15th Triennial World Congress, Barcelona, Spain

BOUNDS ON THE GL_2 NORM FOR SISO SYSTEMS AND THEIR IMPLICATIONS FOR ROBUST PERFORMANCE ANALYSIS

Jun Wang [1] and **David A Wilson** [2]

School of Electronic & Electrical Engineering
University of Leeds, Leeds LS2 9JT, U.K.

Abstract: This paper derives upper and lower bounds on the generalized L_2 (GL_2) norm for SISO systems and investigates the relationship between H_∞, μ, and GL_2 analyses.

Keywords: Generalized L_2 norm, H-infinity control, Single-input / single-output, Structured singular value, Robust performance.

1. INTRODUCTION

Recently, D'Andrea (1999) presented a generalized L_2 (GL_2) framework to deal with robust performance problems involving block structured uncertainty. Some applications (Wang and Wilson, 2001*a*, *b*, *c*; D'Andrea and Istepanian, 2002) have shown that GL_2 synthesis achieves good robust performance and is more computationally tractable than μ synthesis. Wilson (2000) gave a demonstration of a simple relationship between GL_2 and μ analyses of scalar robust tracking and disturbance rejection problems. In this paper, we further the results in (Wilson, 2000) to derive tight bounds for GL_2 robust-performance analysis problems by considering the relationship between H_∞ norm, μ, and GL_2 norm.

The notation is standard and follows (Wilson, 2000) closely. For signals, $\|\cdot\|$ denotes the L_2 norm and for systems it denotes the induced L_2 norm. $G \star K$ stands for the lower linear fractional transformation between G and K.

2. PROBLEM STATEMENT

A system achieves robust performance if only if it is internally stable and the performance can be preserved when the system is perturbed. The perturbation can be modelled as a multiplicative uncertainty, which is widely used and computationally tractable. For an SISO system, many types of uncertainty models, such as an additive uncertainty, can be transformed into the multiplicative ones (Skogestad and Postlethwaite, 1996). As far as robust performance is concerned, there are two typical problems: robust tracking and disturbance rejection, which were shown to be intrinsically equivalent in (Wilson, 2000).

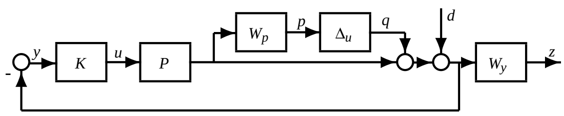

Fig. 1. Robust tracking.

Hence, without loss of generality, we only consider the robust tracking problem subjected to a multiplicative uncertainty Δ_u as shown in Figure 1. A controller K is sought such that the system is robustly stable and achieves robust performance defined by

$$\sup_{\|\Delta_u\|\le 1} \sup_{\|d\|\le 1} \|z\| < 1. \tag{1}$$

The GL_2 framework (D'Andrea, 1999) provides the following equivalent condition for (1) to hold

$$\|G\|_{GL_2} \triangleq \sup_{\|d\|=1} \left(\|G_1 d\| + \|G_2 d\|\right) < 1 \tag{2}$$

where

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} W_p T \\ W_y S \end{bmatrix}$$

[1] E-mail: jwang@ieee.org
[2] E-mail: d.a.wilson@ee.leeds.ac.uk

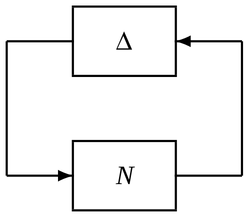

Fig. 2. $N-\Delta$ structure.

and $S=(1+PK)^{-1}$, $T=PK(1+PK)^{-1}$. Note that (2) defines an induced norm for the system G.

Now consider robust performance in terms of μ. Figure 1 can be transformed into the $N-\Delta$ structure required for μ-analysis (Skogestad and Postlethwaite, 1996; Zhou *et al.*, 1996). This is shown in Figure 2, where

$$N=\begin{bmatrix} G_1 & G_1 \\ G_2 & G_2 \end{bmatrix}, \qquad \Delta=\begin{bmatrix} \Delta_u & 0 \\ 0 & \Delta_p \end{bmatrix},$$

and Δ_p is a full uncertainty block associated with the performance transfer function G_2. It is easy to show that, for this simple problem, μ is given by

$$\mu_\Delta(N(j\omega)) \triangleq |G_1(j\omega)|+|G_2(j\omega)|,\ \forall\omega \tag{3}$$

and robust performance requires

$$\sup_\omega \mu_\Delta(N(j\omega)) < 1. \tag{4}$$

3. A RELATIONSHIP BETWEEN μ AND GL_2

Lemma 1.

