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MODEL VALIDATION FOR IQC UNCERTAIN SYSTEMS WITH FIXED INITIAL CONDITIONS [1]

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Abstract: The paper considers a model validation problem for a class of uncertain systems in which the uncertainty is described by an integral quadratic constraint and the uncertain system has zero initial condition. This leads to a method for model validation which is based around a robust Kalman filter type structure.

Keywords: Uncertain Systems; Model Validation; Robust Kalman Filter; Uncertainty Modelling.

1. INTRODUCTION

An important idea in the field of robust control theory is the use of uncertain system models to represent no only the nominal behaviour of a system but also the uncertainty in the dynamic model. An important class of uncertain system models are those in which the uncertainty is modelled via an Integral Quadratic Constraint (IQC). This class of uncertain systems also allows for the tractable solution to problems of minimax optimal guaranteed cost control and set valued state estimation; e.g., see (Petersen *et al.*, 2000; Petersen and Savkin, 1999).

This paper considers a problem of characterizing the set of possible input-output pairs for a given IQC uncertain system model defined over a finite time horizon. The solution to this problem allows one to solve a model validation problem in which it is desired to determine if a given uncertain system model can be invalidated by a measured input-output pair. This process of model validation plays an important role in the construction of an uncertain system model for a given physical process in that it enables the uncertainty bound to be set in such a way that the uncertain system model covers all measured input-output pairs and yet is not so large as to lead to excessive conservatism within the uncertain system model.

Uncertain system model validation problems of the type considered in this paper have previously been considered in the papers (Savkin and Petersen, 1996; Savkin and Petersen, 1997). A critical distinction between the model validation problem considered in this paper and the model validation problem considered in the papers (Savkin and Petersen, 1996; Savkin and Petersen, 1997) is that in this paper, the initial condition of the uncertain system is assumed to be zero whereas in the papers (Savkin and Petersen, 1996; Savkin and Petersen, 1997), the initial condition was assumed to be unknown but bounded as part of the IQC. This is important since in many practical finite horizon model validation experiments, the system is known to be initially at rest. However, extending the results of (Savkin and Petersen, 1996; Savkin and Petersen, 1997) to the case of zero initial conditions introduces a number of technical problems in that the free endpoint optimal tracking problems considered in (Savkin and Petersen, 1996; Savkin and Petersen, 1997) must be replaced by fixed endpoint optimal tracking problems.

An alternative interpretation of the main result of this paper is that it provides a behavioural description of an IQC uncertain system. That is, it describes the set of possible input-output pairs for the uncertain system.

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This behavioural description of an IQC uncertain system has been applied in the papers (Petersen, 2001*b*; Petersen, 2001*a*) to study equivalence relations and inclusion relations between pairs of uncertain systems.

2. PROBLEM STATEMENT

We consider an uncertain system described by the state equations on the finite time interval $[0,\ T]$:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t); \quad x(0) = 0;\\ z(t) &= Kx(t) + Gu(t);\\ y(t) &= Cx(t) + v(t) \end{aligned} \tag{1}$$

where $x(t) \in \mathbf{R}^n$ is the *state*, $w(t) \in \mathbf{R}^p$ and $v(t) \in \mathbf{R}^l$ are the *uncertainty inputs*, $u(t) \in \mathbf{R}^h$ is the *control input*, $z(t) \in \mathbf{R}^q$ is the *uncertainty output* and $y(t) \in \mathbf{R}^l$ is the *measured output*.

System Uncertainty The uncertainty in the above system is required to satisfy the following Integral Quadratic Constraint (IQC). Let $d > 0$ be a given constant, and let Q and R be given positive definite weighting matrices. We will consider uncertainty inputs $w(\cdot)$ and $v(\cdot)$ and initial conditions $x(0)$ such that

$$\begin{aligned}&\int_0^T [w(t)'Qw(t) + v(t)'Rv(t)]dt\\ &\leq d + \int_0^T \|z(t)\|^2 dt. \end{aligned} \tag{2}$$

Here $\|\cdot\|$ denotes the standard Euclidean norm. Further discussion concerning this case of uncertain systems can be found in (Savkin and Petersen, 1995; Savkin and Petersen, 1996; Savkin and Petersen, 1997; Petersen *et al.*, 2000; Petersen and Savkin, 1999).

