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UNIVERSAL OUTPUT-FEEDBACK SISO CONTROLLER

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Abstract: A universal single-input-single-output (SISO) controller is proposed which is based on higher-order sliding modes and requires output measurements only. The uncertain SISO dynamic system to be controlled is required to have a permanent known-in-advance relative degree. Its exact mathematical model is not needed. The proposed controller provides for exact finite-time-convergent output tracking with exact output measurements, otherwise the tracking accuracy is proportional to the magnitude of the measurement noise. The control may be made arbitrarily smooth, thereby removing the chattering effect. *Copyright © 2002 IFAC*

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1. INTRODUCTION

Control under heavy uncertainty conditions remains one of the main subjects of the modern control theory. While a number of advanced methods like adaptation, absolute stability methods or the back-stepping procedure are based on relatively detailed knowledge of the controlled system, the sliding-mode control approach requirements are more moderate. The idea is to react immediately to any deviation of the system from some properly chosen constraint steering it back by a sufficiently energetic effort. Sliding-mode implementation is based on its insensitivity to external and internal disturbances and high accuracy (Utkin, 1992; Zinober, 1994). The main drawback of the standard sliding modes is mostly related to the so-called chattering effect caused by the high-frequency control switching.

Let the constraint be given by the equation $\sigma = s - h(t) = 0$, where s is some available output variable of an uncertain single-input-single-output (SISO) dynamic system and $h(t)$ is an unknown-in-advance smooth input to be tracked in real time. Then the standard sliding-mode control $u = - k \operatorname{sign} \sigma$ may be considered as a universal output controller applicable if the relative degree is 1, i.e. if $\dot\sigma$ explicitly depends on the control u and $\dot\sigma'_u > 0$. Higher-order sliding mode (Levant, 1993; Bartolini, *et al.*, 1999) is applicable for controlling SISO uncertain systems with arbitrary relative degree. The corresponding finite-time-convergent controllers (Levant, 2001a) require actually only the knowledge of the system relative degree. They provide also for higher accuracy with discrete sampling and, properly used, totally remove the chattering effect. The produced control is a discontinuous function of the tracking deviation σ and of a number of its real-time-calculated successive derivatives. The main problem in realization of this approach is the need to measure or calculate these derivatives.

The problem of real-time exact differentiation is mostly considered as ill-posed, for any attempt to differentiate small noises leads to unbounded differentiation errors. Nevertheless, the problem is solvable, some restrictions having been imposed on the unknown basic not-noisy input signal. In particular, *n*th-order exact robust differentiation is possible if the *n*th derivative of the input is assumed to have a known Lipschitz constant (Levant, 1998).

Recently proposed robust exact differentiators with finite-time convergence (Levant, 2001b) allow constructing a universal SISO controller. With relative degree r such a controller is a combination of a sliding controller based on an r-sliding finite-time-convergent sliding mode and of an exact robust $(r-1)$th-order differentiator with finite-time convergence. The resulting controller provides for exact tracking $\sigma = 0$ after a finite-time transient when the measurements of the deviation σ are exact, and for the tracking error proportional to the maximal measurement error otherwise. The corresponding Theorems are formulated and proved in the paper. Simulation examples of control of uncertain systems with the relative degrees 3 and 4 are demonstrated.

2. HIGHER-ORDER SLIDING-MODE CONTROLLERS

While successively differentiating output variable σ along trajectories of a discontinuous system, a discontinuity will be encountered sooner or later in the general case. Thus, sliding modes $\sigma \equiv 0$ may be classified by the number r of the first successive total derivative $\sigma^{(r)}$ which is not a continuous function of the state space variables or does not exist due to some reason like trajectory nonuniqueness. That number is called sliding order (see (Levant, 1993; Fridman and Levant, 1996) for the formal definitions). Hence, the r-th order sliding mode is determined by the equalities

$$\sigma = \dot\sigma = \ddot\sigma = \ldots = \sigma^{(r-1)} = 0 \qquad (1)$$

which impose an r-dimensional condition on the state of the dynamic system (Fig. 1).

