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APPROXIMATE INPUT-OUTPUT LINEARIZATION OF NONMINIMUM PHASE NONLINEAR SYSTEMS WITH LINEAR UNFORCED DYNAMICS

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Abstract: This paper presents an approximate input-output linearization approach for nonminimum phase nonlinear systems with linear unforced dynamics. A special Byrnes Isidori normal form is introduced, where the internal dynamics are not affected by the inputs up to second order such that the corresponding transformation can be obtained by solving linear equations. After a factorization of the internal dynamics into a stable and into an antistable part a quadratic linear and internally stable input-output behaviour is achieved by state feedback in explicit form. *Copyright © 2002 IFAC*

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1. INTRODUCTION AND PROBLEM FORMULATION

In what follows the system under consideration will be a nonminimum phase nonlinear SISO system of nth order with state equations

$$\begin{aligned} \dot{x} &= A\,x + b(x)\,u \\ y &= c^T\,x \end{aligned} \tag{1}$$

having linear unforced dynamics (i.e. the dynamics of the system (1) for $u \equiv 0$ are linear) and equilibrium $(x_0, u_0) = (0,0)$. Since bilinear systems are a special case of (1), one can use any approach for bilinearization of general systems in order to obtain a nonlinear system with linear unforced dynamics (see e.g. Svoronos et al., 1980). In the sequel it is supposed that the continuously differentiable nonlinear vector function $b(x)$ is defined for all x in a neighborhood of $x_0 = 0$. Now consider for $b(x)$ the expansion

$$b(x) = b + b_1(x) \tag{2}$$

where $b = b(0) \neq 0$ and $b_1(0) = 0$. Using (2) the system equations (1) read

$$\begin{aligned} \dot{x} &= A\,x + \big(b + b_1(x)\big)u \\ y &= c^T\,x \end{aligned} \tag{3}$$

In the following it is assumed that the nonlinear system (1) has *relative degree r*, $1 \leq r \leq n$, at $x_0 = 0$ (Isidori, 1995), that is the relations

$$c^T A^{i-1}(b + b_1(x)) = 0 \quad , \quad i = 1(1)r-1 \tag{4}$$

for all x in a neighborhood of x_0 and

$$c^T A^{r-1}(b + b_1(0)) = c^T A^{r-1} b \neq 0 \tag{5}$$

hold in light of (2), where in the case $r = 1$ only (5) has to be evaluated. Since $b_1(0) = 0$ always holds (see (2)) (4) and (5) imply $b = b(0) \neq 0$ if $r \leq n$.
It is shown in (Isidori, 1995), that there always exists a local transformation

$$z = \begin{bmatrix} \xi \\ \cdots \\ \eta \end{bmatrix} = \begin{bmatrix} c^T x \\ \vdots \\ c^T A^{r-1} x \\ \cdots\cdots \\ t_\eta(x) \end{bmatrix} = \begin{bmatrix} T_\xi x \\ \cdots \\ t_\eta(x) \end{bmatrix} = t(x) \quad , \quad t(x_0) = 0 \tag{6}$$

into the normal form

$$\begin{aligned} \dot{\xi} &= J\xi + e_r c^T A^r t^{-1}(\xi,\eta) + e_r c^T A^{r-1} b(1 + n^T b_1(t^{-1}(\xi,\eta)))u \\ \dot{\eta} &= q(\xi,\eta) \\ y &= \xi_1 \end{aligned} \tag{7}$$

where $q(\xi,\eta)$ denotes an $(n-r,1)$ vector of possibly nonlinear functions of the real variables $z_1, \ldots, z_n$ and

$$n^T = \frac{c^T A^{r-1}}{c^T A^{r-1} b} \tag{8}$$

The (r,r) matrix J is a matrix with zero elements except for ones on the first upper secondary diagonal and the $(r,1)$ vector e_r denotes the rth unit vector. The normal form (7) is referred to as *input normalized Byrnes Isidori normal form*, since $q(\xi,\eta)$ does not depend on the input u. Now consider the state feedback (which exists and is well defined for all z in a neighborhood of $z_0 = t(x_0) = 0$; see (5))

$$u = \frac{1}{1+n^T b_1(\xi,\eta)}(-c^T A^r t^{-1}(\xi,\eta) + \bar{u}) \tag{9}$$

in which $\bar{u}$ is a new input for the system (7) yielding the closed loop

$$\begin{aligned} \dot{\xi} &= J\,\xi + e_r \bar{u} \\ \dot{\eta} &= q(\xi,\eta) \\ y &= \xi_1 \end{aligned} \tag{10}$$

with a linear ξ-subsystem of dimension r (a chain of r integrators), which is responsible for the input-output behaviour, and a possibly nonlinear η-subsystem of dimension n-r not affecting the output y. The unforced dynamics of the nonlinear η-subsystem, i.e.