Definition 1. Let $u_0(\cdot)$ and $y_0(\cdot)$ be given vector functions. The input-output pair $[u_0(\cdot), y_0(\cdot)]$ is said to be *realizable* for the uncertain system (1), (2) if there exist $[x(\cdot), w(\cdot), v(\cdot)]$ satisfying conditions (1), (2) with $u(t) = u_0(t)$ and $y(t) = y_0(t)$.

Definition 2. The uncertain system (1), (2) is said to be *verifiable* if the set of all realizable pairs $[u_0(\cdot), y_0(\cdot)]$ is not whole space $\mathbf{L}_2[0,T]$.

Model Validation Problem We will consider the following problem: Given an input-output pair $[u_0(\cdot), y_0(\cdot)]$, determine if this pair is realizable for the uncertain system (1), (2).

3. FIXED ENDPOINT OPTIMAL CONTROL

Our solution to the above model validation problem involves a certain fixed endpoint optimal control problem. In this section, we develop a solution to this fixed endpoint optimal control problem which will be used in our main result presented in the next section.

We consider a fixed endpoint optimal control problem:

$$\phi(x_0) = \inf_u \int_0^T (x'Q_o x + u'R_o u)dt$$

subject to

$$\dot{x} = A_o x + B_o u; \quad x(0) = x_0 \tag{3}$$

and $x(T) = 0$. It is assumed that $R_o > 0$ and Q_o can be written $Q_o = C_o'C_o - K_o'K_o$. Also, it is assumed that the pair (A_o, B_o) is controllable.

Our solution to this fixed endpoint optimal control problem will involve the following Riccati Differential Equation (RDE) which is solved backwards in time:

$$\begin{aligned}\dot{P} &= A_o P + PA_o' + PQ_o P - B_o R_o^{-1} B_o';\\ P(T) &= 0. \end{aligned} \tag{4}$$

Lemma 3. There exists a $\delta > 0$ such that the RDE (4) has a solution $P(t) > 0$ on $[T-\delta, T)$. Furthermore, $P(t)$ is monotone decreasing on $[T-\delta, T)$.

The proof of this lemma will be given in the full version of the paper.

Using the above lemma, we can now define a matrix function $X(t)$ as follows: $X(t) = P(t)^{-1}$ for $t \in [T-\delta, T)$ and for $t < T-\delta$, $X(t)$ is the solution to the following RDE which is solved backwards in time:

$$\begin{aligned}-\dot{X} &= XA_o + A_o'X + Q_o - XB_o R_o^{-1} B_o' X;\\ X(T-\delta) &= P(T-\delta)^{-1}. \end{aligned} \tag{5}$$

If the RDE (5) has no finite escape time on $[0, T-\delta]$, then $X(t)$ is defined on $[0,T]$. If the RDE (5) has a finite escape time at $\gamma \in [0, T-\delta]$, then $X(t)$ is defined on $(\gamma, T]$. Note that it follows from this definition that $X(t)$ satisfies the RDE (5) over the whole range for which it is defined. Moreover, the fact that $P(t)$ is monotone decreasing implies that $X(t)$ is monotone increasing.

Lemma 4. Suppose that the above fixed endpoint optimal control problem is such that $\phi(x_0) \geq 0$ for all x_0. Then the RDE (5) does not have a finite escape time on $[0,T)$ and $\phi(x_0) = x_0'X(0)x_0$.

The proof of this lemma will be given in the full version of the paper.

From the above lemmas, we immediately obtain the following theorem.

Theorem 5. If $\phi(x_0) \geq 0$ for all x_0, then the RDE (4) does not have a finite escape time on $(0,T]$ and $P(t) > 0$ for all $t \in (0,T)$. Furthermore, there exists an $\bar{\varepsilon} > 0$ such that the RDE (5) with boundary condition $X_\varepsilon(T) = \frac{I}{\varepsilon}$ has a solution $X_\varepsilon(t)$ on $[0,T]$ and $\lim_{\varepsilon\to 0} X_\varepsilon(t) = P(t)^{-1} > 0$ for all $t \in (0,T)$.