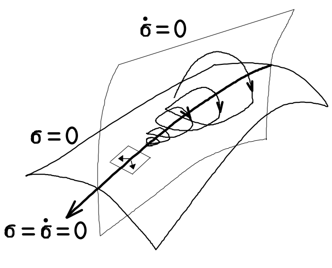

Fig. 1: 2-sliding mode

The standard sliding mode used in the most variable structure systems (VSS) is of the first order ($\dot\sigma$ is discontinuous). Consider a dynamic system of the form

$$\dot x = a(t,x) + b(t,x)u, \quad \sigma = \sigma(t, x), \qquad (2)$$

Here $x \in \boldsymbol{R}^n$, a, b, σ are unknown smooth functions, $u \in \boldsymbol{R}$, n is also uncertain. The relative degree r of the system is assumed to be constant and known. It is supposed that

$$0 < K_m \le L_b L_a^{r-1}\sigma \le K_M, \ \ |L_a^{r}\sigma| \le L \qquad (3)$$

for some K_m, K_M, $L > 0$. Since $L_b L_a^{r-1}\sigma = \frac{\partial}{\partial u}\sigma^{(r)}$, $L_a^{r}\sigma = \sigma^{(r)}|_{u=0}$, conditions (3) are reformulated in terms of input-output relations. It is assumed that trajectories of (2) are infinitely extendible in time for any Lebesgue-measurable bounded control. In practice it means that the system be weakly minimum phase.

Let $p \ge r$, $i = 1,\ldots, r-1$. Denote

$N_{1,r} = |\sigma|^{(r-1)/r}$,

$N_{i,r} = (|\sigma|^{p/r} + |\dot\sigma|^{p/(r-1)} + \ldots + |\sigma^{(i-1)}|^{p/(r-i+1)})^{(r-i)/p}$,

$N_{r-1,r} = (|\sigma|^{p/r} + |\dot\sigma|^{p/(r-1)} + \ldots + |\sigma^{(r-2)}|^{p/2})^{1/p}$.

$\phi_{0,r} = \sigma$,

$\phi_{1,r} = \dot\sigma + \beta_1 N_{1,r}\,\mathrm{sign}(\sigma)$,

$\phi_{i,r} = \sigma^{(i)} + \beta_i N_{i,r}\,\mathrm{sign}(\phi_{i-1,r})$,

where $\beta_1,\ldots, \beta_{r-1}$ are positive numbers. It was proved by Levant (2001) that with properly chosen positive parameters $\beta_1,\ldots, \beta_{r-1}$, α the controller

$$u = -\alpha\,\mathrm{sign}(\phi_{r-1,r}(\sigma, \dot\sigma, \ldots, \sigma^{(r-1)})). \qquad (4)$$

provides for the appearance of r-sliding mode $\sigma \equiv 0$ attracting trajectories in finite time.

Each time some finite-time transient process is mentioned in this paper it means that the transient time is a continuous function of initial conditions.

The positive parameters $\beta_1,\ldots, \beta_{r-1}$ are to be chosen sufficiently large in the index order. Each choice determines a controller family applicable to all systems (2) of relative degree r. Parameter $\alpha > 0$ is to be chosen specifically for any fixed C, K_m, K_M, most conveniently by computer simulation in order to avoid redundantly large estimations of C, K_m, K_M. The proposed controller is easily generalized: coefficients of $N_{i,r}$ may be any positive numbers etc. Obviously, α is to be negative with $\frac{\partial}{\partial u}\sigma^{(r)} < 0$. Following are controllers with $r \le 5$, β_i tested for $r \le 4$, p being the least common multiple of 1, 2, ..., r:

