h.$$

As for segment polynomial (3), when considering the algebraic equation $\tilde{F}(s) = 0$, segments of the characteristic root locus can be drawn on the s-plane. On the other hand, when considering mapping $\tilde{F}(s)$ for a contour $s \in \partial\Gamma$ as shown in Fig. 1, a set of line segments will be drawn on a complex F-plane.

4. CIRCULAR CONTOUR

In this paper, the following circular contour $\partial\Gamma$ (as shown in Fig. 1) is considered:

$$s = \rho e^{j\theta} + \sigma_0 + j\omega_0. \quad (\theta : -\pi \to \pi). \qquad (7)$$

Here, ρ, (σ_0, ω_0) and θ are the radius, the center and the angle of rotation for the specified circle, respectively.

Any point s on circular contour (7) can also be written by the following rational function of real variable α:

$$s = \frac{u + jv\alpha}{1 - j\alpha}, \qquad (8)$$

where $u = \rho + \sigma_0 + j\omega_0$ and $v = \rho - \sigma_0 - j\omega_0$. Here, α can be correspondingly considered as follows:

$$\alpha = \tan(\theta/2). \qquad (9)$$

Circular contours (7) and (8) includes the following extreme case.

(a) In Eqs. (7) and (8), when we consider

$$\rho = R, \quad \sigma_0 = -R \ \ (\text{or} \ \ \sigma_0 = R), \quad \omega_0 = \epsilon$$

and $R \to \infty$, $\epsilon \to 0$, the specified area becomes a left half (or right half) s-plane ($\partial\Gamma_3$ as shown in Fig. 1) In this case, a part of the circular contour approaches the imaginary axis on the s-plane, provided that variable ω is considered in a certain limited range $-\omega_c < \omega < \omega_c$, where

ω_c is a cutoff frequency. Thus, variable α will become a small number, i.e.,

$$\alpha = \frac{\omega}{2R} \ll 1.$$

(b) Similarly, when the parameters are chosen as

$$\rho = R, \quad \sigma_0 = -R, \quad \omega_0 = R\tan\phi_0, \quad R \to \infty,$$

the specified area becomes a problem of a sectorial area in the left half s-plane ($\partial\Gamma_2$ as shown in Fig. 1).

As mentioned above, since Eqs. (7) and (8) can include most of the problems, the following shows the discrimination of the number of roots in a circular area.

When applying the above transformation (8) to Eq. (3), the following numerator polynomial for real variable α can be obtained:

$$(1-j\alpha)^n \tilde{F}(s) = \sum_{i=1}^{m} \tilde{q}_i \left(\sum_{\ell=0}^{n} c_{i,\ell}(u+jv\alpha)^{n-\ell}(1-j\alpha)^{\ell} \right). \tag{10}$$

Since Eq. (10) is a polynomial with complex coefficients, it can be written as the following form:

$$\tilde{\Phi}(j\alpha) = (1-j\alpha)^n \tilde{F}(s) = \tilde{P}(\alpha) + j\tilde{Q}(\alpha), \tag{11}$$

where

$$\tilde{P}(\alpha) = \tilde{a}_{0,0}\alpha^n + \cdots + \tilde{a}_{0,n-1}\alpha + \tilde{a}_{0,n}, \tag{12}$$
$$\tilde{Q}(\alpha) = \tilde{b}_{0,0}\alpha^n + \cdots + \tilde{b}_{0,n-1}\alpha + \tilde{b}_{0,n}. \tag{13}$$

Coefficients in Eqs. (12) and (13) can be calculated from the expansion of Eq. (10).

5. FOUR CORNER POINTS PROBLEM

By using expression (4), Eq. (11) is rewritten as follows:

$$\begin{aligned}\tilde{\Phi}(j\alpha) &= \Phi(j\alpha,\mu) = (1-j\alpha)^n(\mu F^+(s) + (1-\mu)F^-(s)) \\ &= (\mu P^+(\alpha) + (1-\mu)P^-(\alpha)) + j(\mu Q^+(\alpha) + (1-\mu)Q^-(\alpha)), \\ &\mu \in [0,\ 1]\end{aligned} \tag{14}$$

The real and imaginary parts of Eq. (14) correspond to those of Eq. (11), i.e.,

$$\tilde{P}(\alpha) = \mu P^+(\alpha) + (1-\mu)P^-(\alpha),$$
$$\tilde{Q}(\alpha) = \mu Q^+(\alpha) + (1-\mu)Q^-(\alpha).$$