4. THE MAIN RESULT

Our solution to the model validation problem of Section 2 will be given in terms of the following RDE which is solved forward in time:

$$\begin{aligned}\dot{P} &= AP + PA' + P[K'K - C'RC]P \\ &\quad + B_1 Q^{-1} B_1'; \quad P(0) = 0. \end{aligned} \tag{6}$$

Also, our solution to the model validation problem involves a filter defined by the state equations

$$\begin{aligned}\dot{\hat{x}}(t) &= \left[A + P(t)[K'K - C'RC]\right]\hat{x}(t) \\ &\quad + P(t)C'Ry_0(t) + [P(t)K'G + B_2]u_0(t); \\ \hat{x}(0) &= 0 \end{aligned} \tag{7}$$

and a quadratic functional defined by

$$\rho[u_0(\cdot), y_0(\cdot)] \triangleq \int_0^T \left[\begin{array}{l} \|(K\hat{x}(t) + Gu_0(t))\|^2 \\ -(C\hat{x}(t) - y_0(t))'R(C\hat{x}(t) - y_0(t)) \end{array} \right] dt. \tag{8}$$

Note that the filter (7) takes the form of a robust Kalman filter; e.g, see (Petersen and Savkin, 1999).

Theorem 6. Consider the uncertain system (1), (2). Then the following statements hold:

(i) If the system is verifiable then the solution $P(t)$ to Riccati equation (6) is defined on the interval $[0,T)$ and satisfies $P(t) \geq 0$ for $t \in (0,T)$.

(ii) Suppose the uncertain system is verifiable and let $u_0(t)$ and $y_0(t)$ be given input and output signals. Then, the pair $[u_0(\cdot), y_0(\cdot)]$ is realizable if and only if the quantity $\rho[u_0(\cdot), y_0(\cdot)]$ defined in (7), (8) satisfies $\rho[u_0(\cdot), y_0(\cdot)] \geq -d$.

PROOF. (i) Given an input-output pair $[u_0(\cdot), y_0(\cdot)]$, we have by Definition 1 that this pair is realizable if and only if there exist vector functions $x(\cdot), w(\cdot)$ and $v(\cdot)$ satisfying (1) and such that the constraint (2) holds. Now substitution of the equation $y_0(t) = C(t)x(t) + v(t)$ into (2) implies that the pair is realizable if and only if there exist a vector $x_T \in \mathbf{R}^n$ and an uncertainty input $w(\cdot) \in \mathbf{L}_2[0,T]$ such that $x(0) = 0$ and $J[x_T, w(\cdot)] \leq d$. Here $J[x_T, w(\cdot)]$ is defined by

$$J[x_T, w(\cdot)] \triangleq \int_0^T \left(\begin{array}{l} w(t)'Qw(t) - \|(Kx(t) + Gu_0(t))\|^2 \\ +(y_0(t) - Cx(t))'R(y_0(t) - Cx(t)) \end{array} \right) dt \tag{9}$$

and $x(\cdot)$ is the solution to the state equation

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u_0(t); \quad x(T) = x_T \tag{10}$$

with uncertainty input $w(\cdot)$.

Now it follows from Definition 2, that the system (1), (2) is verifiable if and only if there exists a pair $[u_0(\cdot), y_0(\cdot)]$ such that

$$J[x_T, w(\cdot)] > d > 0 \tag{11}$$

for all $x_T \in \mathbf{R}^n$ and for all $w(\cdot) \in \mathbf{L}_2[0,T]$ such that the corresponding solution to (10) satisfies $x(0) = 0$. We now consider the functional $J_0[x_T, w(\cdot)] = J[x_T, w(\cdot)]$ with $u_0(\cdot) \equiv 0$ and $y_0(\cdot) \equiv 0$; i.e.,

$$J_0[x_T, w(\cdot)] = \int_0^T \left(\begin{array}{l} w(t)'Qw(t) - \|(Kx(t)\|^2 \\ +x(t)'C'RCx(t) \end{array} \right) dt.$$

