A DISCRETIZED LYAPUNOV FUNCTIONAL APPROACH TO STABILITY OF LINEAR DELAY-DIFFERENTIAL SYSTEMS OF NEUTRAL TYPE

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Abstract: This paper considers the stability problem of linear delay-differential systems of neutral type. A discretized Lyapunov functional approach is developed. The resulting stability criteria are formulated in the form of a linear matrix inequality. For nominal systems, the analytical results can be approached with fine discretization. For uncertain systems, the new approach is much less conservative. Numerical examples show significant improvement over approaches in the literature. *Copyright © 2002 IFAC*

Keywords: Stability, Time-Delay, Neutral Type, Uncertainty, Linear Matrix Inequality.

1. INTRODUCTION

The problems of stability and stabilization of delay-differential systems of neutral type have received considerable attention in the last two decades, see for example, Byrnes *et al*. (1984); Spong (1985); Hale and Lunel (1993); Logemann and Townley (1996). The practical examples of neutral delay-differential systems include the distributed networks containing lossless transmission lines (Brayton, 1966), and population ecology (Kuang, 1993). Some earlier results are based on matrix measure and matrix norm (Hu and Hu, 1996; Park and Won, 1999) or a simple Lyapunov functional (Slemrod and Infante, 1972). The resulting criteria are independent of delay. Although these criteria are easy to use, they are often overly conservative for practical applications.

Delay-dependent stability results, which take the delay into account, are usually less conservative than the delay-independent stability ones. A model transformation technique is often used to transform a pointwise delay system to a distributed delay system, and delay-dependent stability criteria are obtained, see for example, Pak and Won (2000), Lieu (2001), Niculescu (2001), Ni and Han (2001), Han (2001, 2002). The model transformation may introduce additional dynamics, i.e. additional poles that are not present in the original system, and one of these additional poles may cross the imaginary axis before any of the poles of the original system do as the delay increases from zero (Gu and Niculescu 2000; Kharitonov and Melchor-Aguilar 2000). Moreover, there are no obvious ways to obtain less conservative results even if one is willing to commit more computational effort to the problem.

For a linear system of retarded type with a constant time-delay, it has been proven that the existence of a more general quadratic form Lyapunov-Krasovskii functional is necessary and sufficient for the stability of an uncertainty-free time-delay system (Huang 1989). A discretized Lyapunov functional approach has been proposed to enable one to write the stability criterion in an LMI form (Gu 1997, 1999a, 1999b, 2001). The criteria have shown significant improvements over the existing results even under very coarse discretization. The results in Gu (1999b) have been extended to the cases where the uncertainty is norm-bounded (Han and Gu 2001).

In this paper, we develop the discretized Lyapunov functional approach to study the stability of linear delay-differential systems of neutral type. Stability criteria are formulated in the form of linear matrix inequalities (LMIs). Numerical examples are presented to illustrate the effectiveness of the approach.

Notation. For a symmetric matrix W, "$W>0$" denotes that W is positive definite matrix. Similarly, "≥", "<" and "≤" denote positive semi-definiteness, negative definiteness and negative semi-definiteness. I denotes an identity matrix of appropriate dimension.

2. PROBLEM STATEMENT

Consider the linear delay-differential system of neutral type

$$\dot{x}(t) - C\dot{x}(t-r) = A(t)x(t) + B(t)x(t-r) \qquad (1)$$

with initial condition

$$x(t) = \phi(t)\,,\ \forall t \in [-r, 0] \qquad (2)$$

where $x(t)\in\mathbb{R}^n$ is the state, $r>0$ is a constant time delay, $\varphi(t)$ is the initial condition, $C\in\mathbb{R}^{n\times n}$ is a constant matrix, $A(t)\in\mathbb{R}^{n\times n}$ and $B(t)\in\mathbb{R}^{n\times n}$ are uncertain matrices, which are unkown and possibly time-varying, but are known to be bounded by some compact set Ω, i.e.,

$$\begin{pmatrix} A(t) & B(t)\end{pmatrix} \in \Omega \subset \mathbb{R}^{n\times 2n}\text{, for all } t\in(0,\ \infty) \qquad (3)$$

Define $\mathcal{C}$ as the set of continuous $\mathbb{R}^n$ valued function on the interval $[-r,\ 0]$, and let $x_t \in \mathcal{C}$ be a segment of system trajectory defined as

$$x_t(\theta) = x(t+\theta)\,,\ -r\le\theta\le 0\,. \qquad (4)$$

In this paper, we will attempt to formulate some practically computable criteria for robust stability of uncertain system described by (1) to (3).