$$\dot{\eta} = q(0,\eta) \tag{11}$$

are called *(undriven) zero dynamics* of the nonlinear system (3). A linear stable reference transfer behaviour can be achieved by stabilizing the linear ξ-subsystem by means of state feedback for ξ that decouples the nonlinear η-subsystem from the input-output behaviour. Hence stability of the closed loop only depends on the behaviour of the zero dynamics (11). The nonlinear system is called *(locally) minimum phase* at $(\xi,\eta) = (0,0)$ if the zero dynamics are locally asymptotically stable at $\eta = 0$ and *(locally) nonminimum phase* at $(\xi,\eta) = (0,0)$ if the zero dynamics are locally unstable at $\eta = 0$. Thus exact input-output linearization with internal stability is always possible in the case of minimum phase systems.

For nonminimum phase nonlinear systems an exact input-output linearization with internal stability is not feasible in most cases. A standard approach to the problem of approximate input-output linearization is to introduce new inputs, with respect to which the system has minimum phase characteristics (for an overview see Allgöwer and Doyle, 1998). The problem of this method is that the desired level of linearity achieved with respect to the true outputs cannot be readily taken into acount in the design procedure. In this contribution the approximate input-output linearization approach of Allgöwer (1997) is adopted that computes the approximately linearizing state feedback after a factorization of the zero dynamics into a stable and into an antistable part, where the latter is approximately linearized and decoupled from the stable part. The corresponding controller stabilizes the unstable zero dynamics and increases the level of linearity with respect to the input-output behaviour of the resulting closed loop. For the latter requirement a quadratic linearizing static state feedback is considered in this paper, that is the Taylor series expansion of the state equations related to the input-output behaviour of the compensated system does not contain nonlinear terms up to order two. In the approach of Allgöwer (1997) it is assumed that the system is represented in input normalized Byrnes Isidori normal form (7) that amounts to solve a set of partial differential equations in order to obtain the functions $\eta_i(x)$, $i = 1(1)n$-r, in (6) (for details see Isidori, 1995), which in general is not an easy task to perform. In this contribution the solution of partial differential equations is circumvented by introducing the normal form

$$\begin{aligned} \dot{\xi} &= J\,\xi + e_r c^T A^r t^{-1}(\xi,\eta) + e_r c^T A^{r-1} b(1+n^T b_1(t^{-1}(\xi,\eta)))u \\ \dot{\eta} &= q(\xi,\eta) + O^{[2+]}(\xi,\eta,u) \\ y &= \xi_1 \end{aligned} \tag{12}$$

where $O^{[k+]}(\xi, \eta, u)$ denotes an expression containing arbitrary terms in ξ, η and u of order strictly larger than k. It is shown in the sequel, that the functions $\eta_i(x)$ in (12), needed in the stabilizing state feedback, can be simply computed by solving linear algebraic equations. The normal form (12) will be referred to as *quadratic input normalized Byrnes Isidori normal form*, since u does not affect the differential equation for η up to quadratic terms.

The next section introduces the quadratic input normalized Byrnes Isidori normal form and shows how the approach of Allgöwer (1997) can be applied to a system represented in this form. Section 3 demonstrates the proposed design procedure by means of a simple example.

2. APPROXIMATE INPUT-OUTPUT LINEARIZATION APPROACH

If the following it is assumed that the linear approximation (A,b,c^T) of the nonlinear system (3) has $n_a \geq 1$ zeros in the open right half plane and no zeros on the imaginary axis giving a locally nonminimum phase nonlinear system, i.e. its zero dynamics are locally unstable. Then exact input-output linearization with internal stability is no longer feasible, when using the input-output linearizing state feedback (9). In order to achieve at least an approximately linear input-output behaviour of the controlled system, one can use the approximate input-output linearizing state feedback assuring internal stability developed in this section. The approach proposed consists of the following steps:

(1) Transformation of the system into quadratic input normalized Byrnes Isidori normal form (12) and compensation of the nonlinearities in the direct input-output channel giving a linear ξ-subsystem.

(2) Linear transformation of the system resulting from step (1) such that the η-subsystem is factorized into an antistable η_a-subsystem and into a stable η_s-subsystem in the first approximation.

(3) Using state feedback that decouples the locally stable η_s-subsystem from the input-output behaviour, quadratic linearizes the η_a-subsystem and stabilizes the input-output behaviour comprised of the linear ξ-subsystem and the η_a-subsystem.

2.1 Transformation into quadratic input normalized Byrnes Isidori normal form

The next lemma introduces the nonlinear change of coordinates which transforms the system (3) into quadratic input normalized Byrnes Isidori normal form (12). In the sequel some results from Kronecker calculus are used, which are collected in the appendix.