Then, $J_0[x_T, w(\cdot)]$ is a homogeneous quadratic functional of $[x_T, w(\cdot)] \in \mathbf{R}^n \times \mathbf{L}_2[0,T]$. Also, note that the quantity $J[x_T, w(\cdot)]$ will be a quadratic function of $[x_T, w(\cdot), x_0, u_0(\cdot), y_0(\cdot)]$. In particular, for the given pair $[u_0(\cdot), y_0(\cdot)]$, $J[x_T, w(\cdot)]$ will be a non-homogeneous quadratic functional of $[x_T, w(\cdot)]$. Thus, if the pair $[u_0(\cdot), y_0(\cdot)]$ is realizable, then it follows from (11) that

$$J[x_T, w(\cdot)] \geq 0$$

for all $[x_T, w(\cdot)]$ in the following subspace corresponding to the constraint $x(0) = 0$:

$$\left\{ \begin{array}{l} [x_T, w(\cdot)] \in \mathbf{R}^n \times \mathbf{L}_2[0,T] : \\ e^{-AT}x_T - \int_0^T e^{-A\tau} B_1 w(\tau) d\tau = 0 \end{array} \right\}. \tag{12}$$

Hence, it follows from (11) that the homogeneous part of this quadratic functional must be non-negative for all $[x_T, w(\cdot)]$ in the subspace (12). That is

$$J_0[x_T, w(\cdot)] \geq 0 \tag{13}$$

for all $x_T \in \mathbf{R}^n$ and for all $w(\cdot) \in \mathbf{L}_2[0,T]$ such that the corresponding solution to the state equation

$$\dot{x}(t) = Ax(t) + B_1 w(t); \quad x(T) = x_T \tag{14}$$

satisfies $x(0) = 0$. Given any $x_T \in \mathbf{R}^n$, we now define $V(x_T)$ to be the value of the following constrained optimal control problem:

$$V(x_T) \triangleq \inf_{w(\cdot) \in \mathbf{L}_2[0,T]} J_0[x_T, w(\cdot)] \tag{15}$$

subject to (14) and $x(0) = 0$. It follows from (13) that $V(x_T) \geq 0$ for all $x_T \in \mathbf{R}^n$.

We now apply the Kalman decomposition to the system (14). That is, we assume, without loss of generality that the matrices A and B_1 can be partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}; \quad B_1 = \begin{bmatrix} B_{11} \\ 0 \end{bmatrix} \tag{16}$$

where the pair (A_{11}, B_{11}) is controllable. Also, the matrices K, C and the state vector x are correspondingly partitioned as

$$K = [K_1 \;\; K_2];$$
$$C = [C_1 \;\; C_2];$$
$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{17}$$

Now for any x_T of the form $x_T = \begin{bmatrix} x_{T1} \\ 0 \end{bmatrix}$ and for any $w(\cdot) \in \mathbf{L}_2[0,T]$, the corresponding solution to (14) satisfies $x_2(t) \equiv 0$. Also, $x_1(t)$ is the solution to the state equation

$$\dot{x}_1(t) = A_{11}x_1(t) + B_{11}w(t); \quad x_1(T) = x_{T1}. \tag{18}$$

Hence, the corresponding value of $V(x_T)$ is equal to the value of the following reduced dimension constrained optimal control problem:

$$V(x_T) = V_1(x_{T1}) = \inf_{w(\cdot)\in\mathbf{L}_2[0,T]} J_{01}[x_{T1}, w(\cdot)] \tag{19}$$

subject to (18) and $x_1(0) = 0$. Here

$$J_{01}[x_{T1}, w(\cdot)] = \int_0^T \left(\begin{array}{l} w(t)'Qw(t) - \|K_1x_1(t)\|^2 \\ +x_1(t)'C_1'RC_1x_1(t) \end{array} \right) dt. \tag{20}$$

The constrained optimal control problem (15) can be regarded as an optimal control problem of the form considered in Theorem 5 but operating in reverse time. Hence, using that fact that $V_1(x_{T1}) \geq 0$ for all x_{T1}, it follows from Theorem 5 that the RDE

$$\begin{aligned} \dot{P}_1 = {} & A_{11}P_1 + P_1A_{11}' + P_1[K_1'K_1 - C_1'RC_1]P_1 \\ & +B_{11}Q^{-1}B_{11}'; \quad P_1(0) = 0. \end{aligned} \tag{21}$$

does not have a finite escape time on $[0,T)$ and $P_1(t) > 0$ for all $t \in (0,T)$. From this it is straightforward to verify that the matrix

$$P(t) = \begin{bmatrix} P_1(t) & 0 \\ 0 & 0 \end{bmatrix} \geq 0$$

satisfies the RDE (6) on [0,T). This completes the proof of (i).