3. PRELIMINARIES

Define the operator

$$\mathcal{D}\phi = \phi(0) - C\phi(-r) \qquad (5)$$

Choose a Lyapunov-Krasovskii functional $V(t,\phi)$ of a quadratic form

$$\begin{aligned} &V(t,\phi):\ \mathbb{R}\times\mathcal{C} \mapsto \mathbb{R} \\ &V(t,\phi) = \frac{1}{2}(\mathcal{D}\phi)^T P(\mathcal{D}\phi) + (\mathcal{D}\phi)^T\int_{-r}^{0} Q(\xi)\phi(\xi)d\xi \\ &\qquad + \frac{1}{2}\int_{-r}^{0}\left[\int_{-r}^{0}\phi^T(\xi)R(\xi,\eta)\phi(\eta)d\eta\right]d\xi \\ &\qquad + \frac{1}{2}\int_{-r}^{0} x^T(t+\xi)S(\xi)x(t+\xi)d\xi \end{aligned} \qquad (6)$$

where

$$\begin{aligned} &P\in\mathbb{R}^{n\times n},\ P=P^T \\ &Q:[-r,0]\to\mathbb{R}^{n\times n} \\ &S:[-r,0]\to\mathbb{R}^{n\times n},\ S^T(\xi)=S(\xi) \\ &R:[-r,0]\times[-r,0]\to\mathbb{R}^{n\times n},\ R(\eta,\xi)=R^T(\xi,\eta) \end{aligned}$$

and Q, R and S are Lipschitz matrix functions with piecewise continuous derivatives.

It is well known that (Hale and Lunel 1993)

Theroem 1. *The system (1)-(3) is asymptotically stable if the operator $\mathcal{D}$ is stable and there exists a quadratic Lyapunov-Krasovskii functional V of the form (6) such that for some $c_i>0$ $(i=1,\ 2)$, it satisfies*

$$c_1(\mathcal{D}\phi)^T(\mathcal{D}\phi) \le V(t,\phi) \qquad (7)$$

and its derivative along the solution of (1) satisfies

$$\dot{V}(t,\phi) \le -c_2\phi^T(0)\phi(0) \qquad (8)$$

for any $\phi\in\mathcal{C}$, where

$$\dot{V}(t,\phi) \stackrel{\Delta}{=} \frac{d}{dt}V(t,x_t)\Big|_{x_t=\phi}\,.$$

Proof. See the full version of the paper (Han and Yu 2001).

Choose Q, R and S to be continuous piecewise linear, i.e.,

$$\begin{aligned} &Q^i(\alpha) \stackrel{\Delta}{=} Q(\delta_{i-1}+\alpha h) = (1-\alpha)Q_{i-1} + \alpha Q_i \\ &S^i(\alpha) \stackrel{\Delta}{=} S(\delta_{i-1}+\alpha h) = (1-\alpha)S_{i-1} + \alpha S_i \\ &R(\delta_{i-1}+\alpha h, \delta_{j-1}+\eta h) = R^{ij}(\alpha,\eta) \\ &\quad \stackrel{\Delta}{=} \begin{cases} (1-\alpha)R_{i-1,j-1} + \eta R_{ij} + (\alpha-\eta)R_{i,j-1},\ \alpha\ge\eta \\ (1-\eta)R_{i-1,j-1} + \alpha R_{ij} + (\eta-\alpha)R_{i-1,j},\ \alpha<\eta \end{cases} \end{aligned} \qquad (9)$$

for $0\le\alpha\le 1$, $0\le\eta\le 1$, where

$$\delta_i = -r + ih\,,\ i=0,\ 1,\ 2,\ \cdots,\ N\,;\ h = r/N$$

i.e., N is the number of divisions of the interval $[-r,\ 0]$, and h is the length of each division.

4. LYAPUNOV-KRASOVSKII FUNCTIONAL CONDITION

With the choice of piecewise linear functions, condition (7) can be written in a linear matrix inequality.