1. $u = -\alpha\,\mathrm{sign}\,\sigma$,
2. $u = -\alpha\,\mathrm{sign}(\dot\sigma + |\sigma|^{1/2}\mathrm{sign}\,\sigma)$,
3. $u = -\alpha\,\mathrm{sign}(\ddot\sigma + 2(|\dot\sigma|^{3} + |\sigma|^{2})^{1/6}\,\mathrm{sign}(\dot\sigma + |\sigma|^{2/3}\mathrm{sign}\,\sigma))$,
4. $u = -\alpha\,\mathrm{sign}\{\dddot\sigma + 3(\ddot\sigma^{6} + \dot\sigma^{4} + |\sigma|^{3})^{1/12}\,\mathrm{sign}[\ddot\sigma + (\dot\sigma^{4} + |\sigma|^{3})^{1/6}\,\mathrm{sign}(\dot\sigma + 0.5|\sigma|^{3/4}\mathrm{sign}\,\sigma)]\}$,
5. $u = -\alpha\,\mathrm{sign}(\sigma^{(4)} + \beta_4(|\sigma|^{12} + |\dot\sigma|^{15} + |\ddot\sigma|^{20} + |\dddot\sigma|^{30})^{1/60}\,\mathrm{sign}(\dddot\sigma + \beta_3(|\sigma|^{12} + |\dot\sigma|^{15} + |\ddot\sigma|^{20})^{1/30}\,\mathrm{sign}(\ddot\sigma + \beta_2(|\sigma|^{12} + |\dot\sigma|^{15})^{1/20}\,\mathrm{sign}(\dot\sigma + \beta_1|\sigma|^{4/5}\mathrm{sign}\,\sigma))))$

Controller (4) provides in finite time for the tracking accuracy $|\sigma^{(i)}| < a_i\tau^{r-i}$ with sampling interval τ. That is the best possible accuracy attainable with discontinuous $\sigma^{(r)}$ (Levant, 1993). The following new result shows robustness of controller (4) with respect to measurement errors.

Theorem 1. *Let $\sigma^{(i)}$ be measured with accuracy $\nu_i\varepsilon^{(r-i)/r}$ for some fixed $\nu_i > 0$, $i = 1, ..., r-1$. Then there are such positive constants μ_i that for any $\varepsilon > 0$ the following inequalities are established in finite time*:

$$|\sigma^{(i)}| \le \mu_i\, \varepsilon^{(r - i)/r},\ i = 0, ..., r - 1.$$

3. ROBUST EXACT DIFFERENTIATION

Controller (4) requires real-time exact calculation or direct measurement of $\sigma, \dot\sigma, ..., \sigma^{(r-1)}$. The identity $\sigma^{(r)} = u\frac{\partial}{\partial u}\sigma^{(r)} + L_a^r\sigma$ implies $|\sigma^{(r)}| \le L + \alpha K_M$, which allows implementation of robust $(r-1)$th-order differentiators (Levant, 1998, 2001b).

Let input signal $f(t)$ be a function defined on $[0, \infty)$, consisting of a bounded Lebesgue-measurable noise with unknown features and an unknown base signal $f_0(t)$ with the nth derivative having a known Lipschitz constant $C > 0$.

Denote by $D_{n-1}(f(\cdot),C)$ the $(n-1)$th-order differentiator producing outputs D_{n-1}^i, $i = 0, 1, ..., n-1$, being estimations of $f_0, \dot f_0, \ddot f_0, ..., f_0^{(n-1)}$ for any input $f(t)$ with $f_0^{(n-1)}$ having Lipschitz constant $C > 0$. Then the nth-order differentiator has the outputs $z_i = D_n^i$, $i = 0, 1, ..., n$, defined recursively as follows:

$$\dot z_0 = v,\ v = -\lambda\, |z_0 - f(t)|^{n/(n+1)}\operatorname{sign}(z_0 - f(t)) + z_1,$$
$$z_1 = D_{n-1}^0(v(\cdot), C),\ ...,\ z_n = D_{n-1}^{n-1}(v(\cdot), C).$$

Here $D_0(f(\cdot), C)$ is a simple nonlinear filter

$$D_0:\quad \dot z = -\lambda \operatorname{sign}(z - f(t)),\qquad \lambda > C.$$

Thus, the nth-order differentiator (Levant, 2001b) has the form

$$\begin{aligned}
\dot z_0 &= v_0,\ v_0 = -\lambda_0\, |z_0 - f(t)|^{n/(n+1)}\operatorname{sign}(z_0 - f(t)) + z_1,\\
\dot z_1 &= v_1,\ v_1 = -\lambda_1\, |z_1 - v_0|^{(n-1)/n}\operatorname{sign}(z_1 - v_0) + z_2,\\
&\ \ \dots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5)\\
\dot z_{n-1} &= v_{n-1},\ v_{n-1} = -\lambda_{n-1}|z_{n-1} - v_{n-2}|^{1/2}\operatorname{sign}(z_{n-1} - v_{n-2}) + z_n,\\
\dot z_n &= -\lambda_n \operatorname{sign}(z_n - v_{n-1}),
\end{aligned}$$

where $\lambda_i > 0$ are chosen sufficiently large in the reverse order. Note that it contains actually all the lower-order differentiators and each recursive step requires tuning one parameter only.