(ii) We showed above that a pair $[u_0(\cdot), y_0(\cdot)]$ is realizable if and only if there exist a vector $x_T \in \mathbf{R}^n$ and an uncertainty input $w(\cdot) \in \mathbf{L}_2[0,T]$ such that the corresponding solution to (10) satisfies $x(0) = 0$ and $J[x_T, w(\cdot)] \leq d$. For a given pair $[u_0(\cdot), y_0(\cdot)]$, this leads us to consider the following reverse-time fixed endpoint optimal tracking problem:

$$W(x_T) \stackrel{\Delta}{=} \inf_{w(\cdot)\in\mathbf{L}_2[0,T]} J[x_T, w(\cdot)] \tag{22}$$

subject to (10) and $x(0) = 0$. Thus, the pair $[u_0(\cdot), y_0(\cdot)]$ will be realizable if and only if there exists an x_T such that $W(x_T) \leq d$.

To solve the tracking problem (22), we let $\bar{x}(t)$ be the solution to the state equation

$$\dot{\bar{x}}(t) = A\bar{x}(t) + B_2u_0(t); \quad \bar{x}(0) = 0$$

and define $\tilde{x}(t) = x(t) - \bar{x}(t)$. Then

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + B_1w(t); \quad \tilde{x}(T) = \tilde{x}_T \tag{23}$$

where $\tilde{x}_T = x_T - \bar{x}(T)$. Also $\tilde{x}(0) = x(0) - \bar{x}(0) = 0$. Furthermore, the cost function $J[x_T, w(\cdot)]$ can be rewritten in terms of $\tilde{x}(t)$ as

$$\begin{aligned} & J[x_T, w(\cdot)] \\ & = \tilde{J}[\tilde{x}_T, w(\cdot)] \\ & = \int_0^T \left(\begin{array}{l} w'Qw - \|K(\tilde{x}+\bar{x}) + Gu_0\|^2 \\ +(y_0 - C(\tilde{x}+\bar{x}))'R(y_0 - C(\tilde{x}+\bar{x})) \end{array} \right) dt \end{aligned}$$

Thus, the optimal tracking problem (22) is equivalent to the following optimal tracking problem

$$W(x_T) = \tilde{W}(\tilde{x}_T) \stackrel{\Delta}{=} \inf_{w(\cdot)\in\mathbf{L}_2[0,T]} \tilde{J}[\tilde{x}_T, w(\cdot)] \tag{24}$$

subject to (23) and $\tilde{x}(0) = 0$. In this tracking problem, the vector functions $y_0(\cdot)$, $u_0(\cdot)$ and $\bar{x}(\cdot)$ are all treated as reference inputs.

The tracking problem (24) can be simplified by introducing a state partition as in (16), (17). Then, if we write $\tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$, it follows from $\tilde{x}(0) = 0$ and (23) that $\tilde{x}_2(t) \equiv 0$ and $\tilde{x}_1(t)$ satisfies the state equation

$$\dot{\tilde{x}}_1(t) = A_{11}\tilde{x}_1(t) + B_{11}w(t); \quad \tilde{x}(T) = \tilde{x}_{T1}. \tag{25}$$

Thus, $\tilde{x}_T$ must be of the form $\tilde{x}_T = \begin{bmatrix} \tilde{x}_{T1} \\ 0 \end{bmatrix}$ for $\tilde{W}(\tilde{x}_T)$ to be finite. For such values of $\tilde{x}_T$, we obtain

$$\begin{aligned} & \tilde{J}[\tilde{x}_T, w(\cdot)] = \\ & \tilde{J}_1[\tilde{x}_{T1}, w(\cdot)] = \\ & \int_0^T \left[\begin{array}{l} w'Qw - \|K_1\tilde{x}_1 + K\bar{x} + Gu_0\|^2 + \\ (y_0 - C_1\tilde{x}_1 - C\bar{x})'R(y_0 - C_1\tilde{x}_1 - C\bar{x}). \end{array} \right] dt \end{aligned}$$