Lemma 1 *Consider the nth order nonlinear system (3) with relative degree r at $x_0 = 0$ and the nonlinear transformation*

$$z = \begin{bmatrix} \xi \\ \cdots \\ \eta \end{bmatrix} = \begin{bmatrix} T_\xi \\ \cdots \\ T_\eta \end{bmatrix} x + \tfrac{1}{2}\begin{bmatrix} 0 \\ \cdots\cdots \\ G(H \otimes H) \end{bmatrix} x^{(2)} = T_1\, x + T_2\, x^{(2)} \quad (13)$$

where T_ξ is given by (6) and the (n-r,n) matrix T_η is a solution of

$$T_\eta \begin{bmatrix} b & Ab & \dots & A^{r-1}b \end{bmatrix} = \begin{bmatrix} t_{\eta,1}^T \\ \vdots \\ t_{\eta,n-r}^T \end{bmatrix} \begin{bmatrix} b & Ab & \dots & A^{r-1}b \end{bmatrix} = 0 \quad (14)$$

with $\operatorname{rank}(T_\eta) = n - r$ *and the (n,n) matrix H is given by*

$$H = \begin{bmatrix} H_1 & b \end{bmatrix}^{-1} \quad (15)$$

with H_1 chosen such that $\det[H_1 \quad b] \neq 0$. *The (n-r,n^2) matrix G in (13) is obtained from*

$$G = \begin{bmatrix} q_{r+1}^T \otimes e_n^T + e_n^T \otimes q_{r+1}^T \\ \vdots \\ q_n^T \otimes e_n^T + e_n^T \otimes q_n^T \end{bmatrix} \quad (16)$$

where the (1,n) vectors q_{r+i}^T, i = 1(1)n-r, are resulting from

$$q_{r+i}^T = -t_{\eta,i}^T N H^{-1}(I - \tfrac{1}{2} e_n e_n^T) \quad (17)$$

with the (n,n) Jacobian matrix N

$$N = \frac{\partial}{\partial x^T} b(x)\Big|_{x_0} \quad (18)$$

The nonlinear transformation (13) has a Jacobian matrix which is nonsingular at x_0 and therefore qualifies as a local coordinates transformation in a neighborhood of x_0. The inverse transformation in a neighborhood of x_0 reads

$$x = T_1^{-1} z - T_1^{-1} T_2 (T_1^{-1} \otimes T_1^{-1}) z^{(2)} + O^{[2+]}(z) \quad (19)$$

Proof. In order to show that the Jacobian matrix T_1 of the nonlinear transformation (13) is nonsingular, it is proved that the rows of T_1 are linearly independent. To this end it is verified that the expression

$$\begin{aligned} &k_1 c^T + \ldots + k_{r-1} c^T A^{r-2} + k_r c^T A^{r-1} \\ &\quad + k_{r+1} t_{\eta,1}^T + \ldots + k_n t_{\eta,n-r}^T = 0^T \end{aligned} \quad (20)$$

is only satisfied for scalars $k_1 = ... = k_n = 0$. Postmultiplying (20) with b gives

$$\begin{aligned} &k_1 c^T b + \ldots + k_{r-1} c^T A^{r-2} b + k_r c^T A^{r-1} b \\ &\quad + k_{r+1} t_{\eta,1}^T b + \ldots + k_n t_{\eta,n-r}^T b = 0 \end{aligned} \quad (21)$$

Since the system (3) has relative degree r at $x_0 = 0$ one has

$$c^T A^{i-1} b = 0 \quad , \quad i = 1(1)r-1 \quad (22)$$

implied by (4) such that $k_r = 0$ in view of (5) and (14). Thus (20) becomes

$$\begin{aligned} &k_1 c^T + \ldots + k_{r-2} c^T A^{r-3} + k_{r-1} c^T A^{r-2} \\ &\quad + k_{r+1} t_{\eta,1}^T + \ldots + k_n t_{\eta,n-r}^T = 0^T \end{aligned} \quad (23)$$

By postmultiplying (23) with Ab one obtains

$$\begin{aligned} &k_1 c^T Ab + \ldots + k_{r-2} c^T A^{r-2} b + k_{r-1} c^T A^{r-1} b \\ &\quad + k_{r+1} t_{\eta,1}^T Ab + \ldots + k_n t_{\eta,n-r}^T Ab = 0 \end{aligned} \quad (24)$$

and hence $k_{r-1} = 0$ in light of (5), (14) and (22). Proceeding with this approach (i.e. successively postmultiplying with A^2b up to $A^{r-1}b$) it is straightforward to verify, that $k_1 = ... = k_r = 0$ holds. Then $k_{r+1} = ... = k_n = 0$ follows from the assumption that T_η has full row rank (see Lemma 1). The inverse transformation of (13) can be represented as

$$x = S_1 z + S_2 z^{(2)} + O^{[2+]}(z) \quad (25)$$

in a neighborhood of $z_0 = x_0 = 0$. By introducing (25) in (13) and using (A4) and (A5) one obtains

$$\begin{aligned} z &= T_1 S_1 z + T_1 S_2 z^{(2)} \\ &\quad + T_2((S_1 z + S_2 z^{(2)}) \otimes (S_1 z + S_2 z^{(2)})) + O^{[2+]}(z) \\ &= T_1 S_1 z + T_1 S_2 z^{(2)} + T_2 (S_1 \otimes S_1) z^{(2)} + O^{[2+]}(z) \end{aligned} \quad (26)$$

giving $S_1 = T_1^{-1}$ and $S_2 = -T_1^{-1} T_2 (T_1^{-1} \otimes T_1^{-1})$ which yields (19) in view of (25).

