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PIECEWISE LINEAR CONSTRAINED CONTROL FOR CONTINUOUS-TIME SYSTEMS: THE MAXIMAL ADMISSIBLE DOMAIN

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Abstract: This paper presents an application of a piecewise linear control law in order to obtain the maximal domain of the admissible initial states. This is done in an homothetic way to the initial domain generated by the imposed dynamics on the closed-loop of a linear continuous-time system.

Keywords: Continuous-time linear system, Constrained control, Positive invariance, Piecewise linear control, Maximal domain.

Notations:

- For x, y of $\mathbb{R}^n$, $x \leq y$ (respectively, $x < y$) if $x_i \leq y_i$ (respectively $x_i < y_i$), $i = 1, ..., n$.
- $int(D)$: interior of the set D, whereas ∂D denotes its boundary.
- If x is a vector of $\mathbb{R}^n$, then :

$x_i^+ = \sup(x_i, 0)$, $x_i^- = \sup(-x_i, 0)$, and x^T is the transpose of x.

- For a matrix A of $\mathbb{R}^{n\times n}$:

$\tilde{A} = \begin{bmatrix} A_1 & A_2 \\ A_2 & A_1 \end{bmatrix}$, A_1 and A_2 are two matrices of $\mathbb{R}^{n\times n}$, such that:

$$A_1(i, j) = \begin{cases} A(i, j), \text{ if } i = j \\ \sup(A(i, j), 0), \text{ if } i \neq j \end{cases}$$

$$A_2(i, j) = \begin{cases} 0, \text{ if } i = j \\ \sup(-A(i, j), 0), \text{ if } i \neq j \end{cases}$$

1. INTRODUCTION

Energetic and technological limitations on the dynamic systems often result in linear constraints on the states or /and the control vectors. The introduction of the positive invariance concept plays a fundamental role in the constrained regulator problem.

The application of this concept in the constrained control linear systems leads to obtain a satisfactory result as was presented by several publications (Benzaouia and Burgat, 1988; Benzaouia and Burgat, 1989; Bitsoris, 1991; Benzaouia, 1991; Benzaouia, 1994; Benzaouia and Baddou, 1999)...It is in the same way for this work, which reveals the following situation: given an initial domain D_o generated by a dynamics imposed on the system and an initial state X_o defined outside D_o, our objective is to find a control law which allows the homothetic expansion of the initial domain until it contains the state X_o and the trajectory of the state vector tends asymptotically to the origin while the constraints on the control are respected. Using the piecewise linear control law developed in (Benzaouia and Baddou, 1999), we manage to find a control law not only satisfactory for the problem which we propose to solve, but also useful to find the maximal domain of the initial states with respect to the proposed methodology.
The fact that the computation of the control law is related to the admissibility of the state X_o, it leads to tackle a problem of great interest: it consist of determining the domain of all the initial states for which we can always find an admissible control which emanate the system to convergence and stability. In this context several works were published

such as, (Dorea and Hennet, 1996; Blanchini et al., 1995; Lasserre, 1993; Gutman and Cwikel, 1987), but they represent in the majority of the cases a complexity algorithmic. The interest of this work lies on the simplicity of the computations and the implementation of the final dynamics imposed on the system with respect to the widening of the initial domain.

This paper is structured in the following way: the 1^{st} section is devoted to the preliminary results and position of the problem, the solution of this problem constitutes the object of the 2^{nd} section accompanied by an illustrative example

2. PRELIMINARY RESULTS AND POSITION OF THE PROBLEM

The evolution of the process is described by an equation of the form:

$$\dot{x}(t) = Ax(t) + B\,u(t);\;\; X_o \in \mathbb{R}^n \tag{1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ are the state and the control of the system respectively. Matrices A and B are constant with appropriate dimensions, X_o is the initial state of the system. We assume that :

- (A, B) is stabilizable.
- A possesses at least $(n-m)$ stable eigenvalues.