Remark. It is easy to check that differentiator (5) may be rewritten in the non-recursive form

$$\dot z_i = -\hat\lambda_i\, |z_0 - f(t)|^{(n - i)/(n+1)}\operatorname{sign}(z_0 - f(t)) + z_{i+1}, \qquad (6)$$

where $i = 0,..., n-1$, $\dot z_n = -\hat\lambda_n \operatorname{sign}(z_0 - f(t))$. $\hat\lambda_i$ are calculated on the basis of $\lambda_0, ..., \lambda_n$, $\hat\lambda_n = \lambda_n$.

Following relations are established in finite time with properly chosen parameters, (Levant, 2001b):

1. if $f(t) = f_0(t)$ then
   $z_0 = f_0(t)$; $\quad z_i = v_{i-1} = f_0^{(i)}(t)$, $\quad i = 1, ..., n$;
2. if $|f(t) - f_0(t)| \le \varepsilon$, then for some positive constants μ_i, ν_i depending exclusively on the parameters of differentiator (5)
   $|z_i - f_0^{(i)}(t)| \le \mu_i\, \varepsilon^{(n - i +1)/(n + 1)}$, $i = 0, ..., n$,
   $|v_i - f_0^{(i+1)}(t)| \le \nu_i\, \varepsilon^{(n - i)/(n + 1)}$, $i = 0, ..., n-1$;
3. if $f(t) = f_0(t)$, but $f(t)$ is sampled with constant period $\tau > 0$, then for some $\underline{\mu}_i$, $\underline{\nu}_i$
   $|z_i - f_0^{(i)}(t)| \le \underline{\mu}_i\, \tau^{n - i + 1}$, $i = 0, ..., n$,
   $|v_i - f_0^{(i+1)}(t)| \le \underline{\nu}_i\, \tau^{n - i}$, $i = 0, ..., n - 1$.

Parameters λ_{0i} being tuned for $C = 1$, the parameters are easily recalculated for any value of C by formula $\lambda_i = \lambda_{0i}\, C^{1/(n-i+1)}$. Following are the parameters of the 5th-order differentiator tuned with $C = 1$: $\lambda_0 = 50$, $\lambda_1 = 30$, $\lambda_2 = 16$, $\lambda_3 = 8$, $\lambda_4 = 4$, $\lambda_5 = 2$.

4. UNIVERSAL SISO CONTROLLER

Consider uncertain system (2), (3). Combining controller (4) and differentiator (5) achieve

$$u = -\alpha \operatorname{sign}(\phi_{r-1,r}(z_0, z_1, ..., z_{r-1})), \qquad (7)$$

$$\begin{aligned}
\dot z_0 &= v_0,\ v_0 = -\lambda_0\, |z_0 - \sigma|^{(r-1)/r}\operatorname{sign}(z_0 - \sigma) + z_1,\\
\dot z_1 &= v_1,\ v_1 = -\lambda_1\, |z_1 - v_0|^{(r-2)/(r-1)}\operatorname{sign}(z_1 - v_0) + z_2,\\
&\ \ \dots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (8)\\
\dot z_{r-2} &= v_{r-2},\ v_{r-2} = -\lambda_{r-2}|z_{r-2} - v_{r-3}|^{1/2}\operatorname{sign}(z_{r-2} - v_{r-3}) + z_{r-1},\\
\dot z_{r-1} &= -\lambda_{r-1} \operatorname{sign}(z_{r-1} - v_{r-2}),
\end{aligned}$$

where parameters λ_i of differentiator (8) are chosen according to the condition $|\sigma^{(r)}| \le C$, $C \ge L + \alpha K_M$. As noted above, relations $\lambda_i = \lambda_{0i}\, C^{1/(r-i)}$ may be used where λ_{0i} are chosen in advance for $C = 1$. Hence, in case when L and K_M are known, only one parameter α is really needed to be tuned, otherwise both C and α might be found in computer simulation.