□

By applying the quadratic transformation (13) to system (3) and using a state feedback law, that cancels all nonlinearities in the direct input-output channel one arrives at a system represented in quadratic input normalized Byrnes Isidori normal form with linear ξ-subsystem as given in the following theorem.

Theorem 1 *Consider the nonlinear system (3) with relative degree r at $x_0 = 0$. Then the local change of coordinates (13) and the local state feedback*

$$u = \frac{1}{1 + n^T b_1(x)} \frac{1}{c^T A^{r-1} b} (-c^T A^r x + \bar{u}) \quad (27)$$

both well defined in a neighborhood of x_0 transform the nonlinear system (3) into the quadratic input normalized Byrnes Isidori normal form

$$\begin{aligned} \dot{\xi} &= J\xi + e_r \bar{u} \\ \dot{\eta} &= M\,\eta + P\,\xi + Q_z\, z^{(2)} + O^{[2+]}(z, \bar{u}) \\ y &= \xi_1 \end{aligned} \quad (28)$$

where

$$\begin{bmatrix} P & M \end{bmatrix} = T_\eta A T_1^{-1} \quad (29)$$

and

$$Q_z = (G(HA \otimes H) - T_\eta A T_1^{-1} T_2)(T_1^{-1} \otimes T_1^{-1}) \quad (30)$$

Proof. Consider the system (3) and the transformation (13) of Lemma 1

$$\xi = T_\xi x \quad (31)$$

$$\eta = T_\eta x + \tfrac{1}{2} G(H \otimes H) x^{(2)} \tag{32}$$

The time derivatives of (31) and (32) become with the first line of (7)

$$\dot{\xi} = T_\xi \dot{x} = J\xi + e_r c^T A^r x + e_r c^T A^{r-1} b(1 + n^T b_1(x))u \tag{33}$$

$$\dot{\eta} = (T_\eta + \frac{\partial}{\partial x^T}(\tfrac{1}{2} G(H \otimes H) x^{(2)}))\dot{x} \tag{34}$$

In order to evaluate (34) the ith row of $\frac{1}{2} G(H \otimes H) x^{(2)}$ is written as

$$\begin{aligned} &\tfrac{1}{2}(q_{r+i} \otimes e_n + e_n \otimes q_{r+i})^T (H \otimes H) x^{(2)} \\ &= \tfrac{1}{2}((H^T \otimes H^T)\mathrm{vec}(e_n q_{r+i}^T + q_{r+i} e_n^T))^T x^{(2)} \\ &= \tfrac{1}{2}(\mathrm{vec}(Q_i))^T x^{(2)} = \tfrac{1}{2} x^T Q_i x \end{aligned} \tag{35}$$

where successively the relations (A3), (A10), A(7), A(12) and

$$Q_i = H^T (e_n q_{r+i}^T + q_{r+i} e_n^T) H = H^T \tilde{Q}_i H = Q_i^T \tag{36}$$

for i = 1(1)n-r were used. Along with

$$\frac{\partial}{\partial x^T} \tfrac{1}{2} x^T Q_i x = x^T Q_i \tag{37}$$

and the expansion

$$b(x) = b + Nx + O^{[2+]}(x) \tag{38}$$

around x_0 = 0 with $b = b(0)$ and $N = \partial b(x_0)/\partial x^T$ (34) obtains the form

$$\begin{aligned} \dot{\eta} &= (T_\eta + [Q_1 x \quad \ldots \quad Q_{n-r} x]^T)\dot{x} \\ &= T_\eta A x + T_\eta (b + Nx)u + [Q_1 x \quad \ldots \quad Q_{n-r} x]^T A x \\ &\quad + [Q_1 x \quad \ldots \quad Q_{n-r} x]^T (b + Nx)u + O^{[2+]}(x,u) \\ &= T_\eta A x + Q_x x^{(2)} + O^{[2+]}(x,u) \\ &\quad + (T_\eta b + T_\eta N x + [Q_1 b \quad \ldots \quad Q_{n-r} b]^T x)u \end{aligned} \tag{39}$$

where

$$Q_x = [\mathrm{vec}(Q_1 A) \quad \ldots \quad \mathrm{vec}(Q_{n-r} A)]^T \tag{40}$$