and the optimal tracking problem (24) is equivalent to the following reduced dimension tracking problem:

$$\tilde{W}(\tilde{x}_T) = \tilde{W}_1(\tilde{x}_{T1}) \stackrel{\Delta}{=} \inf_{w(\cdot)\in\mathbf{L}_2[0,T]} \tilde{J}_1[\tilde{x}_{T1}, w(\cdot)] \tag{26}$$

subject to (25) and $\tilde{x}_1(0) = 0$. Thus, the pair $[u_0(\cdot), y_0(\cdot)]$ will be realizable if and only if there exists an $\tilde{x}_{T1}$ such that $\tilde{W}_1(\tilde{x}_{T1}) \leq d$.

In the proof of part (i) above, we used Theorem 5 to conclude that the RDE (21) has a solution $P_1(t) > 0$ on $(0,T)$. Hence, $X_1(t) \stackrel{\Delta}{=} P_1(t)^{-1}$ satisfies the RDE

$$\begin{aligned} -\dot{X}_1(t) = {} & X_1(t)A_{11} + A_{11}'X_1(t) \\ & +X_1(t)B_{11}Q^{-1}B_{11}'X_1(t) \\ & +K_1'K_1 - C_1'RC_1 \end{aligned} \tag{27}$$

on $(0,T)$. Also, it also follows from Theorem 5 that there exists an $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon}]$, the RDE (27) with initial condition $X_1(0) = \frac{I}{\varepsilon}$ has a solution $X_1^{\varepsilon}(t)$ on $[0,T]$. Furthermore $X_1^{\varepsilon}(t) \to X_1(t) = P_1(t)^{-1} > 0$ as $\varepsilon \to 0$ for all $t \in (0,T)$. Moreover, it follows from the continuity of the matrix function $X_1(\cdot)$ that $X_1(T) \geq 0$.

For each of the solutions $X_1^\varepsilon(t)$ to RDE (27), we associate a corresponding free endpoint optimal tracking problem:

$$\tilde{W}_1^\varepsilon(\tilde{x}_{T1}) \triangleq \inf_{w(\cdot)\in\mathbf{L}_2[0,T]} \tilde{J}_1^\varepsilon[\tilde{x}_{T1}, w(\cdot)] \tag{28}$$

subject to (25). Here

$$\begin{aligned}
&\tilde{J}_1^\varepsilon[\tilde{x}_{T1}, w(\cdot)] \triangleq \\
&\frac{\|\tilde{x}_1(0)\|^2}{\varepsilon} + \\
&\int_0^T \left[\begin{array}{l} w'Qw - \|K_1\tilde{x}_1 + K\bar{x} + Gu_0\|^2 + \\ (y_0 - C_1\tilde{x}_1 - C\bar{x})'R(y_0 - C_1\tilde{x}_1 - C\bar{x}) \end{array} \right] dt.
\end{aligned}$$

Now as in (Savkin and Petersen, 1996), we can write

$$\begin{aligned}
\tilde{W}_1^\varepsilon(\tilde{x}_{T1}) = (\tilde{x}_{T1} - \hat{x}_1^\varepsilon(T))'X_1^\varepsilon(T)(\tilde{x}_{T1} - \hat{x}^\varepsilon(T)) \\
-\rho_1^\varepsilon[u_0(\cdot), y_0(\cdot)]
\end{aligned} \tag{29}$$

where

$$\begin{aligned}
\dot{\hat{x}}_1^\varepsilon(t) = &\left[A_{11} + (X_1^\varepsilon(t))^{-1}[K_1'K_1 - C_1'RC_1]\right]\hat{x}_1^\varepsilon(t) \\
&+(X_1^\varepsilon(t))^{-1}[K_1'K - C_1'RC]\bar{x}(t) \\
&+(X_1^\varepsilon(t))^{-1}C_1'R(t)y_0(t) \\
&+(X_1^\varepsilon(t))^{-1}K_1'Gu_0(t); \quad \hat{x}_1^\varepsilon(0) = 0
\end{aligned} \tag{30}$$