The control u is subject to linear constant constraint of the form $u \in \Omega$ where:

$$\begin{array}{l} \Omega = \{u \in \mathbb{R}^m / -u_{\min} \leq u \leq u_{\max}\} \\ \text{with, } u_{\min}, u_{\max} \in int\,\mathbb{R}^m_+ \end{array} \tag{2}$$

The regulator problem for system (1) consists in realizing the following control law :

$$u = F_o x \;\;\text{ where } F_o \in \mathbb{R}^{m\times n} \tag{3}$$

The control is admissible only if the state is constrained to evolve in a specified domain defined by:

$$\begin{array}{l} D_o(F_o, u_{\min}, u_{\max}) = \\ \{x \in \mathbb{R}^n / -u_{\min} \leq F_o x \leq u_{\max}\} \end{array} \tag{4}$$

In such a case, system (3), becomes:

$$\dot{x} = (A + B\,F_o)\,x \tag{5}$$

To apply the results which guarantee the positive invariance of domain (4) with respect to motions of system (1), we must find a matrix H_o such that:

$$\begin{cases} F_o(A + B\,F_o) = H_o F_o \\ \widetilde{H}_o\,U \leq 0 \end{cases} \tag{6}$$

where $U = [u^t_{\max}, u^t_{\min}]^t$

The matrix F_o is solution of the matrix equation:

$$X\,A + X\,B\,X = H_o\,X \tag{7}$$

see for this end (Benzaouia, 1994).

The problem in question is to find a control law to be applied to the system, when X_o is defined outside the domain $D_o(F_o, u_{\min}, u_{\max})$ generated by the regulator F_o characterizing the dynamics imposed on the system. The solution suggested thereafter is an extension of the piecewise linear constrained control for continuous-time systems (Benzaouia and Baddou, 1999). This method consists in changing of the closed-loop dynamics systems during its evolution. We use for this the following control law:

$$u = F\,x \quad \text{with } F = \alpha\,F_o \tag{8}$$

The matrix H, satisfying the equation $F(A + BF) = HF$, is obtained from the following lemma:

Lemma 1. (Benzaouia and Baddou, 1999): If matrix F_o has full rank, then H can be computed from the following equality:

$$H = H_o + (\alpha - 1)\,F_o\,B \tag{9}$$

Remark 1:

If matrix H_o is diagonal, the following equality is satisfied:

$$\widetilde{H}\,U = \widetilde{H}_o U + (\alpha - 1)\,\widetilde{F_o B} U$$

which has a great utility in the sequel.

3. MAIN RESULTS

Consider the following control law:

$$u = F\,x \quad \text{ where } F = \beta\,F_o \tag{10}$$

with $0 < \beta < 1$

According to Lemma 1, we obtain the matrix H from the following equation:

$$H = H_o + (\beta - 1)\,F_o\,B \tag{11}$$

Our objective is to find the scalar β which satisfy the following conditions:

i)

$$X_o \in \partial\,D(F, u_{\min}, u_{\max}) \tag{12}$$

where

$$\begin{array}{l} D(F, u_{\min}, u_{\max}) = \\ \{x \in \mathbb{R}^n / -u_{\min} \leq F\,x \leq u_{\max}\} \end{array}$$

ii)

$$\widetilde{H} U < 0 \tag{13}$$

with

$$H = H_o + (\beta - 1) F_o B$$

Lemma 2. : For any state $X_o \notin D_o(F_o, u_{\min}, u_{\max})$, there always exists a scalar $\beta \in]0, 1[$ such that $X_o \in \partial D(\beta F_o, u_{\min}, u_{\max})$.

<u>Proof:</u>

We look for β such that (10) and (12) are satisfied, that is,

$$-u_{\min} \leq \beta F_o X_o \leq u_{\max}$$

which is equivalent to,

$$\begin{cases} \beta (F_o X_o)^+ \leq u_{\max} \\ \beta (F_o X_o)^- \leq u_{\min} \\ \text{where, } (F_o X_o) = (F_o X_o)^+ - (F_o X_o)^-. \end{cases}$$

this leads to,

$$\beta \leq \min \min_j \left\{ \frac{u^j_{\max}}{(F_o X_o)^+_j}, \frac{u^j_{\min}}{(F_o X_o)^-_j} \right\}$$

Particularly, if we choose

$$\beta = \min \min_j \left\{ \frac{u^j_{\max}}{(F_o X_o)^+_j}, \frac{u^j_{\min}}{(F_o X_o)^-_j} \right\} \tag{14}$$

then $X_o \in \partial D(\beta F_o, u_{\min}, u_{\max})$. □

Lemma 3. : If the scalar β given by (14) satisfies condition (13) then the control $u = \beta F_o x$ is admissible and the system in closed-loop is asymptotically stable for any $X_o \in D(\beta F_o, u_{\min}, u_{\max})$.