in view of

$$\begin{aligned} Q_x x^{(2)} &= [x^T Q_1 A x \quad \ldots \quad x^T Q_{n-1} A x]^T \\ &= [x^T A^T Q_1 x \quad \ldots \quad x^T A^T Q_{n-1} x]^T \end{aligned}$$

follows from (A12). Using successively (36), (A7), (A3) and A(10) the ith row of Q_x in (40) can be written as

$$\begin{aligned} &(\mathrm{vec}(Q_i A))^T = (\mathrm{vec}(H^T \tilde{Q}_i H A))^T \\ &= ((A^T H^T \otimes H^T)\mathrm{vec}(\tilde{Q}_i))^T = (\mathrm{vec}(\tilde{Q}_i))^T (HA \otimes H) \\ &= (q_{r+i} \otimes e_n + e_n \otimes q_{r+i})^T (HA \otimes H) \end{aligned} \tag{41}$$

for i = 1(1)n-r giving

$$Q_x = G(HA \otimes H) \tag{42}$$

in light of (16). Since

$$T_\eta b = 0 \tag{43}$$

is implied by (14) and $T_\eta N + [Q_1 b \quad \ldots \quad Q_{n-r} b]^T$ vanishes since the ith row can be simplified to

$$\begin{aligned} &t_{\eta,i}^T N + b^T Q_i = t_{\eta,i}^T N + b^T H^T (e_n q_{r+i}^T + q_{r+i} e_n^T) H \\ &= t_{\eta,i}^T N + e_n^T (e_n q_{r+i}^T + q_{r+i} e_n^T) H \\ &= t_{\eta,i}^T N + (q_{r+i}^T + q_{r+i}^T e_n e_n^T) H \\ &= t_{\eta,i}^T N + q_{r+i}^T (I + e_n e_n^T) H \\ &= t_{\eta,i}^T N - t_{\eta,i}^T N H^{-1} (I - \tfrac{1}{2} e_n e_n^T)(I + e_n e_n^T) H = 0^T \end{aligned} \tag{44}$$

for i = (1)n-r, where successively (36), (15) and (17) were used, (39) takes the form

$$\dot{\eta} = T_\eta A x + Q_x x^{(2)} + O^{[2+]}(x,u) \tag{45}$$

As the system (3) has relative degree r at x_0 = 0 the state feedback law (27) is well defined (see (5)) and can be applied to (33) and (45) giving

$$\begin{aligned} \dot{\xi} &= J\xi + e_r \bar{u} \\ \dot{\eta} &= T_\eta A x + Q_x x^{(2)} + O^{[2+]}(x,\bar{u}) \end{aligned} \tag{46}$$

By substituting the inverse transformation (19) in (46) one finally obtains (28).

□

2.2 Linear transformation of the η-subsystem

Suppose the Jacobi linearization of the zero dynamics of the system (3) has no eigenvalues with zero real part. Then the η-subsystem of system (28) can be factorized into an antistable η_a-subsystem and into a stable η_s-subsystem in the first approximation by means of the linear transformation

$$\begin{bmatrix} \xi \\ \eta_a \\ \eta_s \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & T_l \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} \tag{47}$$

where the nonsingular (n-r,n-r) matrix T_l satisfies

$$\tilde{M} = T_l M T_l^{-1} = \begin{bmatrix} M_a & \underset{(n_a, n-r-n_a)}{0} \\ N_s & M_s \end{bmatrix} \tag{48}$$

such that Re$\lambda(M_a) > 0$ and Re$\lambda(M_s) < 0$ with $\lambda(\cdot)$ denoting the spectrum of a matrix. Since the transformation (47) leaves the system states ξ unchanged, the transformation of the matrix P (see (29)) reads

$$\tilde{P} = \begin{bmatrix} P_a \\ P_s \end{bmatrix} = T_l P \tag{49}$$

and by using (A4) it is straightforward to show that the coefficient matrix of the quadratic term takes the form

$$\begin{bmatrix} Q_a \\ Q_s \end{bmatrix} = T_l\, Q_z\, \mathrm{diag}(I_r, I_r \otimes T_l^{-1}, T_l^{-1} \otimes I_r, T_l^{-1} \otimes T_l^{-1}) \tag{50}$$

The resulting system has the special quadratic input normalized Byrnes Isidori normal form with state variables $z'^T = [\xi^T \quad \eta_a^T \quad \eta_s^T]$

$$\begin{aligned} \begin{bmatrix} \dot{\xi} \\ \dot{\eta}_a \\ \dot{\eta}_s \end{bmatrix} &= \begin{bmatrix} J & 0 & 0 \\ P_a & M_a & 0 \\ P_s & N_s & M_s \end{bmatrix} \begin{bmatrix} \xi \\ \eta_a \\ \eta_s \end{bmatrix} + \begin{bmatrix} e_r \\ 0 \\ 0 \end{bmatrix} \bar{u} + \begin{bmatrix} 0_r \\ Q_a \\ Q_s \end{bmatrix} z'^{(2)} + \begin{bmatrix} 0_r \\ O^{[2+]}(z',\bar{u}) \\ O^{[2+]}(z',\bar{u}) \end{bmatrix} \\ y &= \xi_1 \end{aligned} \tag{51}$$

A Jacobi linearization of (51) directly shows that the η_a-subsystem of order n_a is locally antistable (i.e. its first approximation has all its eigenvalues in the open right half plane) and the η_s-subsystem of order n-r-n_a is locally asymptotically stable.