Proposition 1. *For piecewise linear Q, S and R as described by (9), there exists a* $c_1 > 0$ *such that the Lyapunov-Krasovskii functional satisfies (7) if*

$$\tilde{S} > 0 \tag{10}$$

and

$$\begin{bmatrix} P & \tilde{Q} \\ \tilde{Q}^T & \frac{1}{h}\tilde{S} + \tilde{R} \end{bmatrix} > 0 \tag{11}$$

where

$$\tilde{S} = diag(S_0\ S_1 \cdots S_{N-1}\ S_N)$$

$$\tilde{R} = \begin{bmatrix} R_{00} & R_{01} & \cdots & R_{0N} \\ R_{10} & R_{11} & \cdots & R_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{N0} & R_{N1} & \cdots & R_{NN} \end{bmatrix}$$

$$\tilde{Q} = (Q_0,\ Q_1,\ \cdots,\ Q_N)$$

Proof. See the full version of the paper (Han and Yu 2001).

5. LYAPUNOV-KRASOVSKII DERIVATIVE CONDITION

Similar to (7), condition (8) can also be written in a linear matrix inequality. We have the following proposition.

Proposition 2. *For piecewise linear Q, S and R as described by (9), equation (8) is satisfied for some* $c_2 > 0$, *and arbitrary* $\phi \in \mathcal{C}$ *if*

$$\Xi(t) = \begin{bmatrix} G_{11}(t) & -G_{12}(t) & \Gamma_1^s & \Gamma_1^a \\ -G_{12}^T(t) & G_{22}(t) & \Gamma_2^s & \Gamma_2^a \\ \Gamma_1^{sT} & \Gamma_2^{sT} & \frac{1}{h}S_d + R_d & 0 \\ \Gamma_1^{aT} & \Gamma_2^{aT} & 0 & \frac{3}{h}S_d \end{bmatrix} > 0 \tag{12}$$

for all $(A(t) \quad B(t)) \in \Omega$, *where*

$$G_{11}(t) = -PA(t) - A^T(t)P - S_N - Q_N - Q_N^T$$

$$G_{12}(t) = PB(t) - A^T(t)PC - Q_N^T C - Q_0$$

$$G_{22}(t) = C^T PB(t) + B^T(t)PC - C^T Q_0 - Q_0^T C + S_0$$

$$S_d = diag(S_{d1}\ S_{d2}\ \cdots\ S_{dN})$$

$$S_{di} = \frac{1}{h}(S_i - S_{i-1})$$

$$R_d = \begin{bmatrix} R_{d11} & R_{d12} & \cdots & R_{d1N} \\ R_{d21} & R_{d22} & \cdots & R_{d2N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{dN1} & R_{dN2} & \cdots & R_{dNN} \end{bmatrix}$$

$$R_{dij} = \frac{1}{h}(R_{ij} - R_{i-1,j-1});\ i, j = 1,\ 2,\ \cdots,\ N$$

$$\Gamma_j^s = [\Gamma_{j1}^s\ \ \Gamma_{j2}^s\ \ \cdots\ \ \Gamma_{jN}^s];\ \ j = 1,\ 2;$$

$$\Gamma_{1i}^s = -\frac{1}{2}A^T(t)(Q_i + Q_{i-1}) + \frac{1}{h}(Q_i - Q_{i-1}) - \frac{1}{2}(R_{i,N}^T + R_{i-1,N}^T);\ i = 1, 2, \cdots, N$$

$$\Gamma_{2i}^s = -\frac{1}{2}B^T(t)(Q_i + Q_{i-1}) - \frac{1}{h}C^T(Q_i - Q_{i-1}) + \frac{1}{2}(R_{i,0}^T + R_{i-1,0}^T);\ i = 1, 2, \cdots, N$$

$$\Gamma_j^a = [\Gamma_{j1}^a\ \ \Gamma_{j2}^a\ \ \cdots\ \ \Gamma_{jN}^a];\ \ j = 1,\ 2;$$

$$\Gamma_{1i}^a = \frac{1}{2}A^T(t)(Q_i - Q_{i-1}) + \frac{1}{2}(R_{i,N}^T - R_{i-1,N}^T)$$

$$\Gamma_{2i}^a = \frac{1}{2}B^T(t)(Q_i - Q_{i-1}) - \frac{1}{2}(R_{i,0}^T + R_{i-1,0}^T).$$

$i = 1, 2, \cdots, N$

Proof. See the full version of the paper (Han and Yu 2001).