$$\sup_\omega \mu_\Delta(N(j\omega)) \le \|G\|_{GL_2} \le \sqrt{2}\|G\| \tag{5}$$

PROOF. Firstly, we prove the left hand side of (5). The proof uses the fact (Desoer and Vidyasagar, 1975) that, for any frequency ω_0, it is possible to find a sequence of finite energy signals tending to a signal d_0 such that

$$\frac{\|G d_0\|}{\|d_0\|} = |G(j\omega_0)|.$$

Suppose the supremum of $(|G_1|+|G_2|)$ occurs at a finite ω_0. Then,

$$\begin{aligned} &\sup_{\|d\|=1} (\|G_1 d\|+\|G_2 d\|) \\ &= \sup_{d\neq 0}\left(\frac{\|G_1 d\|+\|G_2 d\|}{\|d\|}\right) \\ &\ge |G_1(j\omega_0)|+|G_2(j\omega_0)| \\ &= \sup_\omega(|G_1|+|G_2|). \end{aligned}$$

Therefore,

$$\sup_\omega(|G_1|+|G_2|) \le \sup_{\|d\|=1} (\|G_1 d\|+\|G_2 d\|)$$

i.e. $\sup_\omega \mu_\Delta(N) \le \|G\|_{GL_2}$. Similarly, if $\omega_0=\infty$, the conclusion follows by letting $\omega_0\to\infty$ (Zhou *et al.*, 1996).

Secondly, we prove the right hand side of (5).

$$\begin{aligned} &\left(\|G_1 d\|+\|G_2 d\|\right)^2 \\ &\le 2\left(\|G_1 d\|^2+\|G_2 d\|^2\right) \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\left[|G_1(j\omega)d(j\omega)|^2+|G_2(j\omega)d(j\omega)|^2\right]d\omega \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\left(|G_1(j\omega)|^2+|G_2(j\omega)|^2\right)|d(j\omega)|^2\,d\omega. \end{aligned}$$

Then,

$$\begin{aligned} &\|G\|_{GL_2} \\ &= \sup_{\|d\|=1}\left(\|G_1 d\|+\|G_2 d\|\right) \\ &\le \sqrt{2}\sup_{\|d\|=1} \\ &\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(|G_1(j\omega)|^2+|G_2(j\omega)|^2\right)|d(j\omega)|^2\,d\omega\right\}^{\frac{1}{2}} \\ &= \sqrt{2}\sup_\omega\left(|G_1(j\omega)|^2+|G_2(j\omega)|^2\right)^{\frac{1}{2}} \\ &= \sqrt{2}\|G\|. \end{aligned}$$

Therefore, $\|G\|_{GL_2} \le \sqrt{2}\|G\|$. □

Remark 2. This lemma shows that if a system has H_∞ robust performance subject to the uncertainty

$$\|\left[\Delta_u\ \Delta_p\right]\| \le \sqrt{2},$$

it has the GL_2 robust performance defined in Section 2. It also shows that GL_2 synthesis is sufficient to guarantee robust performance in term of μ (Wilson, 2000).

Remark 3. From the triangle inequality,

$$\begin{aligned} &\sqrt{2}\sup_\omega\left(|G_1(j\omega)|^2+|G_2(j\omega)|^2\right)^{\frac{1}{2}} \\ &\le \sqrt{2}\sup_\omega\left(|G_1(j\omega)|+|G_2(j\omega)|\right). \end{aligned}$$

Therefore,

$$\sup_\omega \mu_\Delta(N) \le \|G\|_{GL_2} \le \sqrt{2}\sup_\omega \mu_\Delta(N),$$

i.e. the maximum relative error between μ and the GL_2 norm, in this simple case, is $\sqrt{2}-1$.