2.3 Quadratic input-output linearizing state feedback

Finally the quadratic input-output linearizing state feedback law assuring internal stability can be obtained from the following theorem.

Theorem 2 *Consider the nth order system (3) with relative degree r at $x_0 = 0$ represented in the quadratic input normalized Byrnes Isidori normal form (51). The input-output behaviour of the system (3) can be linearized up to second order using a static state feedback if*

(i) the Jacobi linearization of the system (3) is controllable,

(ii) there exists an $(n_a,1)$ vector $\Phi^{[2]}(\bar{\xi},\eta')$ of homogeneous polynomials of second degree in $\bar{\xi}^T = [\xi_1 \ \ldots \ \xi_{r-1}]$ and $\eta'^T = [\eta_a^T \ \ \eta_s^T]$ satisfying

$$\frac{\partial}{\partial\bar{\xi}^T}\Phi^{[2]}(\bar{\xi},\eta')\bar{J}\,\xi + \frac{\partial}{\partial\eta'^T}\Phi^{[2]}(\bar{\xi},\eta')\begin{bmatrix}\tilde{P} & \tilde{M}\end{bmatrix}z' - M_a\Phi^{[2]}(\bar{\xi},\eta') = -Q_a z'^{(2)} \tag{52}$$

where $\bar{J} = [I_{r-1} \quad 0]\,J$.

The quadratic linearizing state feedback law for the system (3) is given by

$$u = \frac{1}{1+n^T b_1(x)}\frac{1}{c^T A^{r-1} b}(-c^T A^r x - k_\xi^T \xi(x) - k_{\eta_a'}^T \eta_a'(x) + lw) \tag{53}$$

with

$$\eta_a'(x) = \eta_a(x) + \Phi^{[2]}(\bar{\xi}(x),\eta'(x)) \tag{54}$$

$$\eta'(x) = \begin{bmatrix}\eta_a(x)\\ \eta_s(x)\end{bmatrix} = T_l T_\eta x + T_l G(H\otimes H)x^{(2)}$$

$$\xi(x) = T_\xi x$$

The $(r+n_a)$th order input-output behaviour of the system (3) with feedback (53) is represented by

$$\begin{bmatrix}\dot{\xi}\\ \dot{\eta}_a'\end{bmatrix} = \begin{bmatrix}J - e_r k_\xi^T & -e_r k_{\eta_a'}^T\\ P_a & M_a\end{bmatrix}\begin{bmatrix}\xi\\ \eta_a'\end{bmatrix} + \begin{bmatrix}e_r l\\ 0\end{bmatrix}w + \begin{bmatrix}0_r\\ O^{[2+]}(x,w)\end{bmatrix}$$
$$y = \xi_1 \tag{55}$$

System (3) with feedback (53) is internally stable if and only if

$$\operatorname{Re}\lambda\left(\begin{bmatrix}J - e_r k_\xi^T & -e_r k_{\eta_a'}^T\\ P_a & M_a\end{bmatrix}\right) < 0 \tag{56}$$

Furthermore the feedback vectors k_ξ^T and $k_{\eta_a'}^T$ can always be chosen such that (56) holds.

Proof. Consider the transformation

$$\eta_a' = \eta_a + \Phi^{[2]}(\bar{\xi},\eta') \tag{57}$$

for the coordinates η_a of system (51) giving

$$\dot{\eta}_a' = \dot{\eta}_a + \frac{\partial}{\partial\bar{\xi}^T}\Phi^{[2]}(\bar{\xi},\eta')\dot{\bar{\xi}} + \frac{\partial}{\partial\eta'^T}\Phi^{[2]}(\bar{\xi},\eta')\dot{\eta}'$$
$$= P_a\xi + M_a\eta_a + Q_a z'^{(2)} + \frac{\partial}{\partial\bar{\xi}^T}\Phi^{[2]}(\bar{\xi},\eta')\bar{J}\xi + O^{[2+]}(z',\bar{u})$$
$$+\frac{\partial}{\partial\eta'^T}\Phi^{[2]}(\bar{\xi},\eta')\left(\begin{bmatrix}\tilde{P} & \tilde{M}\end{bmatrix}z' + \begin{bmatrix}Q_a\\ Q_s\end{bmatrix}z'^{(2)} + O^{[2+]}(z',\bar{u})\right)$$
$$= P_a\xi + M_a\eta_a' + O^{[2+]}(z',\bar{u})$$
$$+\frac{\partial}{\partial\bar{\xi}^T}\Phi^{[2]}(\bar{\xi},\eta')\bar{J}\xi + \frac{\partial}{\partial\eta'^T}\Phi^{[2]}(\bar{\xi},\eta')\begin{bmatrix}\tilde{P} & \tilde{M}\end{bmatrix}z' \tag{58}$$
$$-M_a\Phi^{[2]}(\bar{\xi},\eta') + Q_a z'^{(2)}$$