6. STABILITY CRITERION

Now we consider the stability of the operator $\mathcal{D}$. According to Definition 3.1. in Chapter 9 in Hale and Lunel (1993), $\mathcal{D}$ is stable if the considered difference system $\mathcal{D}\phi = \phi(0) - C\phi(-r) = 0$ is asymptotically stable. A sufficient condition is that there exists a matrix $K > 0$ such that

$$C^T KC - K < 0 \tag{13}$$

From the above discussion, we now state and establish the following theorem.

Theroem 2. *The system (1)-(3) is asymptotically stable if there exist real matrices* $P = P^T > 0$, Q_i, S_i, *(* $i = 0,\ 1,\ 2,\ \cdots,\ N$ *),* R_{ij} *(* $i, j = 0,\ 1,\ 2,\ \cdots,\ N$ *) and* $K > 0$ *such that* $S_0 > 0$, *(11), (13) and (12) for all* $(A(t) \quad B(t)) \in \Omega$ *are satisfied.*

Proof. See the full version of the paper (Han and Yu 2001).

When $C \equiv 0$, system (1) becomes the following system of retarded type

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-r) \tag{14}$$

(12) reduces to

$$\begin{bmatrix} G_{11}^{retarded}(t) & -G_{12}^{retarded}(t) & \Gamma_1^s & \Gamma_1^a \\ -G_{12}^{retardedT}(t) & G_{22}^{retarded}(t) & \Gamma_2^s & \Gamma_2^a \\ \Gamma_1^{sT} & \Gamma_2^{sT} & \frac{1}{h}S_d + R_d & 0 \\ \Gamma_1^{aT} & \Gamma_2^{aT} & 0 & \frac{3}{h}S_d \end{bmatrix} > 0 \tag{15}$$

which is (46) in Gu (2001), where

$$G_{11}^{retarded}(t) = -PA(t) - A^T(t)P - S_N - Q_N - Q_N^T$$

$$G_{12}^{retarded}(t) = PB(t) - Q_0$$

$G_{22}^{retarded}(t) = S_0$.

From Theorem 2, the following corollary is easily obtained.

Corollary 1 (Gu 2001). *System (14) (2) and (3) is asymptotically stable there exist real matrices* $P = P^T > 0$, Q_i, S_i, *(* $i = 0, 1, 2, \cdots, N$ *) and* R_{ij} *(* $i, j = 0, 1, 2, \cdots, N$ *) such that (11) and (15) for all* $\left(A(t) \quad B(t) \right) \in \Omega$ *for all are satisfied.*

Proof. See the full version of the paper (Han and Yu 2001).

7. STABILITY OF SYSTEMS WITH NORM BOUNDED UNCERTAINTY

For polytopic uncertainty, it is clearly sufficient that (12) only needs to be satisfied at all the vertices. Now we consider the norm bounded uncertainty described by

$$A(t) = A + \Delta A(t)\,,\ B(t) = B + \Delta B(t) \qquad (16)$$

where

$$[\Delta A(t) \quad \Delta B(t)] = LF(t)[E_a \quad E_b] \qquad (17)$$

where $F(t) \in \mathbb{R}^{p\times q}$ is an unknown real and possibly time-varying matrix with Lebesgue measurable elements satisfying

$$\sigma_{\max}(F(t)) \le 1 \qquad (18)$$

and L, E_a and E_b are known real constant matrices which characterize how the uncertainty enters the nominal matrices *A* and *B*.

Now we can state and establish the following result.

Theroem 3. *The system described by (1) and (2), with uncertainty by (16) to (18) is asymptotically stable if there exist real* $n \times n$ *matrices* $X = X^T > 0$, Y_i, $W_i = W_i^T$ *(* $i = 0, 1, 2, \cdots, N$ *),* $Z_{ij} = Z_{ji}^T$ *(* $i, j = 0, 1, 2, \cdots, N$ *),* $K > 0$ *and a scalar* $\lambda > 0$ *such that* $W_0 > 0$, *(13) and the following LMIs are satisfied*

$$\begin{bmatrix} X & \tilde{Y} \\ \tilde{Y}^T & \frac{1}{h}\tilde{W} + \tilde{Z} \end{bmatrix} > 0 \qquad (19)$$