Remark 4. The unit balls for $|G_1|+|G_2| \le 1$ and $\sqrt{2}\left(|G_1|^2+|G_2|^2\right)^{\frac{1}{2}} \le 1$ are shown in Figure 3. Since $\sup_\omega \mu_\Delta(N)=\sup_\omega(|G_1|+|G_2|)$ and $\sqrt{2}\|G\|=\sup_\omega\sqrt{2}\left(|G_1|^2+|G_2|^2\right)^{\frac{1}{2}}$, the boundary of the unit ball for $\|G\|_{GL_2}\le 1$ must lie in the shaded area in Figure 3.

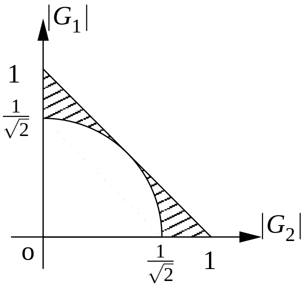

Fig. 3. Unit balls.

We are now in a position to give sufficient conditions under which GL_2 and μ robust performance analyses are equivalent.

Theorem 5. (Sufficient Condition 1). Given an LTI system $G = \begin{bmatrix} G_1(j\omega) \\ G_2(j\omega) \end{bmatrix}$, if $|G_1(j\omega)|$ and $|G_2(j\omega)|$ achieve their suprema at the same frequency ω_0, then

$$\|G\|_{GL_2} = \sup_{\omega} \mu_{\Delta}(N) = |G_1(j\omega_0)| + |G_2(j\omega_0)|. \quad (6)$$

PROOF.

$$\begin{aligned} & \sup_{\|d\|=1} \left(\|G_1 d\| + \|G_2 d\| \right) \\ \le & \sup_{\|d\|=1} \|G_1 d\| + \sup_{\|d\|=1} \|G_2 d\| \\ = & |G_1(j\omega_0)| + |G_2(j\omega_0)| \\ = & \sup_{\omega} |G_1(j\omega)| + \sup_{\omega} |G_2(j\omega)| \\ = & \sup_{\omega} \left(|G_1(j\omega)| + |G_2(j\omega)| \right). \end{aligned}$$

Hence,

$$\sup_{\|d\|=1} \left(\|G_1 d\| + \|G_2 d\| \right) \le \sup_{\omega} \left(|G_1(j\omega)| + |G_2(j\omega)| \right).$$

From Lemma 1, we get

$$\sup_{\|d\|=1} \left(\|G_1 d\| + \|G_2 d\| \right) \ge \sup_{\omega} \left(|G_1(j\omega)| + |G_2(j\omega)| \right).$$

Therefore,

$$\|G\|_{GL_2} = \sup_{\omega} \mu_{\Delta}(N) = |G_1(j\omega_0)| + |G_2(j\omega_0)|. \quad \square$$

Remark 6. This theorem appears too restrictive to be useful. However, since the μ and GL_2 syntheses always try to flatten the magnitudes of G_1 and G_2, it is quite possible that $|G_1|$ and $|G_2|$ achieve their suprema at the same frequency. A simple case will be demonstrated in Section 4.

Definition 7. (Boyd and Barratt, 1991) A function f on $\mathscr{X}$ is quasi-concave if for $\forall x_1, x_2 \in \mathscr{X}$ and $\lambda \in [0,1]$,

$$f(\lambda x_1 + (1-\lambda)x_2)) \ge \min\{f(x_1), f(x_2)\}.$$

Theorem 8. (Sufficient Condition 2). Let $|G_1(j\omega)|$ and $|G_2(j\omega)|$ be bounded quasi-concave functions. If $|G_1(j\omega)|$ and $|G_2(j\omega)|$ have their suprema at ω_1 and ω_2 respectively (say $\omega_1 \le \omega_2$), then

$$\|G\|_{GL_2} = \inf_{0<y<1} \left\{ y^{-1}|G_1(j\omega_0)|^2 + (1-y)^{-1}|G_2(j\omega_0)|^2 \right\}^{\frac{1}{2}}$$

for some $\omega_0 \in [\omega_1, \omega_2]$. Furthermore, if $|G_1(j\omega)|$ and $|G_2(j\omega)|$ are concave functions for $\omega \in [\omega_1, \omega_2]$, then $\|G\|_{GL_2} = \sup_{\omega} \mu_{\Delta}(N)$.

