15th Triennial World Congress, Barcelona, Spain

MINIMIZING RADAR EXPOSURE IN AIR VEHICLE PATH PLANNING[1]

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Abstract: An aircraft exposed to illumination by a tracking radar is considered and the problem of determining an optimal planar trajectory connecting two prespecified points is addressed. An analytic solution yielding the trajectory that minimizes the radar energy reflected from the target is derived using the Calculus of Variations. The solution is shown to exist only if the angle θ_f, formed by the lines connecting the radar to the two prespecified trajectory end points, is less than 60^o. In addition, expressions are given for the path length and optimal cost.

Keywords: Aircraft Control, Geometry, Path Planning, Optimization Problem, Optimal Trajectory

1. INTRODUCTION

Given a radar located at the origin O of the Euclidean plane, it is desired to find the optimal aircraft trajectory that connects two prespecified points A and B in the plane such that the Radio Frequency (RF) energy reflected from the aircraft is minimized; see, e.g., Fig. 1. According to the “Radar Transmission Equation" in Skolnik (1990), the ratio of the received RF power to the transmitted RF power reflected from the target is inversely proportional to R^4, where R is the slant range from the target to the monostatic radar. The cost to be minimized is then

$$\int_0^{\frac{l}{v}} \frac{1}{R^4(t)} dt$$

where v is the (constant) speed of the aircraft and l is the path length. Now, consider the trajectory in Fig. 1 to be given in polar form, as $R = R(\theta)$. Furthermore, $v = \frac{ds}{dt}$ i.e., $dt = \frac{ds}{v}$, and ds, the element of arc length, is given in polar coordinates by

$$ds = \sqrt{\left(\frac{dR}{d\theta}\right)^2 + R^2}\, d\theta$$

Substituting into the cost equation we then obtain the functional

$$J[R(\theta)] = \int_0^{\theta_f} \frac{\sqrt{\dot{R}^2 + R^2}}{R^4}\, d\theta \tag{1}$$

The boundary conditions are

$$R(0) = R_o \tag{2}$$

$$R(\theta_f) = R_f, \quad 0 < \theta \leq \theta_f. \tag{3}$$

[1] The views expressed in this article are those of the authors and do not reflect the official policy of the U.S. Air Force, Department of Defense, or the U.S. Government.
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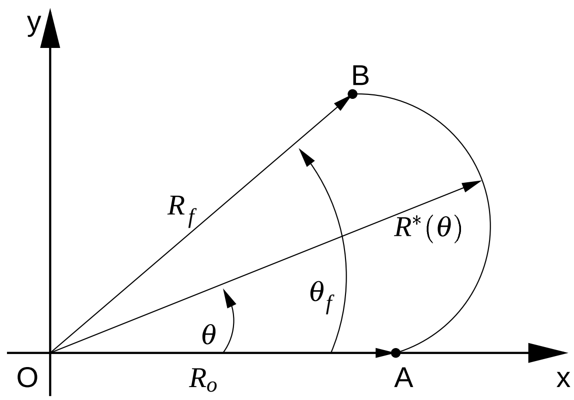

Fig. 1. Optimal Trajectory

2. OPTIMAL PATH

Without loss of generality, assume $R_f \geq R_o$ and $0 < \theta_f \leq \pi$, see, e.g., Fig. 1. Polar coordinates are used. We have the following:

Theorem 1. *The optimal trajectory which connects points A and B at a distance R_o and R_f from the radar located at the origin O, where θ_f is the angle $\angle AOB$, and minimizes the exposure to the radar according to Eqs. (1)-(3), is*

$$R^*(\theta) = R_o \sqrt[3]{\frac{\sin(3\theta+\phi)}{\sin\phi}},\ 0 < \theta \leq \theta_f \tag{4}$$

where

$$\phi = \text{Arctan}\left(\frac{\sin 3\theta_f}{\left(\frac{R_f}{R_o}\right)^3 - \cos 3\theta_f}\right) \tag{5}$$

Moreover, the length of the optimal path is given by the integral

$$l^* = \frac{R_o}{\sqrt[3]{\sin\phi}} \int_0^{\theta_f} [\sin(3\theta+\phi)]^{-\frac{2}{3}}\ d\theta$$

and the cost function explicitly evaluates to

$$J^* = \frac{1}{3{R_o}^3} \frac{\sin 3\theta_f}{\sin(3\theta_f+\phi)}$$

This result holds provided $0 < \theta_f < \frac{\pi}{3}$. However, if $\frac{\pi}{3} \leq \theta_f \leq \pi$, then an optimal path does not exist and a constraint on the path length, l, must be included to render the optimization problem well posed.