and

$$\begin{aligned}
&\rho_1^\varepsilon[u_0(\cdot), y_0(\cdot)] \triangleq \\
&\int_0^T \left[\begin{array}{l} \|(K\bar{x} + K_1\hat{x}_1^\varepsilon + Gu_0)\|^2 - \\ (C\bar{x} + C_1\hat{x}_1^\varepsilon - y_0)'R(C\bar{x} + C_1\hat{x}_1^\varepsilon - y_0) \end{array} \right] dt;
\end{aligned} \tag{31}$$

see also (Bertsekas and Rhodes, 1971; Lewis, 1995). Therefore, it follows as in the proof of Theorem II.4.2 of (Clements and Anderson, 1978) that $\tilde{W}_1(\tilde{x}_{T1})$ in (26) satisfies $\tilde{W}_1(\tilde{x}_{T1}) = \lim_{\varepsilon\to 0}\tilde{W}_1^\varepsilon(\tilde{x}_{T1})$. Hence, taking the limit as $\varepsilon \to 0$ in (29), (30) and (31), we obtain

$$\begin{aligned}
\tilde{W}_1(\tilde{x}_{T1}) = (\tilde{x}_{T1} - \hat{x}_1(T))'X_1(T)(\tilde{x}_{T1} - \hat{x}(T)) \\
-\rho_1[u_0(\cdot), y_0(\cdot)]
\end{aligned} \tag{32}$$

where $\hat{x}_1(T)$ is defined by

$$\begin{aligned}
\dot{\hat{x}}_1(t) = &\left[A_{11} + P_1(t)[K_1'K_1 - C_1'RC_1]\right]\hat{x}_1(t) \\
&+P_1(t)[K_1'K - C_1'RC]\bar{x}(t) \\
&+P_1(t)C_1'R(t)y_0(t) + P_1(t)K_1'Gu_0(t); \\
\hat{x}_1(0) = 0 &
\end{aligned} \tag{33}$$

and $\rho_1[u_0(\cdot), y_0(\cdot)]$ is defined by

$$\begin{aligned}
&\rho_1[u_0(\cdot), y_0(\cdot)] \triangleq \\
&\int_0^T \left[\begin{array}{l} \|(K\bar{x} + K_1\hat{x}_1 + Gu_0)\|^2 \\ -(C\bar{x} + C_1\hat{x}_1 - y_0)'R(C\bar{x} + C_1\hat{x}_1 - y_0) \end{array} \right] dt.
\end{aligned} \tag{34}$$

Since, $X_1(T) \geq 0$, it follows that the pair $[u_0(\cdot), y_0(\cdot)]$ will be realizable if and only if $\rho_1[u_0(\cdot), y_0(\cdot)] \geq -d$.

Now let $\hat{x} = \begin{bmatrix} \hat{x}_1 \\ 0 \end{bmatrix} + \bar{x}$ and $P(t) = \begin{bmatrix} P_1(t) & 0 \\ 0 & 0 \end{bmatrix}$. Hence, $P(t)$ is the solution to (6). Also

$$\begin{aligned}
\dot{\hat{x}} &= \begin{bmatrix} \dot{\hat{x}}_1 \\ 0 \end{bmatrix} + \dot{\bar{x}} \\
&= \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}\begin{bmatrix} \hat{x}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}\bar{x} + B_2u_0 \\
&\quad + \begin{bmatrix} P_1(t) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} K_1' \\ K_2' \end{bmatrix}[K_1 \ \ K_2]\begin{bmatrix} \hat{x}_1 \\ 0 \end{bmatrix} \\
&\quad - \begin{bmatrix} P_1(t) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} C_1' \\ C_2' \end{bmatrix}R[C_1 \ \ C_2]\begin{bmatrix} \hat{x}_1 \\ 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} P_1(t) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} K_1' \\ K_2' \end{bmatrix}K\bar{x} \\
&\quad - \begin{bmatrix} P_1(t) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} C_1' \\ C_2' \end{bmatrix}RC\bar{x} \\
&\quad + \begin{bmatrix} P_1(t) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} C_1' \\ C_2' \end{bmatrix}Ry_0 \\
&\quad + \begin{bmatrix} P_1(t) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} K_1' \\ K_2' \end{bmatrix}Gu_0 \\
&= A\hat{x} + B_2u_0 + P(t)K'K\hat{x} - P(t)C'RC\hat{x} \\
&\quad + P(t)C'Ry_0 + P(t)K'Gu_0
\end{aligned}$$