PROOF. Based on the definition of quasi-concave function, for any $\omega_a \le \omega_b \le \omega_1$,

$$|G_1(j\omega_b)| \ge \min\{|G_1(j\omega_a)|, |G_1(j\omega_1)|\} = |G_1(j\omega_a)|$$

i.e. $|G_1(j\omega_a)| \le |G_1(j\omega_b)|$. Hence $|G_1(j\omega)|$ is a monotone increasing function over the frequency $(-\infty, \omega_1]$. Similarly $|G_2(j\omega)|$ is also a monotone increasing function over the frequency $(-\infty, \omega_1]$. In addition, $|G_1(j\omega)|$ and $|G_2(j\omega)|$ are monotone decreasing functions over the frequency $[\omega_2, \infty)$.

Note that $\|G\|_{GL_2} = \sup_{\|d\|=1}(\|G_1 d\| + \|G_2 d\|)$. By the GL_2 analysis theorem (D'Andrea, 1999; Wang and Wilson, 2001*a*),

$$\|G\|_{GL_2} = \inf_{y_1+y_2 \ge 1} \|Y^{-\frac{1}{2}} G\|$$

where $Y = \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix}$ and $y_1, y_2 \in \mathbb{R}^+$.

Let $\bar{G} = Y^{-\frac{1}{2}} G = \begin{bmatrix} y_1^{-\frac{1}{2}} G_1(j\omega) \\ y_2^{-\frac{1}{2}} G_2(j\omega) \end{bmatrix}$, then

$$\begin{aligned} \|\bar{G}\| &= \sup_{\omega} \sigma_{\max}(\bar{G}) \\ &= \sup_{\omega} \lambda_{\max}^{\frac{1}{2}}(\bar{G}^* \bar{G}) \\ &= \sup_{\omega} (y_1^{-1}|G_1(j\omega)|^2 + y_2^{-1}|G_2(j\omega)|^2)^{\frac{1}{2}}. \end{aligned}$$

Thus,

$$\|G\|_{GL_2}^2 = \inf_{y_1+y_2 \ge 1} \sup_{\omega} \left\{ y_1^{-1}|G_1(j\omega)|^2 + y_2^{-1}|G_2(j\omega)|^2 \right\}.$$

It is clear that, for fixed y_1 and y_2,

$$\omega \mapsto y_1^{-1}|G_1(j\omega)|^2 + y_2^{-1}|G_2(j\omega)|^2$$

is monotone increasing in $(-\infty, \omega_1]$ and monotone decreasing in $[\omega_2, \infty)$.
Therefore,

$$y_1^{-1}|G_1(j\omega)|^2 + y_2^{-1}|G_2(j\omega)|^2$$

can only achieve its supremum in $[\omega_1, \omega_2]$, i.e.

$$\|G\|_{GL_2} = \inf_{0<y<1} \left\{y^{-1}|G_1(j\omega_0)|^2 + (1-y)^{-1}|G_2(j\omega_0)|^2\right\}^{\frac{1}{2}}$$

for some $\omega_0 \in [\omega_1, \omega_2]$.

Furthermore, if $|G_1(j\omega)|$ and $|G_2(j\omega)|$ are concave functions over the frequency domain $\omega \in [\omega_1, \omega_2]$, by using the Minimax Theorem (Balakrishnan, 1981), we obtain

$$\begin{aligned}
&\|G\|_{GL_2}^2 \\
&= \inf_{y_1+y_2\geq 1} \sup_{\omega\in[\omega_1,\omega_2]} (y_1^{-1}|G_1(j\omega)|^2 + y_2^{-1}|G_2(j\omega)|^2) \\
&= \sup_{\omega\in[\omega_1,\omega_2]} \inf_{y_1+y_2=1} (y_1^{-1}|G_1(j\omega)|^2 + y_2^{-1}|G_2(j\omega)|^2) \\
&= \sup_{\omega\in[\omega_1,\omega_2]} \inf_{y\in[0,1]} \\
&\qquad \left\{y^{-1}|G_1(j\omega)|^2 + (1-y)^{-1}|G_2(j\omega)|^2\right\}.
\end{aligned}$$

Now fix $\omega \in [\omega_1, \omega_2]$ and define

$$f_\omega(y) = y^{-1}|G_1(j\omega)|^2 + (1-y)^{-1}|G_2(j\omega)|^2$$

with $y \in [0,1]$. Then $f'_\omega(y_0) = 0$ when

$$y_0 = \frac{|G_1(j\omega)|}{|G_1(j\omega)| + |G_2(j\omega)|} \in [0,1]$$

and for any ω, $f''_\omega(y_0) > 0$.