$$\begin{bmatrix} H_{00} & -H_{01} & H_{02} & \Pi_0^s & \Pi_0^a \\ -H_{01}^T & H_{11} & -H_{12} & \Pi_1^s & \Pi_1^a \\ H_{02}^T & -H_{12}^T & H_{22} & \Pi_2^s & \Pi_2^a \\ \Pi_0^{sT} & \Pi_1^{sT} & \Pi_2^{sT} & \frac{1}{h}W_d + Z_d & 0 \\ \Pi_0^{aT} & \Pi_1^{aT} & \Pi_2^{aT} & 0 & \frac{3}{h}W_d \end{bmatrix} > 0 \quad (20)$$

where

$$X = \lambda P\,,\ Y_i = \lambda Q_i\,,\ W_i = \lambda S_i\,,\ (i = 0, 1, 2, \cdots, N)$$

$$Z_{ij} = \lambda R_{ij} \ \ (i, j = 0, 1, 2, \cdots, N)$$

and

$$G_{ij}^0 = G_{ij}(t)\big|_{F(t)=0}\ ;\ i, j = 1, 2$$

$$\Gamma_j^{os} = \Gamma_j^s\big|_{F(t)=0}\ ;\ \Gamma_j^{0a} = \Gamma_j^a\big|_{F(t)=0}\ ;\ j = 1, 2$$

$$\tilde{Z} = \lambda\tilde{R}\ ;\ \tilde{Y} = \lambda\tilde{Q}$$

$$H_{00} = I\ ;\ H_{01} = L^T X\ ;\ H_{02} = L^T XC$$

$$H_{11} = \lambda G_{11}^0 - E_a^T E_a\ ;\ H_{12} = \lambda G_{12}^0 + E_a^T E_b$$

$$H_{22} = \lambda G_{22}^0 - E_b^T E_b$$

$$W_d = \lambda S_d\ ;\ Z_d = \lambda R_d$$

$$\Pi_j^s = [\Pi_{j1}^s,\ \Pi_{j2}^s,\ \cdots,\ \Pi_{jN}^s]\ ;\ j = 0, 1, 2$$

$$\Pi_{0i}^s = -\frac{1}{2}L^T(Y_i + Y_{i-1})\ ;\ \Pi_{1i}^s = \lambda\Gamma_{1i}^{0s}\ ;\ \Pi_{2i}^s = \lambda\Gamma_{2i}^{0s}$$

$$\Pi_j^a = [\Pi_{j1}^a\ \ \Pi_{j2}^a\ \ \cdots\ \ \Pi_{jN}^a]\ ;\ j = 0, 1, 2$$

$$\Pi_{0i}^a = \frac{1}{2}L^T(Y_i - Y_{i-1})$$

$$\Pi_{1i}^a = \lambda\Gamma_{1i}^{0a}\ ;\ \Pi_{2i}^a = \lambda\Gamma_{2i}^{0a}\ .$$

Proof. See the full version of the paper (Han and Yu 2001).

8. EXAMPLES

To illustrate the effectiveness of the approach, three numerical examples are presented.

Example 1. Consider the system

$$\dot{x}(t) - c\dot{x}(t-r) = -bx(t-r) \qquad (21)$$

where *b* and *c* are scalars, $|c| < 1$, $b > 0$. In Niculescu (2001), it was concluded that the system is asymptotically stable for

$$r < r_{\max}^{Niculescu} = \frac{1-|c|}{b}\ . \qquad (22)$$

The exact stability limit was analytically calculated as (Niculescu 2001)

$$r_{\max}^{analytical} = \frac{\sqrt{1-c^2}}{b}\arctan\sqrt{\frac{1}{c^2} - 1}\ . \qquad (23)$$

Let $c = 0.2$, $b = 1$, applying the discretized Lyapunov functional approach, the resulting stability limits obtained for different *N* are listed in the following table, along with the analytical limit $r_{\max}^{analytical}$ calculated using (23) and the estimated limit $r_{\max}^{Niculescu}$ obtained using (22).