Theorem 2. *Let σ be measured with a Lebesgue-measurable noise η, $|\eta| \le \varepsilon$. Then with properly chosen parameters of controller* (7), (8) *the following inequalities are established in finite time in the closed system* (2), (7), (8) *for some positive μ_i*:

$$|\sigma^{(i)}| \le \mu_i\, \varepsilon^{(r - i)/r},\ i = 0, ..., r - 1.$$

Theorem 1 means that with exact measurements ($\varepsilon = 0$) an r-sliding mode $\sigma \equiv 0$ is established in the closed system globally attracting trajectories in finite time.

Theorem 3. *Let* $\tau > 0$ *be the constant input sampling interval and the noises be absent. Then the following inequalities are established in finite time for some positive constants* μ_i:

$$|\sigma^{(i)}| \le \mu_i \tau^{r-i}, \; i = 0, ..., r-1.$$

5. THE PROOFS

All the proofs are based on homogeneity reasoning. Following is the proof of Theorem 2. The equality of the relative degree of (2) to r implies that

$$\sigma^{(r)} = L_a^r \sigma + u L_b L_a^{r-1} \sigma = L_a^r \sigma + u \frac{\partial}{\partial u} \sigma^{(r)},$$

As follows from (3)

$$\sigma^{(r)} \in [-C, C] + [K_m, K_M]\, u \,. \tag{9}$$

Denote $\xi_i = z_i - \sigma^{(i)}$. Taking into account $\eta \in [-\varepsilon, \varepsilon]$ and form (6) of the differentiator achieve

$$u = -\alpha \,\mathrm{sign}(\phi_{r-1,r}(\sigma+\xi_0, \dot\sigma+\xi_1, ..., \sigma^{(r-1)} + \xi_{r-1})), \tag{10}$$

$$\begin{aligned}
\dot\xi_0 &\in -\hat\lambda_0 \,|\, \xi_0 + [-\varepsilon, \varepsilon] \,|^{(r-1)/r} \,\mathrm{sign}(\xi_0 + [-\varepsilon, \varepsilon]) + \xi_1,\\
\dot\xi_1 &\in -\hat\lambda_1 \,|\, \xi_0 + [-\varepsilon, \varepsilon]|^{(r-2)/r} \,\mathrm{sign}(\xi_0 + [-\varepsilon, \varepsilon]) + \xi_2,\\
&\;\;...\\
\dot\xi_{r-2} &\in -\hat\lambda_{r-2} \,|\, \xi_0 + [-\varepsilon, \varepsilon]|^{1/r} \mathrm{sign}(\xi_0 + [-\varepsilon, \varepsilon]) + \xi_{r-1},\\
\dot\xi_{r-1} &\in -\hat\lambda_{r-1} \,\mathrm{sign}(\xi_0 + [-\varepsilon, \varepsilon]).
\end{aligned} \tag{11}$$

The right-hand set (9) - (11) is understood in the natural way. The differential inclusion itself is understood in the Filippov sense (1988), which means that the right-hand vector set is enlarged in a special way in order to satisfy certain convexity and continuity conditions. This inclusion does not "remember" anything on system (2) except the constants r, C, K_m, K_M, ε. Thus, providing for the convergence of $\sigma^{(i)}$, ξ_i to zero or to some small vicinity of the origin, the tracking problem is simultaneously solved for all systems (2) with the same parameters.

The dynamics of the differentiator is described by (11) and is completely independent of the system dynamics (9), (10). Thus, with $\varepsilon = 0$ derivative deviations ξ_i vanish in finite time (Levant, 2001b). Starting from that moment the controller receives exact values of $\sigma^{(i)}$, and in its turn provides for finite time convergence to the r-sliding mode $\sigma^{(i)} = 0$, $i = 0, ..., r$-1 (Levant, 2001a). Hence, trajectories of (9) - (11) converge to 0 in finite time with $\varepsilon = 0$.