Proof. We have obtained a variational problem with an integrand which is not explicitly dependent on the independent variable θ. In this case, the Euler equation of the Calculus of Variations - see, e.g., Gelfand and Fomin (1963) - can be reduced to a first order differential equation

$$\frac{1}{R^2} = C\sqrt{\dot{R}^2 + R^2} \tag{6}$$

where C is a constant. Thus,

$$\dot{R} = \pm \frac{\sqrt{1/C^2 - R^6}}{R^2}$$

where $\frac{1}{C^2} > R^6 > 0$. Hence, we have obtained the non-linear ordinary differential equation

$$\frac{dR}{d\theta} = \pm \frac{\sqrt{1/C^2 - R^6}}{R^2}, \quad R(0) = R_o \equiv |OA|.$$

The integration constant C will be determined by the terminal condition,

$$R(\theta_f) = R_f \equiv |OB|.$$

Obviously, $R(\theta)$ is unimodal on $0 < \theta \leq \theta_f$. Hence, $\exists\ \bar{\theta} \in (0, \theta_f]$ such that $R(\theta)$ is monotonically increasing (decreasing) on $(0, \bar{\theta}]$, and is monotonically decreasing (increasing) on $[\bar{\theta}, \theta_f]$. At $\theta = \bar{\theta}$, $R(\theta)$ is maximal and $\frac{dR}{d\theta}\Big|_{\bar{\theta}} = 0$. Let $R(\theta)$ be monotonically increasing on $0 < \theta \leq \bar{\theta}$ and let $R(\theta)$ be monotonically decreasing on $\bar{\theta} \leq \theta \leq \theta_f$.

Consider $0 < \theta \leq \bar{\theta}$ where $R(\theta)$ is monotonically increasing, and

$$\frac{dR}{d\theta} = \frac{\sqrt{1/C^2 - R^6}}{R^2}$$

Thus,

$$d\theta = \frac{R^2}{\sqrt{1/C^2 - R^6}}\ dR$$

The solution of this ODE entails an integration. To this end, define the new variable

$$u = CR^3, \text{ i.e., } du = 3CR^2\ dR$$

Hence,

$$d\theta = \frac{1}{3} \frac{du}{\sqrt{1-u^2}}$$

Integration yields $u = \sin(3\theta + \phi)$, where ϕ is the integration constant. Hence,

$$R^3(\theta) = \frac{1}{C} \sin(3\theta + \phi)$$

Therefore on $0 < \theta \leq \bar{\theta}$,

$$R(\theta) = \frac{1}{\sqrt[3]{C}} \sqrt[3]{\sin(3\theta+\phi)},\ 0 \leq \phi \tag{7}$$

Similarly, on $\bar{\theta} \leq \theta \leq \theta_f$,

$$R(\theta) = -\frac{1}{\sqrt[3]{C}} \sqrt[3]{\sin(3\theta-\psi)},\ 0 \leq \psi \tag{8}$$

where $C > 0$.

We have three unknowns: ϕ, ψ, and $\bar{\theta}$, and three conditions: $R(0) = R_o$, $R(\theta_f) = R_f$ and $R(\bar{\theta}) = \max_{0<\theta\leq\theta_f} R(\theta)$. The latter yields:

$$3\bar{\theta} + \phi = \frac{\pi}{2} \tag{9}$$

and

$$3\bar{\theta} - \psi = -\frac{\pi}{2}$$

see, e.g., Eqs. (7) and (8).

Thus, combining Eqs. (9) and (2) yields

$$\phi + \psi = \pi$$

i.e.,

$$\psi = \pi - \phi \tag{10}$$

Hence, for $\bar{\theta} < \theta \leq \theta_f$, inserting Eq. (10) into (8) yields

$$\begin{aligned} R(\theta) &= -\frac{1}{\sqrt[3]{C}}\sqrt[3]{\sin(3\theta + \phi - \pi)} \\ &= \frac{1}{\sqrt[3]{C}}\sqrt[3]{\sin(3\theta + \phi)} \end{aligned}$$

Therefore, the formula

$$R^*(\theta) = \frac{1}{\sqrt[3]{C}}\sqrt[3]{\sin(3\theta + \phi)}$$

applies on the complete domain of definition of $R(\theta)$, viz., it applies for $0 < \theta \leq \theta_f$.