and $\hat{x}(0) = 0$. That is, $\hat{x}(t)$ is defined as in (7). Moreover

$$\begin{aligned}
&\rho_1[u_0(\cdot), y_0(\cdot)] \\
&= \int_0^T \left\| K\bar{x} + [K_1 \ \ K_2]\begin{bmatrix} \hat{x}_1 \\ 0 \end{bmatrix} + Gu_0 \right\|^2 dt \\
&\quad - \int_0^T \begin{bmatrix} C\bar{x} + \\ [C_1 \ C_2]\begin{bmatrix} \hat{x}_1 \\ 0 \end{bmatrix} \\ -y_0 \end{bmatrix}' R \begin{bmatrix} C\bar{x} + \\ [C_1 \ C_2]\begin{bmatrix} \hat{x}_1 \\ 0 \end{bmatrix} \\ -y_0 \end{bmatrix} dt \\
&= \int_0^T \left[\begin{array}{l} \|(K\hat{x} + Gu_0)\|^2 - (C\hat{x} \\ -y_0)'R(C\hat{x} - y_0) \end{array} \right] dt \\
&= \rho[u_0(\cdot), y_0(\cdot)]
\end{aligned}$$

as defined in (8). Therefore, the pair $[u_0(\cdot), y_0(\cdot)]$ will be realizable if and only if $\rho[u_0(\cdot), y_0(\cdot)] \geq -d$. This completes the proof of the theorem.

□

Remark 7. (i) As in (Savkin and Petersen, 1995; Savkin and Petersen, 1996), we can also use the above theorem to obtain a set valued state estimate of the state of the uncertain system (1), (2). Indeed, if we define a quantity $\check{x}_1 = \hat{x}_1 + \bar{x}_1$ then it is straightforward to verify that $\check{x}_1$ is the solution to the filter state equations

$$\dot{\check{x}}_1 = [A_{11} + P_1(t)[K_1'K_1 - C_1'RC_1]]\,\check{x}_1(t) + [A_{12} + P_1(t)[K_1'K_1 - C_1'RC_1]\bar{x}_2(t) + P_1(t)C_1'R(t)y_0(t) + P_1(t)K_1'Gu_0(t)$$
$$\check{x}_1(0) = 0.$$

Also, $\bar{x}_2(t)$ is the solution to the state equations

$$\dot{\bar{x}}_2 = A_{22}\bar{x}_2 + B_{22}u_0; \quad \bar{x}_2(0) = 0.$$

Furthermore, the quantity $\rho_1[u_0(\cdot), y_0(\cdot)]$ can be re-written as

$$\rho_1[u_0(\cdot), y_0(\cdot)] = \int_0^T \left[\begin{array}{l} \|(K_1\check{x}_1 + K_2\bar{x}_2 + Gu_0)\|^2 - \\ (C_1\check{x}_1 + C_2\bar{x}_2 - y_0)' \\ \times R(C_1\check{x}_1 + C_2\bar{x}_2 - y_0) \end{array} \right] dt.$$

Hence, using the fact that $x_T = \tilde{x}_T + \bar{x}(T) = \begin{bmatrix} \check{x}_1 \\ \bar{x}_2 \end{bmatrix}$, it follows from (32) that the set of possible values for the state of the uncertain system (1), (2) at time T is given by

$$\left\{ \begin{array}{l} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : (x_1 - \check{x}_1)'X_1(T)(x_1 - \check{x}_1) \\ \leq d + \rho_1[u_0(\cdot), y_0(\cdot)] \;\&\; x_2 = \bar{x}_2(T) \end{array} \right\}.$$

(ii) An alternative interpretation of Theorem 6 is that the state equations (7) and the condition $\rho[u_0(\cdot), y_0(\cdot)] \geq -d$ provide a behavioural characterization of the uncertain system (1), (2). Indeed, from this point of view, the set of realizable input-output pairs is defined as a level set of the quadratic functional $\rho[u_0(\cdot), y_0(\cdot)]$. In this description, there is no need to distinguish between the input $u_0(\cdot)$ and the output $y_0(\cdot)$ and thus this description of the uncertain system can be regarded as a behavioural description.

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