If there exists a solution $\Phi^{[2]}(\bar{\xi},\eta')$ of (52) system (51) takes the form

$$\begin{bmatrix}\dot{\xi}\\ \dot{\eta}_a'\\ \dot{\eta}_s\end{bmatrix} = \begin{bmatrix}J & 0 & 0\\ P_a & M_a & 0\\ P_s & N_s & M_s\end{bmatrix}\begin{bmatrix}\xi\\ \eta_a'\\ \eta_s\end{bmatrix} + \begin{bmatrix}e_r\\ 0\\ 0\end{bmatrix}\bar{u} + \begin{bmatrix}0_r\\ 0_{n_a}\\ Q_s''\end{bmatrix}z''^{(2)} + \begin{bmatrix}0_r\\ O^{[2+]}(z'',\bar{u})\\ O^{[2+]}(z'',\bar{u})\end{bmatrix}$$
$$y = \xi_1 \tag{59}$$

in the new coordinates $z''^T = [\xi^T \quad \eta_a'^T \quad \eta_s^T]$. Applying the feedback

$$\bar{u} = -k_\xi^T\xi(x) - k_{\eta_a'}^T\eta_a'(x) + lw \tag{60}$$

to the system (59) yields

$$\begin{bmatrix}\dot{\xi}\\ \dot{\eta}_a'\\ \dot{\eta}_s\end{bmatrix} = \begin{bmatrix}J - e_r k_\xi^T & -e_r k_{\eta_a'}^T & 0\\ P_a & M_a & 0\\ P_s & N_s & M_s\end{bmatrix}\begin{bmatrix}\xi\\ \eta_a'\\ \eta_s\end{bmatrix} + \begin{bmatrix}e_r l\\ 0\\ 0\end{bmatrix}w + \begin{bmatrix}0_r\\ 0_{n_a}\\ Q_s''\end{bmatrix}z''^{(2)}$$
$$+\begin{bmatrix}0_r\\ O^{[2+]}(x,w)\\ O^{[2+]}(x,w)\end{bmatrix} \tag{61}$$
$$y = \xi_1$$

where the stable η_s-subsystem is decoupled from the input-output behaviour (55) up to second order. In view of the Jacobian matrix of (61) it follows from the Principle of Stability in the First Approximation that the compensated system is internally stable if (56) holds, since $\operatorname{Re}\lambda(M_s) < 0$ by assumption. Transformation (13) and nonsingular feedback (27) do not change the controllability of the Jacobi linearization of the system. Therefore condition (i) directly implies controllability of the Jacobian linearization of (59), such that there exist feedback vectors k_ξ^T and $k_{\eta_a'}^T$ which assign an arbitrary set of eigenvalues to the system (55) in the first approximation.

□

Remark In order to find the monomial coefficients of the vector $\Phi^{[2]}(\bar{\xi},\eta')$ satisfying (52) one must solve a set of linear algebraic equations. In general the number of equations is greater than the number of unknowns, such that a solution may not exist. However, in this case one can seek an approximate solution of (52) in some least square sense (for more details see [Ka1]).

3. EXAMPLE

Consider the nonminimum phase bilinear system

$$\dot{x} = \begin{bmatrix}0 & 1\\ \frac{3}{4} & -1\end{bmatrix}x + \left(\begin{bmatrix}0\\ 1\end{bmatrix} + \begin{bmatrix}6 & 4\\ 0 & 0\end{bmatrix}x\right)u$$
$$y = \begin{bmatrix}-1 & 1\end{bmatrix}x \tag{62}$$

Chosing $t_\eta^T = [1 \quad 0]$ satisfying (14) and $H = I$ (see (15)) the matrix G in (16) can be computed from (17) giving the transformation (13)

$$z = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -3 & -3 & -2 \end{bmatrix} x^{(2)} \qquad (63)$$