Therefore,

$$\begin{aligned}
&\inf_{y\in[0,1]} [y^{-1}|G_1(j\omega)|^2 + (1-y)^{-1}|G_2(j\omega)|^2] \\
&= y_0^{-1}|G_1(j\omega)|^2 + (1-y_0)^{-1}|G_2(j\omega)|^2 \\
&= (|G_1(j\omega)| + |G_2(j\omega)|)^2.
\end{aligned}$$

Hence, we get

$$\|G\|_{GL_2}^2 = \sup_\omega (|G_1(j\omega)| + |G_2(j\omega)|)^2,$$

i.e., $\|G\|_{GL_2} = \sup_\omega \mu(N)$. Note the proof does not rely on knowing $\omega_0 \in [\omega_1, \omega_2]$ □

Proposition 9. If the scaling matrix Y is allowed to be dynamic, more specifically, if

$$Y = diag\left\{\frac{|G_1(j\omega)|}{|G_1(j\omega)| + |G_2(j\omega)|}, \frac{|G_2(j\omega)|}{|G_1(j\omega)| + |G_2(j\omega)|}\right\},$$

then

$$\|G\|_{GL_2} = \|Y^{\frac{1}{2}}G\| = \sup_\omega \mu_\Delta(N).$$

PROOF. This proposition is a direct result from the proof of Theorem 8, therefore the proof is omitted here. □

Remark 10. When the scaling matrix Y is dynamic, the GL_2 synthesis problem will become non-convex and need "Y-K" iterations, similar to the so-called "D-K" iterations in μ synthesis.

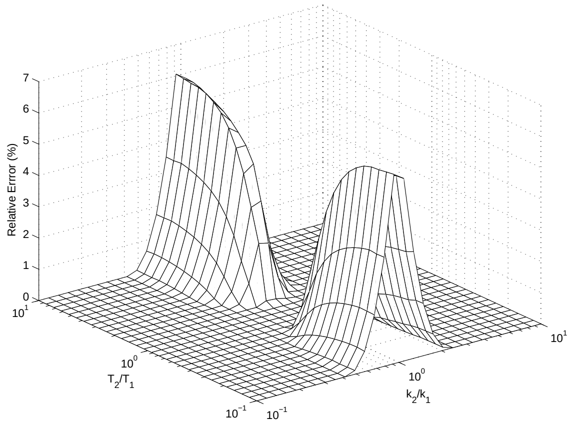

Fig. 4. The relative error between GL_2 norm and μ.

4. NUMERICAL EXAMPLES

Example 1: How far could GL_2 be from μ?

Suppose

$$G_1 = k_1 \frac{T_1^2 s^2 + 2\xi_z T_1 s + 1}{T_1^2 s^2 + 2\xi_p T_1 s + 1}$$

and

$$G_2 = k_2 \frac{T_2^2 s^2 + 2\xi_z T_2 s + 1}{T_2^2 s^2 + 2\xi_p T_2 s + 1}$$

where $\xi_z = 0.7$, $\xi_p = 0.3$, $T_1 = 1$, $k_1 = 1$, $k_2 \in [0.1, 10]$ and $T_2 \in [0.1, 10]$. So, $|G_i|, i \in \{1,2\}$ is a quasi-concave function with peak value at just below the frequency $\frac{1}{T_i}$.

Let $G = \begin{bmatrix} G_1(j\omega) \\ G_2(j\omega) \end{bmatrix}$ and $N = \begin{bmatrix} G & G \end{bmatrix}$.