Consider now $\varepsilon > 0$. It is easy to see that the transformation

$$(t, \sigma^{(i)}, \xi_i) \mapsto (\kappa t, \kappa^{r-i}\sigma^{(i)}, \kappa^{r-i}\xi_i), \; i = 0, ..., r-1 \tag{12}$$

transfers trajectories of (9) - (11) into trajectories of (9) - (11) but with the changed noise magnitude $\kappa^r\varepsilon$.

Consider some closed ball B_1 with the center at the origin. There is such $T > 0$ that with $\varepsilon = 0$ any trajectory starting in B_1 enters the origin to stay there forever. With sufficiently small ε_0 the trajectories will gather in some small ball B' around the origin in time T. Also any trajectory starting in B' returns back in time T. As follows from the continuous dependence on the right-hand side and initial conditions of the differential inclusion, these trajectories do not leave some larger but still small ball B_0. Thus, any trajectory starting in B_1 enters B_0 in time T to stay there forever.

It is easy to build such a sequence of ellipsoids B_i, $B_i \subset B_{i+1}$, using transformation (12), that their union covers the whole space and all trajectories of (9) - (11) starting in B_{i+1} enter B_i, $i = 0, 1, ...$, in finite time and stay there forever. Thus, B_0 is a finite-time attracting set. The convergence time is easily estimated. Now taking any $\varepsilon > 0$ and applying transformation (12) with $\kappa = (\varepsilon/\varepsilon_0)^{1/r}$ achieve the desired asymptotics of the attracting set. ■

Theorems 1, 3 are proved in a similar way.

6. SIMULATION EXAMPLE

Consider a variable-length pendulum control problem. There is no friction. All motions are restricted to some vertical plane. A load of known mass m is moving along the pendulum rod (Fig. 1). Its distance from O equals $R(t)$ and is not measured. An engine transmits a torque w which is considered as control. The task is to track some function x_c given in real time by the angular coordinate x of the rod.

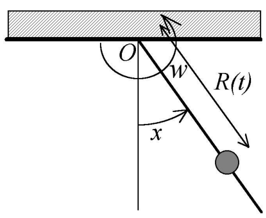

Fig. 2: Illustrative example

The system is described by the equation

$$\ddot\sigma = -2\, \frac{\dot R}{R}\, \dot x - g \frac{1}{R} \sin x + \frac{1}{mR^2}\, w, \tag{13}$$

where $g = 9.81$ is the gravitational constant, $m = 1$ was taken. Let $0 < R_m \le R \le R_M$, $\dot R$, $\ddot R$, $\dot x_c$ and $\ddot x_c$ be bounded, $\sigma = x - x_c$ be available. The initial conditions are $x(0) = \dot x(0) = 0$. The relative degree of the system is 2, but $L_a^2\sigma$ depends on $\dot x$ and is not uniformly bounded. Nevertheless all requirements of the Theorems are satisfied in any bounded vicinity of the origin, which provides for the local application of the method. Following are the functions R and x_c considered in the simulation:

$$R = 1 + 0.25 \sin 4t + 0.5 \cos t,$$
$$x_c = 0.5 \sin 0.5t + 0.5 \cos t\,.$$

While parameters of the controllers demonstrated further may be evaluated with respect to the above-mentioned restrictions on unknown functions $R(t)$, $x_c(t)$, their derivatives and some chosen bound on $\dot x$, they are usually excessively large in this case. The better way is to tune the parameters during simulation. Surely, the controlled class is somewhat smaller, but it still allows significant disturbances of the considered realizations of R and x_c.

6.1 3-sliding pendulum control

Introduce a new control $u = \dot w$ in order to produce a continuous Lipschitzian torque w, $w(0) = 0$. The resulting system has now relative degree $r = 3$. The corresponding parameters of controller (7), (8) were chosen as follows:

$$\begin{aligned}
\dot w = u &= -10\,\mathrm{sign}(z_2 + 2\,(|z_1|^3 + z_0^2)^{1/6}\\
&\qquad \mathrm{sign}(z_1 + z_0^{2/3}\,\mathrm{sign}\, z_0)),\\
\dot z_0 &= v_0,\ v_0 = -10\,|z_0 - \sigma|^{2/3}\,\mathrm{sign}(z_0 - \sigma) + z_1, \qquad (14)\\
\dot z_1 &= v_1,\ v_1 = -30\,|z_1 - v_0|^{1/2}\,\mathrm{sign}(z_1 - v_0) + z_2,\\
\dot z_2 &= -50\,\mathrm{sign}(z_2 - v_1),
\end{aligned}$$

also the controller with direct measurements was considered (the third standard controller from Section 2):

$$\dot w = u = -10\,\mathrm{sign}(\ddot\sigma + 2\,(|\dot\sigma|^3 + |\sigma|^2)^{1/6}\,\mathrm{sign}(\dot\sigma + |\sigma|^{2/3}\mathrm{sign}\,\sigma)). \qquad (15)$$