Here, the extreme polynomials are expressed as follows:

$$\begin{aligned} P^+(\alpha) &= a^+_{0,0}\alpha^n + \cdots + a^+_{0,n-1}\alpha + a^+_{0,n}, \\ Q^+(\alpha) &= b^+_{0,0}\alpha^n + \cdots + b^+_{0,n-1}\alpha + b^+_{0,n}, \end{aligned} \tag{15}$$

$$\begin{aligned} P^-(\alpha) &= a^-_{0,0}\alpha^n + \cdots + a^-_{0,n-1}\alpha + a^-_{0,n}, \\ Q^-(\alpha) &= b^-_{0,0}\alpha^n + \cdots + b^-_{0,n-1}\alpha + b^-_{0,n}. \end{aligned} \tag{16}$$

Thus, the following four corner points (vertices)

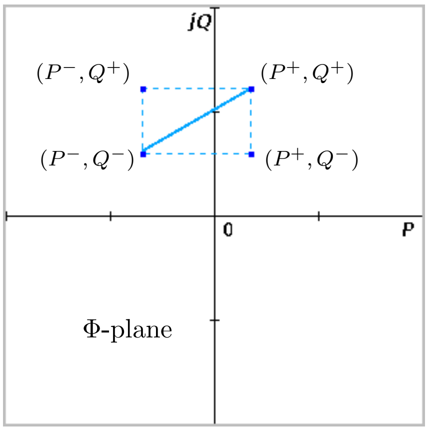

Fig. 2. Four corner points and rectangle.

can be given, and a rectangle together with a line segment (edge) can be drawn in the Φ-plane as shown in Fig. 2:

$$\mathbf{V}_1 = (P^+,\ Q^+), \quad \mathbf{V}_2 = (P^-,\ Q^-),$$
$$\mathbf{V}_3 = (P^-,\ Q^+), \quad \mathbf{V}_4 = (P^+,\ Q^-),$$

where the latter two points are additional ones. (In these expressions, note that polynomials P, Q and coefficients $a_{0,\ell}$, $b_{0,\ell}$ with superscript $+$ do not always denote larger values than those with superscript $-$). Then, the following four pairs of polynomials can be defined for $i = 1, 2, 3, 4$:

$$P^{(i)}(\alpha) = a^{(i)}_{0,0}\alpha^n + \cdots + a^{(i)}_{0,n-1}\alpha + a^{(i)}_{0,n}, \tag{17}$$
$$Q^{(i)}(\alpha) = b^{(i)}_{0,0}\alpha^n + \cdots + b^{(i)}_{0,n-1}\alpha + b^{(i)}_{0,n}, \tag{18}$$

where

$$\begin{aligned} P^{(1)}(\alpha) &= P^+(\alpha), & Q^{(1)}(\alpha) &= Q^+(\alpha), \\ P^{(2)}(\alpha) &= P^-(\alpha), & Q^{(2)}(\alpha) &= Q^-(\alpha), \\ P^{(3)}(\alpha) &= P^-(\alpha), & Q^{(3)}(\alpha) &= Q^+(\alpha), \\ P^{(4)}(\alpha) &= P^+(\alpha), & Q^{(4)}(\alpha) &= Q^-(\alpha). \end{aligned}$$

As an expression of polynomials with complex coefficients, the following can be given:

$$\Phi^{(i)}(j\alpha) = P^{(i)}(\alpha) + jQ^{(i)}(\alpha), \quad (i = 1, 2, 3, 4) \tag{19}$$

(Note that as for the edges in the F-plane, two additional polynomials with constant coefficients cannot be determined in general).

As for a polynomial with complex coefficients (19), the following lemma can be shown by using Sturm's theorem.

[Lemma-1] When coefficient ratios

$$\frac{b^{(i)}_{0,0}}{a^{(i)}_{1,1}}, \ \frac{b^{(i)}_{1,1}}{a^{(i)}_{2,2}}, \ \cdots, \ \frac{b^{(i)}_{n-1,n-1}}{a^{(i)}_{n,n}} \tag{20}$$

are calculated for an extreme polynomial $\Phi^{(i)}(\alpha)$ $(i = 1, 2)$, the number of ratios (20) to be negative μ is equal to the number of characteristic roots for the polynomial in the specified circle. In these