Hence,

$$\sup_\omega \mu_\Delta(N) = \sup_\omega \left(|G_1(j\omega)| + |G_2(j\omega)|\right)$$

and

$$\|G\|_{GL_2} = \sup_{\|d\|=1} \left(\|G_1 d\| + \|G_2 d\|\right).$$

When $T_2 = T_1$ and $k_2 \in [0.1, 10]$, from Theorem 5, we observe that $\sup_\omega \mu_\Delta(N) = \|G\|_{GL_2}$. When $k_2 = k_1$ and T_2 is very close to T_1, we obtain that $\sup_\omega \mu_\Delta(N) = \|G\|_{GL_2}$ from Theorem 8. Then, how far is $\sup_\omega \mu_\Delta(N)$ from $\|G\|_{GL_2}$ when the parameters T_2 and k_2 varies in the domain $[0.1, 10]$? From Lemma 1, we only know the supremum of the relative error is $\sqrt{2} - 1$. As a complement to Lemma 1 and Theorem 8, in this example, we show in Figure 4 the relative error between $\|G\|_{GL_2}$ and $\sup_\omega \mu_\Delta(N)$, i.e.,

$$\frac{\|G\|_{GL_2} - \sup_\omega \mu_\Delta(N)}{\sup_\omega \mu_\Delta(N)} \times 100\%$$

as $T_2 \in [0.1, 10]$ and $k_2 \in [0.1, 10]$.

Figure 4 shows that the relative error is nearly zero in a wide area

$$\{(T_2, k_2) | T_2 \in [0.7, 1.5], \text{ or } k_2 \in [0.1, 0.4] \cup [2.5, 10]\}$$

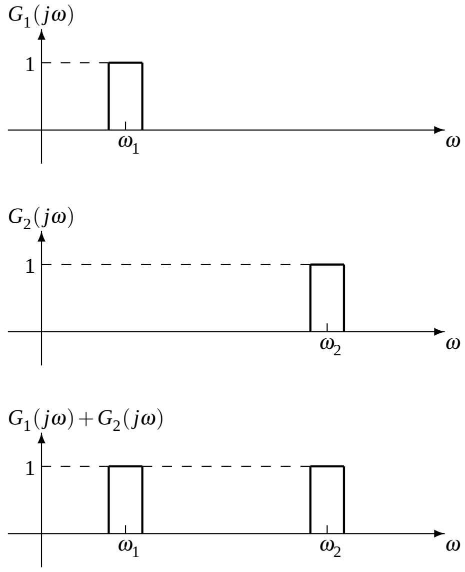

Fig. 5. The magnitude of $G_1(j\omega)$ and $G_2(j\omega)$.

and the maximum error is about 6% when $\{(T_2,k_2)|T_2 \in \{0.1,10\}, k_2 = 1\}$. Hence, in this case, although μ and GL_2 robust performance analyses are not equivalent everywhere, they are very close to each other.

Example 2: The worst case [3] *where*

$$\|G\|_{GL_2} = \sqrt{2}\|G\| = \sqrt{2}\sup_{\omega}\mu_\Delta(N).$$

Here we give a worst-case example. Let

$$G_i(j\omega) = \begin{cases} 1 & \omega \in [\omega_i - \varepsilon, \omega_i + \varepsilon] \\ 0 & \text{otherwise} \end{cases}$$

where $i \in \{1,2\}$ and $\varepsilon \in \mathbb{R}^+ \to 0$. The specific forms of G_1 and G_2 are shown in Figure 5.

Define

$$d_{\omega_i,T}(t) \triangleq \begin{cases} A_{\omega_i,T}\cos\omega_i t & t \in [-T,T] \\ 0 & \text{otherwise} \end{cases}$$

where $A_{\omega_i,T} = \frac{1}{\sqrt{T(1+\frac{\sin 2\omega_i T}{2\omega_i T})}}$.

Let $d_{\omega_i}(t) \triangleq \lim_{T\to\infty} d_{\omega_i,T}(t)$, then (Zhou *et al.*, 1996)

$$\frac{\|g * d_{\omega_i}\|^2}{\|d_{\omega_i}\|} = |G(j\omega_i)|.$$

We construct a signal

$$d \triangleq d_1 + d_2 \triangleq \tfrac{1}{2}d_{\omega_1}(t) + \tfrac{1}{2}d_{\omega_2}(t),$$

where $\|d\| = 1$, $\|d_1\| = \frac{1}{\sqrt{2}}$, and $\|d_2\| = \frac{1}{\sqrt{2}}$. In addition, $G_1 d = G_1 d_1$ and $G_2 d = G_2 d_2$.