The integration was carried out according to the Euler method (the only integration method possible with discontinuous dynamics), the sampling step being equal to the integration step $\tau = 10^{-4}$. In the absence of noises the tracking accuracies $|\sigma| \le 5.7{\cdot}10^{-10}$, $|\dot\sigma| \le 8.1{\cdot}10^{-7}$, $|\ddot\sigma| \le 2.8{\cdot}10^{-3}$ were attained after application of the controller with direct measurements (15). Noises in the measurements of σ, $\dot\sigma$, $\ddot\sigma$ of the magnitudes 0.001, 0.01, 0.1 respectively, lead to the tracking accuracies $|\sigma| \le 1.0{\cdot}10^{-3}$, $|\dot\sigma| \le 5.4{\cdot}10^{-3}$, $|\ddot\sigma| \le 0.17$. That demonstrates high robustness of 3-sliding controller (15) with respect to measurement noises. Fast convergence of the differentiator outputs to the directly calculated derivatives of σ in the noised system is demonstrated in Fig. 3d (the differentiator was not used in control).

In the absence of output noises the tracking accuracies $|\sigma| \le 1.2{\cdot}10^{-7}$, $|\dot\sigma| \le 2.6{\cdot}10^{-5}$, $|\ddot\sigma| \le 0.016$ were attained after application of controller (14) (Fig. 3a). The corresponding graph of the torque is shown in Fig. 3c. In the presence of a measurement noise of the magnitude 10^{-6} the accuracies $4.1{\cdot}10^{-4}$, $6.8{\cdot}10^{-3}$, 0.33 were respectively attained. After the noise magnitude was changed to 10^{-5} the accuracies 0.0038, 0.030, 0.69 were attained, which corresponds to Theorem 2 (Fig. 3b).

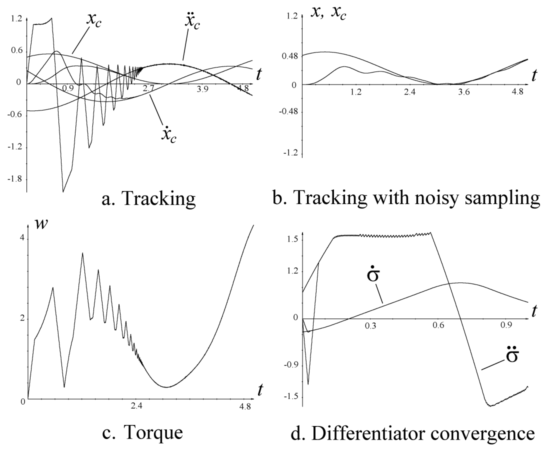

Fig. 3: 3-sliding pendulum control

6.2 4-sliding pendulum control

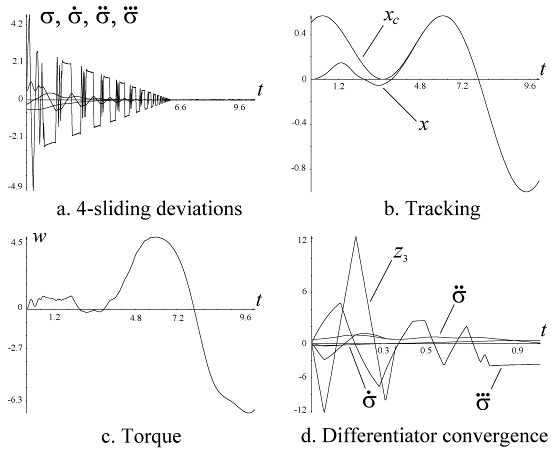

Fig. 4: 4-sliding control of the pendulum with an actuator

Introduce an actuator in dynamic system (13):