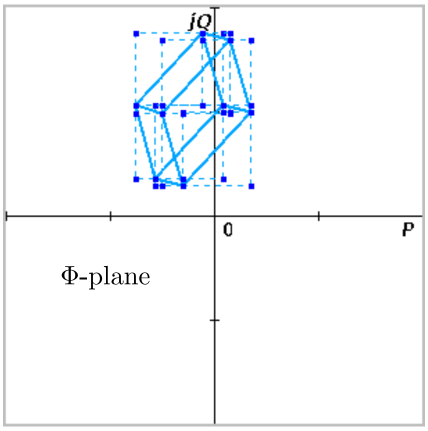

Fig. 3. Parallelotope and rectangles.

ratios (20), $a^{(i)}_{q,q}$, $b^{(i)}_{q-1,q-1}$ $(q = 1, 2, \cdots, n)$ are calculated by using a division algorithm as shown in APPENDIX.

(Proof) This lemma is a necessary and sufficient condition in regard to the existing area of characterisitic roots for the fixed polynomial. In the appendix, the proof will be given systematically by using Sturm's theorem. □

Based on the above premise, the following theorem is obtained as to the above four pairs of polynomials:

[Theorem-1] If the number of ratios (20) to be negative is not changed for the four corner polynomials, the dynamic system that is characterized by segment polynomial Eq. (3) has a robust performance in regard to the invariance of the number of characteristic roots in the specified circle. When considering only one root, for instance, a dominant root, the circle $\partial\Gamma$ (i.e., disc Γ) gives a sufficient condition for the characterisic root area of the dynamic system with an interval set parameter.

(Proof) This theorem is a sufficient condition in regard to the existing area of characterisitic roots for the segment polynomial. The proof is obvious from the zero exclusion of the Kalitonov-like rectangle that is composed of the four corner points (17) and (18). That is, any edge of the rectangle does not pierce the origin. As a natural consequence, the line segment in the Φ-plane and also in the F-plane does not pierce the origin (Bartlett, 1988). □

6. MULTIPLE UNCERTAINTIES

Theorem-1 can also be applied to control systems with multiple uncertainties such as the characteristic polynomials which are shown in Eq. (2) in general. When complex variable s is fixed (frozen), a view of (hyper)polyhedron (a parallelotope) is drawn on the Φ-plane as shown in Fig 3. As for polynomials expressed by Eq. (2), the number of vertices is 2^m, and the number of edges becomes $m \cdot 2^{m-1}$. Obviously, the number of additional vertces is given by $2 \times m \cdot 2^{m-1}$. Thus, the number of total vertices which should be checked for interval polynomial (2) is given by

$$2^m + 2m \cdot 2^{m-1} = (m + 1) \cdot 2^m. \qquad (21)$$

Based on the above premise, the following theorem is derived as to the above number (21) of the corner polynomials:

[Theorem-2] If the number of ratios (20) to be negative is not changed for all the corner polynomials (21), the dynamic system that is characterized by interval polynomial Eq. (2) has a robust performance in regard to the invariance of the number of characteristic roots in the specified circle. When considering only one root, for instance, a dominant root, the circle $\partial\Gamma$ (i.e., disc Γ) gives a sufficient condition for the characterisic root area of the dynamic system with interval set parameters.

(Proof) This theorem is a sufficient condition in regard to the existing area of characterisitic roots for the interval polynomial. The proof is obvious from the result in Theorem-1 in which any edge of the rectangles does not pierce the origin. Consequently, as mentioned in the proof of Theorem-1, any edge of the parallelotope (a set of line segments) in the Φ-plane and in the F-plane does not pierce the origin. □

Although the above results (Theorem-1,2) are only a sufficient condition (i.e., sufficient for edge theorems), the discrimination method proposed in this paper will be useful in robust control systems design(Okuyama *et al.*, 2000).

7. NUMERICAL EXAMPLES

[Example-1] Consider an interval system expressed by Eq. (1) as:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -2 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -2 & 0 \end{bmatrix}, \text{ and } \tilde{q}_1 \in [-0.2,\ 0.2].$$

The characteristic polynomial can be given by a segment polynomial as follows:

$$\tilde{F}(s) = s^3 + 2s^2 + 2s + 1 + [-0.2,\ 0.2](2s + 1). \quad (22)$$

When a circle with a center of $(-0.5, 0.8j)$ and a radius of $r = 0.3$ was specified as shown in Fig. 4, the line segments, four corner points and rectangles are drawn in the Φ-plane as shown in Fig. 5. As is obvious from the figure, the series of four corner points (rectangles) well exclude the origin.