So,

[3] This example was originally suggested in the correspondence with Dr R. D'Andrea, Cornell University, USA.

$$\begin{aligned} &\|G_1 d\| + \|G_2 d\| \\ &= \|G_1 d_1\| + \|G_2 d_2\| \\ &= |G_1(j\omega_1)|\cdot\|d_1\| + |G_2(j\omega_2)|\cdot\|d_2\| \\ &= \sqrt{2}. \end{aligned}$$

Hence, $\|G\|_{GL_2} = \sqrt{2}$.

From Figure 5, it is clear that

$$\sup_{\omega}\mu_\Delta(N) = \sup_{\omega}\{|G_1(j\omega)| + |G_2(j\omega)|\} = 1$$

and

$$\|G\| = \sup_{\omega}\{|G_1(j\omega)|^2 + |G_2(j\omega)|^2\}^{\frac{1}{2}} = 1.$$

Therefore, in this case,

$$\|G\|_{GL_2} = \sqrt{2}\|G\| = \sqrt{2}\sup_{\omega}\mu_\Delta(N).$$

Example 3: Synthesis problem

So far, we have only considered the robust performance analysis problems. We now give an example of synthesis problem.

Suppose the plant is

$$P = \frac{0.1s+1}{s+1}$$

with a performance weight W_y and an uncertainty weight W_p given by

$$W_y = \frac{1}{s^2+1.4s+1}, \qquad W_p = \frac{s+3}{s+30}.$$

The system diagram is shown in Figure 1 and the generalized system is

$$\begin{bmatrix} p \\ z \\ \cdots \\ y \end{bmatrix} = \begin{bmatrix} 0 & \vdots & W_p P \\ W_y & \vdots & W_y P \\ \cdots & \cdots & \cdots \\ -1 & \vdots & -P \end{bmatrix} \begin{bmatrix} q+d \\ \cdots \\ u \end{bmatrix}.$$

The minimal state-space realization of the above transfer function matrix is

$$G_{gen} = \left[\begin{array}{ccc|ccc} A_p & 0 & 0 & 0 & \vdots & B_p \\ B_2C_p & A_2 & 0 & 0 & \vdots & B_2D_p \\ B_1C_p & 0 & A_1 & B_1 & \vdots & D_p \\ \hline D_2C_p & C_2 & 0 & 0 & \vdots & D_2D_p \\ D_1C_p & 0 & C_1 & D_1 & \vdots & D_1D_p \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ C_p & 0 & 0 & -1 & \vdots & -D_p \end{array}\right].$$

By using the μ Analysis and Synthesis Toolbox (Balas *et al.*, 1998), we obtained a μ controller

$$K_\mu = \frac{4202.456(s+93.54)(s+30)(s+2.316)(s+1.771)(s+1)}{(s+2043)(s+93.51)(s+10)(s+2.172)(s^2+1.4s+1)}$$

and $\sup_\omega \mu_\Delta(G_{gen} \star K_\mu) = 0.1711$. Note that, we chose a second-order scaling matrix D in $\inf \bar{\sigma}(DND^{-1})$ (Balas *et al.*, 1998).

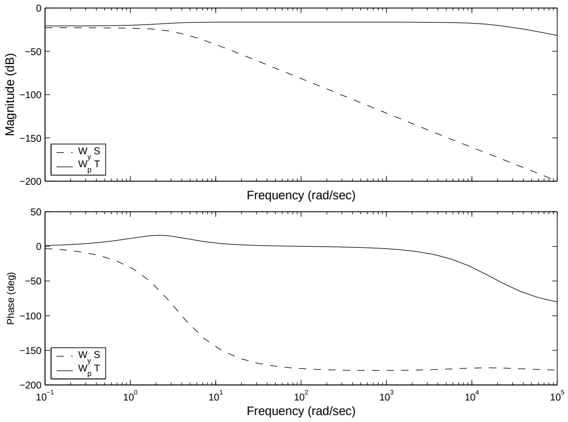

Fig. 6. The Bode plot of W_yS and W_pT with $K = K_{GL_2}$.

By using the LMI Control Toolbox (Gahinet *et al.*, 1995), we designed a GL_2 controller

$$K_{GL_2} = \frac{26476.73(s+30.03)(s+2.342)(s+1)}{(s+1.467e4)(s+10)(s^2+1.4s+1)}$$

and $\|G_{gen} \star K_{GL_2}\|_{GL_2} = 0.1785$.