Remark 2:

- The homothetic expansion of the initial set $D_o(F_o, u_{\min}, u_{\max})$ until it contains the state X_o causes a deceleration of the system dynamics, but the fact of choosing β such that $X_o \in \partial D(\beta F_o, u_{\min}, u_{\max})$ makes it possible not too much slow down the system.

- Once the domain which contains X_o is obtained, we apply the piecewise linear control (see Benzaouia and Baddou, 1999) for the case $N = 1$, with the rate of improvement $\alpha = \frac{1}{\beta}$, so that we can respect the final dynamics imposed on the system.

The homothetic expansion of the domain D_o until it contains the initial state X_o, leads us to look for the maximal domain $D_{\max}$ of the admissible initial states, for which we can always find an admissible control law allowing the stability of the linear system with constraints. In this context two procedures are proposed.

3.1 *1^{st} procedure*

According to the fact that the existence of maximal domain is related to the existence of the scalar β in the interval $]0, 1[$, we seek so that the matrix H_o, according to remark 1, satisfies the following inequality:

$$\widetilde{H}_o U < -\frac{\varepsilon}{\beta} U \tag{15}$$

$$\text{where } \quad 0 < \beta < 1, \text{ and } \varepsilon > 0$$

and the corresponding matrix H obtained from (9) satisfies the condition

$$\widetilde{H} U < -\varepsilon U \tag{16}$$

Lemma 4. : For system (1), with matrix F_o solution of equation (7) there exists a scalar $\beta \in]0, 1[$ such that (15) and (16) are satisfied.

Proof:

According to equation (9),

$H = H_o + (\beta - 1) F_o B$ implies

$\widetilde{H} U = \widetilde{H}_o U + (1 - \beta) \widetilde{G} U$ with $G = -F_o B$.

Taking into account (15) ;

$$\widetilde{H} U < -\frac{\varepsilon}{\beta} U + (1 - \beta) \widetilde{G} U$$

Thus, $\widetilde{H} U < -\varepsilon U$ if $-\frac{\varepsilon}{\beta} U + (1 - \beta) \widetilde{G} U < -\varepsilon U$

this latter leads us to an equation of the 2^{nd} order, given by:

$\beta^2 (\widetilde{G} U)_j - \beta((\widetilde{G} U)_j + \varepsilon U_j) + \varepsilon U_j > 0$ for any $j \in \{1, .., 2m\}$,

having as solution:

$$\Rightarrow \begin{cases} \beta_1 = 1 \\ \beta_2 = \dfrac{\varepsilon U_j}{(\widetilde{G} U)_j} < 1 \end{cases} \text{; if } (\widetilde{G} U)_j > \varepsilon U_j, \text{ then}$$

$$\beta \in]0\,,\ \min_j (\frac{\varepsilon U_j}{(\widetilde{G} U_j)})[$$

or well

$$\Rightarrow \begin{cases} \beta_1 = \dfrac{\varepsilon U_j}{(\widetilde{G} U)_j} > 1 \\ \beta_2 = 1 \end{cases} \text{; if } (\widetilde{G} U)_j \leq \varepsilon U_j, \text{ then}$$

$$\beta \in]0,\, 1[$$

Therefore $\forall j \in \{1, .., 2m\}$;

$$\beta \in]0, \ \min(1, \min_j (\frac{\varepsilon U_j}{(\widetilde{G} U)_j}))[\qquad \square$$

To find the maximal domain, we try to determine the minimal value of the scalar β in the interval $]0, 1[$ which gives the maximal domain of the initial states where the condition (16) remains satisfied. In this case we propose the following algorithm:

Algorithm 1:

- step 1: for the matrix H_o chosen at the beginning, we pose $\frac{\varepsilon}{\beta} = s$, and we seek the maximal value of $s > 0$ for which $\widetilde{H}_o U \leq -sU$ is satisfied. The maximal value of s is noted $s_{\max}$, in other words: $\forall\ s > s_{\max}$, $\widetilde{H}_o U \nleq -sU$.