The number of the characteristic roots in the specified circle becomes $\mu = 1$ for all the corner polynomials $\Phi^{(i)}(j\alpha)$, $(i = 1, 2, 3, 4)$. With

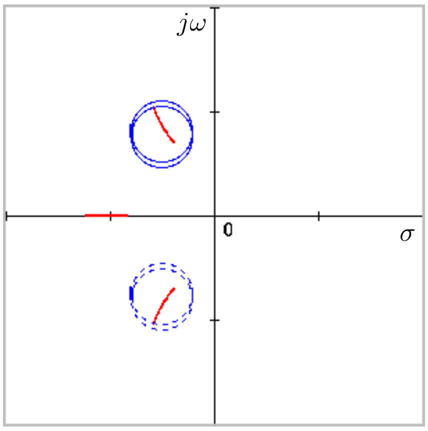

Fig. 4. Circles and segments of root locus for Example-1.

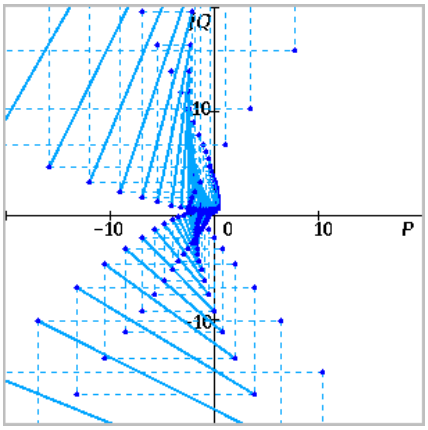

Fig. 5. Rectangles in the Φ-plane for Example-1.

respect to pairs of polynomials $(P^+(\alpha), Q^+(\alpha))$, $(P^-(\alpha), Q^-(\alpha))$, $(P^-(\alpha), Q^+(\alpha))$, $(P^+(\alpha), Q^-(\alpha))$, ratios (20) are calculated, e.g.,

$$\frac{b_{0,0}^{(1)}}{a_{1,1}^{(1)}} = \frac{0.544}{1.214}, \quad \frac{b_{1,1}^{(1)}}{a_{2,2}^{(1)}} = \frac{-0.760}{-0.507}, \quad \frac{b_{2,2}^{(1)}}{a_{3,3}^{(1)}} = \frac{0.093}{-1.123},$$

$$\frac{b_{0,0}^{(3)}}{a_{1,1}^{(3)}} = \frac{0.304}{1.402}, \quad \frac{b_{1,1}^{(3)}}{a_{2,2}^{(3)}} = \frac{0.445}{0.211}, \quad \frac{b_{2,2}^{(3)}}{a_{3,3}^{(3)}} = \frac{-3.26}{1.68}.$$

The number of roots in the specified circle did not change. However, when a circle with a center of $(-0.5, 0.75j)$ and a radius of $r = 0.3$ was specified, $\mu = 0$ appeared in the case of $i = 1$.

[Example-2] Consider the same system as described in Example-1, but there are three interval set parameters:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -2 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \tilde{q}_1 \in [-0.2,\ 0.2],$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix}, \quad \tilde{q}_2 \in [-0.1,\ 0.1],$$

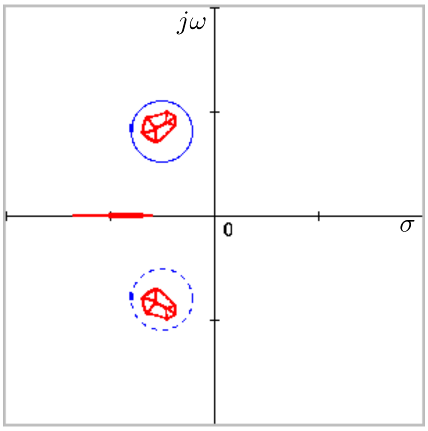

Fig. 6. Circles and root areas for Example-2.