Note that the controllers K_μ and K_{GL_2} are very similar if we ignore the real zeros far away from the original point and cancel a pair of pole and zero close to each other. The relative error of $\|G_{gen} \star K_{GL_2}\|_{GL_2}$ to $\sup_\omega \mu_\Delta(G_{gen} \star K_\mu)$ is 4.32%.

It is of interest that $\sup_\omega \mu_\Delta(G_{gen} \star K_{GL_2}) = 0.1785$, which is equal to $\|G_{gen} \star K_{GL_2}\|_{GL_2}$ to the 4th decimal place. This is not surprising when we observe the Bode plot of systems $|W_pT|$ and $|W_yS|$ as shown in Figure 6. Here we reset the controller $K \triangleq K_{GL_2}$ and all the notation follows that in Section 2.

It is well known that H_∞, μ, and GL_2 syntheses try to minimize the peak value in frequency domain, and therefore flatten the magnitude of the system. Hence, the conditions given in Theorem 5 and Theorem 8 are common in a GL_2 synthesis problem.

5. CONCLUSIONS

GL_2 control is a natural extension of H_∞ control and can be close or equivalent to μ. This paper investigates their relationship resulting in a tight bound on GL_2 robust-performance analysis problems for SISO systems. Although the work is mainly concerned with analysis problems, it is helpful in synthesis problems, as demonstrated by the numerical example. It will be interesting if (some of) the results in this work can be extended into MIMO robust performance problems. In addition, if a dynamical model can be incorporated into the scaling matrix of GL_2 synthesis, the GL_2 could be more close to μ.

ACKNOWLEDGEMENTS

The first author would like to thank the Committee of Vice-Chancellors & Principals of the University of UK (CVCP) and the Great Britain – China Educational Trust for their funding.

6. REFERENCES

Balakrishnan, A. V. (1981). *Applied Functional Analysis*. second ed.. Springer-Verlag.

Balas, G. J., J. C. Doyle, K. Glover, A. Packard and R. Smith (1998). *μ-Analysis and Synthesis Toolbox*. The Math Works, Inc.

Boyd, S. P. and C. H. Barratt (1991). *Linear Controller Design: Limits of Performance*. Prentice-Hall, Inc.

D'Andrea, R. (1995). H_∞ optimization with spatial constraints. In: *Proceedings of IEEE Conference on Decision and Control*. pp. 4327–4332.

D'Andrea, R. (1999). Generalized l_2 synthesis. *IEEE Transactions on Automatic Control* **44**(6), 1145–1156.

D'Andrea, R. and R. S. H. Istepanian (2002). Design of full state feedback finite precision controllers. *To appear in the International Journal of Robust and Nonlinear Control*.

Desoer, C. A. and M. Vidyasagar (1975). *Feedback Systems: Input-Output Properties*. Academic Press.

Gahinet, P., A. Nemirovski, A. J. Laub and M. Chilali (1995). *LMI Control Toolbox for Use with MATLAB*. The Math Works Inc.

Packard, A. and J. Doyle (1993). The complex structured singular value. *Automatica* **29**(1), 71–109.

Skogestad, S. and I. Postlethwaite (1996). *Multivariable Feedback Control – Analysis and Design*. John Wiley & Sons.

Wang, J. and D. A. Wilson (2001*a*). Application of generalized L_2 synthesis to decoupled active suspension systems. In: *Proceedings of the American Control Conference*. pp. 2664–2669.

Wang, J. and D. A. Wilson (2001*b*). Mixed GL_2 / H_2 / GH_2 control with pole placement and its application to vehicle suspension systems. *International Journal of Control* **74**(13), 1353–1369.

Wang, J. and D. A. Wilson (2001*c*). Multi-objective control of decoupled vehicle suspension systems. In: *Proceedings of IEEE Conference on Decision and Control*. pp. 535–540.

Wilson, D. A. (2000). A relationship between μ and generalized L_2 synthesis for robust tracking and disturbance rejection problems. In: *Proceedings of the American Control Conference*. pp. 1281–1282.

Zhou, K., J. C. Doyle and K. Glover (1996). *Robust and Optimal Control*. Prentice-Hall.