By applying the transformation (63) and the feedback law (27) to (62) one obtains the quadratic input normalized Byrnes Isidori normal form

$$\begin{aligned} \dot{\xi} &= \bar{u} \\ \dot{\eta} &= \eta + \xi + 9\eta\xi + \tfrac{9}{2}\eta^2 + O^{[2+]}(\xi, \eta, \bar{u}) \\ y &= \xi \end{aligned} \qquad (64)$$

given by (28). In order to achive a quadratic linearization in (64) solve (52) for $\Phi^{[2]}(\eta) = 0.5q\eta^2$. This yields the solution $q = -9$ giving the new system state

$$\eta' = \eta + \Phi^{[2]}(\eta) = \eta - \tfrac{9}{2}\eta^2 \qquad (65)$$

such that the state space representation

$$\begin{aligned} \dot{\xi} &= \bar{u} \\ \dot{\eta}' &= \eta' + \xi + O^{[2+]}(\xi, \eta', \bar{u}) \\ y &= \xi \end{aligned} \qquad (66)$$

has a quadratic linear η'-subsystem (see (59)). Thus the state feedback achieving an approximately linear input-output behaviour follows from (53) as

$$\begin{aligned} u &= \tfrac{1}{1-6x_1-4x_2}\left(-\tfrac{3}{4}x_1 + 2x_2 - k_\xi \xi - k_{\eta'}\eta' + lw\right) \\ &= \tfrac{1}{1-6x_1-4x_2}\left(-\tfrac{3}{4}x_1 + 2x_2 - k_\xi c^T x - k_{\eta'}(\eta - \tfrac{9}{2}\eta^2) + lw\right) \end{aligned} \qquad (67)$$

where

$$\eta = x_1 - 6x_1x_2 - 2x_2^2 \qquad (68)$$

in view of $c^T b = 1$ and (63).

4. CONCLUSIONS

A method for approximate input-output linearization of nonminimum phase systems with linear unforced dynamics was considered in this paper. The proposed approach is based on the transformation of the system to be controlled into quadratic input normalized Byrnes Isidori normal form first, where the corresponding transformation is obtained by solving a set of linear algebraic equations. The η-subsystem of the normal form is factorized into a stable and into an antistable part in the first approximation, where the latter part is quadratic linearized and the stable part is decoupled from the input-output behaviour giving a quadratic linear reference transfer behaviour. The corresponding feedback law was obtained in explicit form. A simple example demonstrated the design procedure.

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APPENDIX

Consider the (m,n) matrix $A = [a_{ij}]$ and the (p,q) matrix B, then the *Kronecker product* of A and B is defined as the (mp,nq) matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} \qquad (A1)$$

A special case of (A1) is the second degree $(n^2,1)$ *vector monomial* $x^{(2)}$ related to the n dimensional state vector $x^T = [x_1 \quad \ldots \quad x_n]$

$$x^{(2)} = x \otimes x = \begin{bmatrix} x_1 x^T & \ldots & x_n x^T \end{bmatrix}^T \qquad (A2)$$

Basic properties of the Kronecker product are

$$(\underset{(m,n)}{A} \otimes \underset{(p,q)}{B})^T = (A^T \otimes B^T) \qquad (A3)$$

$$(\underset{(m,n)}{A} \otimes \underset{(p,q)}{B})(\underset{(n,r)}{C} \otimes \underset{(q,s)}{D}) = AC \otimes BD \qquad (A4)$$

$$\begin{aligned} &(\underset{(n,m)}{A} + \underset{(n,m)}{H}) \otimes (\underset{(p,q)}{B} + \underset{(p,q)}{R}) \\ &= A \otimes B + A \otimes R + H \otimes B + H \otimes R \end{aligned} \qquad (A5)$$

Let $X = [x_{ij}]$ be an (n,p) matrix and let vec(X) denote the vector formed by stacking the columns of X into one $(np,1)$ vector according to

$$\text{vec}(X) = \begin{bmatrix} e_1^T X^T & \ldots & e_p^T X^T \end{bmatrix}^T \qquad (A6)$$

then for any matrices A, B and X one has

$$\text{vec}(\underset{(m,n)}{A}\underset{(n,p)}{X}\underset{(p,q)}{B}) = (B^T \otimes A)\,\text{vec}(X) \qquad (A7)$$

Some properties of the vec-function are

$$\text{vec}(\underset{(n,m)}{A} + \underset{(n,m)}{H}) = \text{vec}(A) + \text{vec}(H) \qquad (A8)$$

$$\text{trace}(\underset{(n,m)}{A}^T \underset{(n,m)}{H}) = (\text{vec}(A))^T \text{vec}(H) \qquad (A9)$$

$$\text{vec}(\underset{(n,1)}{a}\ \underset{(m,1)}{b}^T) = b \otimes a \qquad (A10)$$

where the *trace* of an (n,n) matrix $M = [m_{ij}]$ is defined as

$$\text{trace}(M) = \sum_{i=1}^{n} m_{ii} \qquad (A11)$$

A compact representation of quadratic forms in terms of the vector monomial $x^{(2)}$ is given by

$$\begin{aligned} x^T Q x &= \text{trace}(Q\,xx^T) = (\text{vec}(Q^T))^T \text{vec}(xx^T) \\ &= (\text{vec}(Q^T))^T x^{(2)} \end{aligned} \qquad (A12)$$

where (A2), (A9) and (A10) were used.