- step 2: for $s_{\max}$ found at step 1; the following function is minimized :

$$\min_{0<\beta<1} \beta, \text{such that } 0 < s < s_{\max} \text{ and } \varepsilon = \beta\, s$$

under constraints:

$$\begin{cases} H = H_o + (\beta - 1)F_o B \\ \widetilde{H}U < -\varepsilon U \end{cases} \quad (17)$$

the obtained β from step 2 generates the maximal domain of the initial states

$$D_{\max}(\beta F_o, u_{\min}, u_{\max}).$$

3.2 *2^st^ procedure*

Let us choose H_o (according to remark 1) such as

$$\widetilde{H}_o U < 0 \quad (18)$$

We seek so that the matrix computed from (9) checks the following condition :

$$\widetilde{H}\, U < 0 \quad (19)$$

Lemma 5. : For system (1) with matrix F_o solution of equation (7) there exists a scalar $\beta \in]0, 1[$ such that (18) and (19) are satisfied.

Proof:

Using equation (9) we can write

$\widetilde{H}U = \widetilde{H}_o U + (1 - \beta)\widetilde{G}U$ with $G = -F_o B$.

$\widetilde{H}U < 0$ if and only if $\widetilde{H}_o U + (1 - \beta)\widetilde{G}U < 0$

If $\widetilde{G}U \leq 0$ then $\forall\, \beta \in]0\ 1[$ we have $\widetilde{H}U < 0$

If there exists $j \in J = \{1, ..., 2m\}$ such that $(\widetilde{G}U)_j > 0$

then

$$\beta > \max_{j\in J}\left(\frac{(\widetilde{H}_o U)_j + (\widetilde{G}U)_j}{(\widetilde{G}U)_j}\right) = \beta_3 \quad (20)$$

two cases can arise:

* If $\beta_3 < 0$, we are free to choose β in the interval $]0, 1[$.
* If $\beta_3 > 0$, then $\beta \in]\beta_3, 1[$. □

To determine the maximal domain, we try to find the minimal value of the scalar β in the interval $]0, 1[$ which gives the maximal domain of the initial states where the condition (19) is satisfied.

- If $\beta_3 > 0$; then $D_{\max} = D(\beta_3 F_o, u_{\min}, u_{\max})$

such that: $\forall\, X_o \in intD_{\max}$, there exists an admissible control law $u \in \Omega$.

- If $\beta_3 < 0$; then, the scalar $\beta_{\min}$ which characterize the maximal domain

$$D_{\max} = D(\beta_{\min} F_o, u_{\min}, u_{\max})$$

is obtained by the minimization of the function : $\min_{0<\beta<1} \beta$, under the following constraints:

$$\begin{cases} H = H_o + (\beta - 1)F_o B \\ \widetilde{H}\, U < 0 \end{cases}$$

Illustrative example

Let us consider the continuous-time system (1) with:

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 0.45 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 0.5 & 1 \end{bmatrix},$$
$$X_o = \begin{bmatrix} -0.2 & 0.8 \end{bmatrix}^T$$

$$\sigma(A) = \{1; 0.45\}$$

$$u_{\min} = [2.5;\ 3]^T;\ u_{\max} = [3;\ 2]^T \text{ and}$$
$$U = [3;\ 2;\ 2.5;\ 3]^T$$

We hope to assign $\{-6; -5\}$ as the spectrum of the closed-loop. That is

$$H_o = \begin{bmatrix} -6 & 0 \\ 0 & -5 \end{bmatrix}$$

Using equation (7), we obtain

$$F_o = \begin{bmatrix} -7.1857 & 2.3952 \\ 3.0358 & -6.4619 \end{bmatrix}$$

the State X_o is defined outside the domain :

$$D(F_o,\ u_{\min},\ u_{\max}) =$$
$$\{x \in \mathbb{R}^n / -u_{\min} \leq F_o x \leq u_{\max}\}$$

from (14), and according to lemma (3):

$$\beta_1 = 0.5193$$

With $\beta = 0.5193$, the matrix H computed from (9) is given by:

$$H = \begin{bmatrix} -3.1215 & -1.1514 \\ 0.0938 & -1.8938 \end{bmatrix};$$
$$\sigma(H) = \{-3.0261; -1.9891\}$$

$$\text{and} \quad F = \beta F_o = \begin{bmatrix} -3.7315 & 1.2438 \\ 1.5765 & -3.3557 \end{bmatrix}$$

To determine the maximal domain of the initial states, we apply each of the two procedures proposed above.