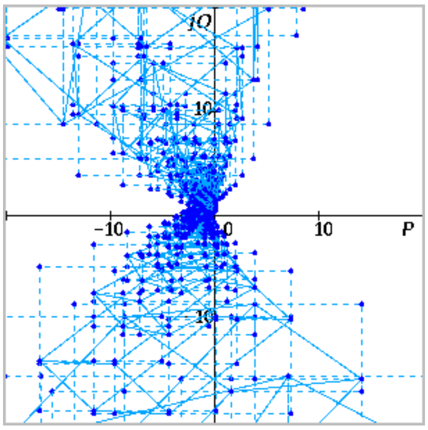

Fig. 7. Rectangles in the Φ-plane for Example-2.

$$A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad \tilde{q}_3 \in [-0.1,\ 0.1],$$

$$\tilde{F}(s) = s^3 + 2s^2 + 2s + 1 + [-0.1,\ 0.1]2s^2 + [-0.1,\ 0.1]2s + [-0.2,\ 0.2]. \tag{23}$$

When a circle with a center of $(-0.5, 0.85j)$ and a radius of $r = 0.3$ is specified as shown in Fig. 6, the parallelotopes and rectangles with $(m + 1) \cdot 2^m = 32$ corners are drawn in the Φ-plane as shown in Fig. 7. The series of 32 corner points (sets of rectangles) also excludes the origin well. The number of roots in the specified circle did not change. That is, the robust performance was guaranteed in regard to the existing area of dominant roots for the interval system.

8. CONCLUSIONS

This paper examined the robust performance evaluation of a control system with interval set parameters. In particular, in this paper, a sufficient condition for the roots area which is enclosed by a specified circle on the s-plane was given by applying Sturm's theorem to the four corners of a segment polynomial. The concept of finite calculations based on the division algorithm in regard to the coefficients of the segment polynomial was extended to general interval polynomials with multiple uncertain parameters.

9. APPENDIX

When notations $f_0^{(i)}(\alpha) = P^{(i)}(\alpha)$ and $f_1^{(i)}(\alpha) = Q^{(i)}(\alpha)$ $(i = 1, 2, 3, 4)$ are used for the corner polynomials (17) and (18), the following division algorithm can be executed (Takagi, 1965):

$$\begin{aligned} f_{2\kappa-2}^{(i)}(\alpha) &= f_{2\kappa-1}^{(i)}(\alpha) q_{2\kappa-1}^{(i)}(\alpha) - f_{2\kappa}^{(i)}(\alpha), \\ f_{2\kappa-1}^{(i)}(\alpha) &= f_{2\kappa}^{(i)}(\alpha) q_{2\kappa}^{(i)}(\alpha) - f_{2\kappa+1}^{(i)}(\alpha), \\ &\kappa = 1, 2, \cdots, n. \end{aligned} \tag{24}$$

If $f_0^{(i)}(\alpha)$ and $f_1^{(i)}(\alpha)$ are of the n-th order for α, $f_2^{(i)}(\alpha), f_3^{(i)}(\alpha), \cdots, f_{2n}^{(i)}$ are expressed as:

$$\begin{aligned} & f_2^{(i)}(\alpha) = a_{1,1}^{(i)}\alpha^{n-1} + \cdots + a_{1,n}^{(i)} \\ & f_3^{(i)}(\alpha) = b_{1,1}^{(i)}\alpha^{n-1} + \cdots + b_{1,n}^{(i)} \\ & \quad \cdots \\ & f_{2n-2}^{(i)}(\alpha) = a_{n-1,n-1}^{(i)}\alpha + a_{n-1,n}^{(i)} \\ & f_{2n-1}^{(i)}(\alpha) = b_{n-1,n-1}^{(i)}\alpha + b_{n-1,n}^{(i)} \\ & f_{2n}^{(i)} = a_{n,n}^{(i)}. \end{aligned} \tag{25}$$

Here, each coefficient can be given by the following sequential operations:

$$\begin{aligned} & a_{1,p}^{(i)} = b_{0,p}^{(i)} \left(\frac{a_{0,0}^{(i)}}{b_{0,0}^{(i)}} \right) - a_{0,p}, \\ & b_{1,p}^{(i)} = a_{1,p+1}^{(i)} \left(\frac{b_{0,0}^{(i)}}{a_{1,1}^{(i)}} \right) - b_{0,p}, \\ & \quad (p = 1, 2, \cdots, n) \\ & \quad \cdots \\ & a_{q,p}^{(i)} = b_{q-1,p}^{(i)} \left(\frac{a_{q-1,q-1}^{(i)}}{b_{q-1,q-1}^{(i)}} \right) - a_{q-1,p}^{(i)}, \\ & b_{q,p}^{(i)} = a_{q,p+1}^{(i)} \left(\frac{b_{q-1,q-1}^{(i)}}{a_{q,q}^{(i)}} \right) - b_{q-1,p}^{(i)}, \\ & \quad (p = q, \cdots, n) \\ & \quad \cdots \\ & a_{n,n}^{(i)} = b_{n-1,n}^{(i)} \left(\frac{a_{n-1,n-1}^{(i)}}{b_{n-1,n-1}^{(i)}} \right) - a_{n-1,n}^{(i)}, \\ & \quad (a_{q,n+1}^{(i)} = 0). \end{aligned} \tag{26}$$