$r_{\max}^{analytical}$	$r_{\max}^{Niculescu}$	$r_{\max}^{N=1}$	$r_{\max}^{N=2}$	$r_{\max}^{N=3}$
1.3418	0.8	1.3407	1.3417	1.3418

It is clear that the stability limit obtained by the discretized Lyapunov functional approach is less conservative than the result in Niculescu (2001) and it converges to analytical solution as N increases.

Example 2. Consider the nominal delay-differential system of neutral type

$$\dot{x}(t) - C\dot{x}(t-r) = Ax(t) + Bx(t-r) \qquad (24)$$

where

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix},\ B = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$

$$C = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix},\ |c| < 1.$$

The exact stability limit can be analytically calculated as follows.

(i) For $|c| < 1$ and $c \neq 0$,

$$r_{\max}^{analytical} = \frac{1}{\omega}\arccos\left(\frac{c\omega^2 - 0.9}{1 + c^2\omega^2}\right) \qquad (25)$$

where

$$\omega = \sqrt{\frac{-1 + 1.19c^2 + \sqrt{1 - 1.62c^2 + 0.6561c^4}}{2(c^2 - c^4)}}.$$

(ii) For $c = 0$, $r_{\max}^{analytical} = 6.17258$

Let $c = 0.5$, the maximum time-delay for stability as judged by the criterion in Han (2002) and the discretized Lyapunov functional approach, are estimated in the following table, along with the analytical limit $r_{\max}^{analytical}$ calculated using (25).

$r_{\max}^{analytical}$	$r_{\max}^{Han}$	$r_{\max}^{N=1}$	$r_{\max}^{N=2}$	$r_{\max}^{N=3}$
4.7388	3.62	4.6850	4.7357	4.7381

It is again to show that the stability limit obtained by the discretized Lyapunov functional approach is less conservative than the result in Han (2002) and it converges to analytical solution as N increases.

We now consider the effect of parameter c on the maximum time-delay for stability $r_{\max}$. The following table gives the $r_{\max}$ by the criteria in Han (2002) and this paper for $N = 3$. It is clear that the new criterion here significantly improve the estimate of stability limit over the results in Han (2002).

c	0.00	0.10	0.30
Han (2002)	4.35	4.33	4.10
This paper	6.17	6.03	5.54
c	0.50	0.70	0.90
Han (2002)	3.62	2.73	0.99
This paper	4.73	3.50	1.57

Example 3. Consider the following uncertain delay-differential system of neutral type

$$\dot{x}(t) - C\dot{x}(t-r) = [A + \Delta A(t)]x(t) + [B + \Delta B(t)]x(t-r)$$

where A, B and C are the same as Example 2, and $\Delta A(t)$ and $\Delta B(t)$ are unknown matrices satisfying $\|\Delta A(t)\| \le \alpha$ and $\|\Delta B(t)\| \le \alpha$, $\forall t$. The system above is of the form of Eqs. (16) and (18) with $L = \alpha I$ and $E_a = E_b = I$.

For $\alpha = 0.2$, the maximum value $r_{\max}$ is listed by the criterion in Han (2002) and the discretized Lyapunov functional method for $N = 3$ in the following table for various parameter c. As c increases, $r_{\max}$ decreases.

c	0.05	0.10	0.15	0.20
Han (2002)	1.63	1.48	1.33	1.16
This paper	2.98	2.83	2.66	2.49
c	0.25	0.30	0.35	0.40
Han (2002)	0.98	0.79	0.59	0.37
This paper	2.31	2.12	1.93	1.73

For $c = 0.10$, the effect of uncertainty bound α on the maximum time-delay for stability $r_{\max}$ is studied by the criteria in Han (2002) and this paper for $N = 3$. The following table illustrates the numerical results for different α. We can see that $r_{\max}$ decreases as α increases.

α	0.05	0.10	0.15
Han (2002)	3.61	2.90	2.19
This paper	4.93	4.05	3.36
α	0.20	0.25	0.30
Han (2002)	1.48	0.77	0.03
This paper	2.83	2.40	2.06

It is clear that the stability limit obtained by the discretized Lyapunov functional approach is less conservative than the result in Han (2002).

9. CONCLUSION

The stability problem of linear delay-differential systems of neutral type has been investigated. The discretized Lyapunov functional approach has been developed. Stability criteria have been obtained. Numberical examples have shown that the results derived by these new criteria significantly improve the estimate of stability limit over the existing results in the literature.

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