1^st^ procedure:

With the aim of determining $D_{\max}$, we apply the algorithm 1, which gives the following result:

$$\widetilde{H}_o U \leq -5U \text{ ; with } s_{\max} = 5$$

by minimization of the function $f(\beta) = \beta$ under constraints (17) we obtain as realizable solution $\beta = 0.2843$.

Therefore, relative to the suggested method $D_{\max} = D(\beta F_o, u_{\min}, u_{\max})$

2^{nd} procedure :

For our example $(\widetilde{G}U)_j > 0$ for any $j \in \{1, ..., 2m\}$

Fulfiling (20) this implies

$$\beta > 0.2843$$

Therefore, for every $X_o \in intD_{\max}$, where $\beta = 0.2843$ there exists an admissible control law $u \in \Omega$.

Remark 3

In the case of our example, where $X_o = [-0.2\ 0.8]^t$, the value of $\beta = 0.5193$ makes it possible to have X_o on the boundary of the induced domain; on the other hand, if we approach to the value 0.28, the dynamics of the system becomes slower and the state X_o does not belong to the border of the induced polyhedral domain. **Figure 1** and **2** illustrate the difference between two values of β: a smaller domain is the initial domain, the full feature correponds to $\beta = 0.5193$ and dotted line to $\beta = 0.2843$.

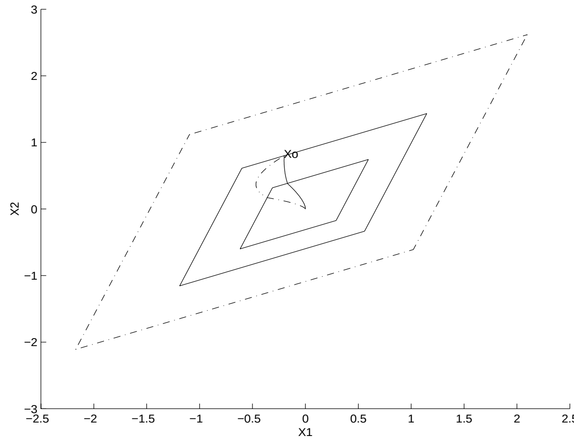

Fig. 1. representation of the trajectory of the state as well as the polyhedral domains.

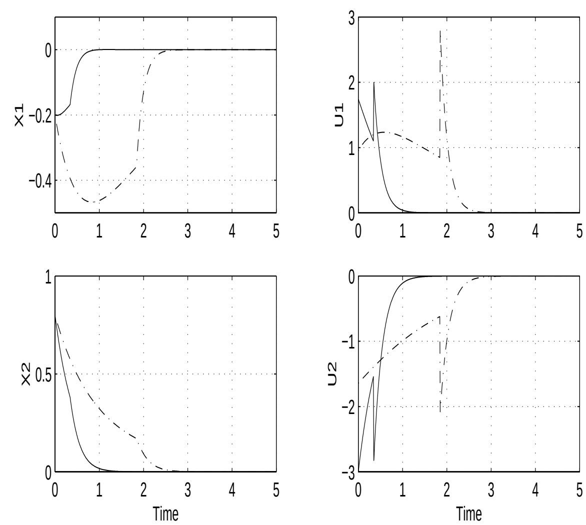

Fig. 2. Trajectories of the components of the state X and the control U using the two values of β.

4. CONCLUSION

In this paper we give another context of application of the piecewise linear control , it is question of maximal domain of the initial states from which it is possible to guarantee the admissibility of the trajectory with respect to the constraints. New results are obtained leading to built the maximal set of initial admissible state and positive invariance. An algorithm together with an example are also presented.

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