$$\ddot w = -5\,\dot w + 6\,w + \eta + u,\quad |\eta| \le 10. \qquad (16)$$

Here η is some input noise. The relative degree of the system (13), (16) with the output σ is 4 and controller (7), (8) was realized as follows:

$$\begin{aligned}
u &= -70\,\mathrm{sign}\{z_3 + 3\,(z_2^6 + z_1^4 + |z_0|^3)^{1/12}\,\mathrm{sign}[z_2 +\\
&\qquad (z_1^4 + |z_0|^3)^{1/6}\,\mathrm{sign}(z_1 + 0.5\,|z_0|^{3/4}\mathrm{sign}\, z_0)]\},\\
\dot z_0 &= v_0,\ v_0 = -10\,|z_0 - \sigma|^{3/4}\,\mathrm{sign}(z_0 - \sigma) + z_1, \qquad (17)\\
\dot z_1 &= v_1,\ v_1 = -10\,|z_1 - v_0|^{2/3}\,\mathrm{sign}(z_1 - v_0) + z_2,\\
\dot z_2 &= v_2,\ v_2 = -15\,|z_2 - v_1|^{1/2}\,\mathrm{sign}(z_2 - v_1) + z_3,\\
\dot z_3 &= -150\,\mathrm{sign}(z_3 - v_2).
\end{aligned}$$

The fourth standard controller from Section 2 with

α = 70 is used here. Parameter 150 of the differentiator cannot be considered as a high gain here, for it is seen from Fig. 4d that the variation rates of z_3 and $\dddot{\sigma}$ are comparable. The system dynamics requires relatively large magnitude of control and the corresponding differentiator parameters.

In the absence of output noises the tracking accuracies $|\sigma| \le 3.4 \cdot 10^{-9}$, $|\dot{\sigma}| \le 2.6 \cdot 10^{-7}$, $|\ddot{\sigma}| \le 7.1 \cdot 10^{-5}$, $|\dddot{\sigma}| \le 0.057$ were attained after application of controller (17) (Fig. 4a, b) with $\tau = 2 \cdot 10^{-5}$. The corresponding torque signal is shown in Fig. 4c. It is seen from Fig. 4d that the embedded third-order differentiator provides for exact estimations of output derivatives. The tracking accuracies 0.032, 0.036, 0.39, 6.7 respectively are attained with the input-noise magnitude 10^{-7}. It is seen that 4-sliding controller (17) is much more sensitive to input noises than 3-sliding controller (14) due to natural sensitivity of the higher-order differentiation.

7. CONCLUSIONS

Arbitrary-order real-time exact differentiation together with the arbitrary-order sliding controllers provide for full SISO control based on the input measurements only, when the only information on the controlled uncertain process is actually its relative degree. The obtained controller is locally applicable to general case SISO systems; it is also globally applicable if the relative degree is constant and few boundedness restrictions hold globally. The controller is proved to provide for extremely high tracking accuracy in the absence of noises. The resulting accuracy is proportional to τ^r, τ being a sampling period and r being the relative degree. That is the best possible accuracy with discontinuous control (Levant, 1993). It may be further improved increasing the relative degree artificially, which produces arbitrarily smooth control and removes the chattering effect.

The higher-order sliding controllers are shown to be very robust with respect to errors in direct measurements of successive input derivatives. Also the proposed closed-loop output-feedback controller is robust with respect to output noises. At the same time this robustness of the output-feedback controller decreases drastically with the growing relative degree. The reason is not an unsuccessful differentiator structure, but the very nature of the higher order differentiation (Levant, 2001b).

Thus, direct measurements of successive output derivatives are redundant if the output measurements are sufficiently accurate. It is important that the only needed information on the measurement noise is its magnitude, no frequency considerations are relevant.

It is easy to see that most practically important problems of output control are covered by the cases when relative degree r equals 2, 3 and 4, rarely 5. Indeed, according to the Newton law, the relative degree of a spatial variable with respect to a force, being understood as a control, is r = 2. Taking into account some dynamic actuator, achieve relative degree 3 or 4. If the actuator input is needed to be a continuous Lipschitz function, the relative degree has to be artificially increased to 4 or 5.

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