Thus, argument change $2\mu\pi$ for polynomial $\tilde{F}(s)$ becomes $(2\mu - n)\pi$ for $\tilde{\tilde{\Phi}}(\alpha)$ by adding change $-n\pi$ in the argument of $(1-j\alpha)^n$. When $P^{(i)}/Q^{(i)}$ (or $-Q^{(i)}/P^{(i)}$) is considered, the number of sign changes which cross zero for $\alpha : -\infty \to +\infty$ is $n - 2\mu$.

If the number of sign changes which cross the zero of $f_0^{(i)}(\alpha)/f_1^{(i)}(\alpha)$ for $\alpha : \alpha_1 \to \alpha_2$ is expressed as $N^{(i)}(\alpha_1, \alpha_2)$ and the number of sign changes of sequence $f_0^{(i)}(\alpha),\ f_1^{(i)}(\alpha),\ \cdots,\ f_{2n}^{(i)}$ is expressed as $V^{(i)}(\alpha)$, the following relationship is obtained:

$$N^{(i)}(\alpha_1, \alpha_2) = V^{(i)}(\alpha_1) - V^{(i)}(\alpha_2). \tag{27}$$

Since the condition is $N^{(i)}(-\infty, +\infty) = n - 2\mu$,

$$V^{(i)}(-\infty) - V^{(i)}(+\infty) = n - 2\mu \tag{28}$$

is obtainable. The condition of Eq. (28) corresponds to observing whether the following ratios are negative or not (The details were described in (Okuyama *et al.*, 2001)):

$$\lim_{\alpha \to +\infty} \frac{f_1^{(i)}(\alpha)}{|\alpha| f_2^{(i)}(\alpha)}, \quad \cdots, \quad \lim_{\alpha \to +\infty} \frac{f_{2n-1}^{(i)}(\alpha)}{|\alpha| f_{2n}^{(i)}(\alpha)}.$$

Suppose that the number of negative ratios is N and the number of positive ratios is P. Thus, $P - N = n - 2\mu$ can be obtained from Eq. (28). Since $P + N = n$, $N = \mu$ is given.

10. REFERENCES

Kharitonov, V.L. (1979). Asymptotic Stability of an Equilibrium Position of a Family of Systems of Linear Differential Equations. *Differential Equations*, **14**, pp. 1483-1485.

Ackermann, J. (1993). *Robust Control Systems with Uncertain Physical Parameters.* Springer Verlag.

Barmish, B. R. (1994). *New Tools for Robustness of Linear Systems.* Macmillan.

Bhattacharyya, S. P., H. Chapellat and L. H. Keel (1996). *Robust Control, The Parametric Approach.* Prentice Hall.

Bartlett, A.C., C.V. Hollot and L. Huang (1988). Root Location of an Entire Polytope of Polynomials: it suffices to check the edges. *Mathematics of Control, Signal and Systems*, **1**, pp. 61-71.

Soh, Y.C., R.J. Evans, I.R. Petersen and R.E. Betz (1997). Robust Pole Placement. *Automatica*, **27**, pp. 711-715.

Okuyama, Y., F. Takemori and H. Chen (1999). Discriminance of Characteristic Roots Area for Interval Systems, *Proc. of the 14th IFAC World Congress* (Eds. H-F Chen *et al.*). Pergamon Press, **G**, pp. 13-18.

Okuyama, Y. and F. Takemori (2000). Evaluation of Robust Performance for Interval Systems Based on Characteristic Roots Area. *Robust Control Design 2000* (Eds. V. Kucera and M. Sebek), Pergamon Press, **1**, pp. 101-106.

Takagi, T. (1965). *Lecture Notes in Algebra* (Daisugaku Kougi). Kyoritsu Shuppan (in Japanese).

Okuyama, Y. and F. Takemori (2001). Robust Performance Evaluation for Interval Systems Based on Characteristic Roots Area. *Proc. of the 2001 American Control Conference*, Arlington, USA, pp. 2963-2969.