Finally, we use the boundary conditions $R^*(0) = R_o$ and $R^*(\theta_f) = R_f$ to determine C and ϕ, respectively, viz.,

$$R_o = R^*(0) = \frac{1}{\sqrt[3]{C}}\sqrt[3]{\sin\phi}$$

Solving for C yields

$$C = \frac{\sin\phi}{{R_o}^3} \tag{11}$$

Thus, the extremizing trajectory is explicitly given by

$$R^*(\theta) = R_o\sqrt[3]{\frac{\sin(3\theta + \phi)}{\sin\phi}}$$

In addition,

$$ {R_f}^3 = R^*(\theta_f)^3 = {R_o}^3\left(\frac{\sin(3\theta_f + \phi)}{\sin\phi}\right)$$

which yields

$$\phi = \operatorname{Arctan}\left(\frac{\sin 3\theta_f}{\left(\frac{R_f}{R_o}\right)^3 - \cos 3\theta_f}\right)$$

It can be shown that the extremal given in Eq. (4) satisfies the necessary and sufficient conditions for a weak local minimum - see, e.g., Hebert (2001).

Once the optimal path $R^*(\theta)$ has been explicitly determined, it is possible to calculate the path length of the trajectory. The path length is given by

$$l = \int_0^{\theta_f}\sqrt{\dot{R}^2(\theta) + R^2(\theta)}\, d\theta \tag{12}$$

Substituting (6) into (12) yields

$$l^* = \int_0^{\theta_f}\frac{1}{CR^2(\theta)}\, d\theta \tag{13}$$

Using Eqs. (4) and (11) yields the optimal path length

$$\begin{aligned} l^* &= \int_0^{\theta_f}\frac{{R_o}^3}{\sin\phi}\left(R_o\sqrt[3]{\frac{\sin(3\theta + \phi)}{\sin\phi}}\right)^{-2} d\theta \\ &= \frac{R_o}{\sqrt[3]{\sin\phi}}\int_0^{\theta_f}\left[\sin(3\theta + \phi)\right]^{-\frac{2}{3}}\, d\theta \end{aligned} \tag{14}$$

It can be shown that the path length integral (14) evaluates into an expression consisting of elliptic integrals of the first kind.

The cost function Eq. (1) can be simplified by substituting Eq. (6) to obtain

$$J = \frac{1}{C}\int_0^{\theta_f}\frac{1}{R^6}\, d\theta$$

Substituting for the previously determined integration constant (11), and optimal trajectory (4), yields

$$\begin{aligned} J^* &= \frac{\sin\phi}{{R_o}^3}\int_0^{\theta_f}\frac{1}{\sin^2(3\theta + \phi)}\, d\theta \\ &= \frac{\sin\phi}{3{R_o}^3}\int_{\phi}^{3\theta_f + \phi}\frac{1}{\sin^2(x)}\, d\theta \\ &= \frac{\sin\phi}{3{R_o}^3}\left[\cot\phi - \cot(3\theta_f + \phi)\right] \\ &= \frac{1}{3{R_o}^3}\frac{\sin 3\theta_f}{\sin(3\theta_f + \phi)} \end{aligned} \tag{15}$$

□

Several interesting special cases concerning the trajectory given by Eq. (4) are now considered.

In the case where the the origin O and points A and B are colinear, viz., $\theta_f = 0$ and $R_f > R_o$, the optimal trajectory is a straight line, as shown in Fig. 2.

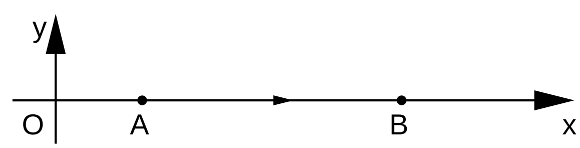

Fig. 2. Optimal Trajectory for the Special Case where $\theta_f = 0$

Also, the following holds for the symmetric case where $R_o = R_f$.

Corollary 1. *The optimal trajectory which connects points A and B at a distance $R_o = R_f$ from the radar located at the origin O, minimizing the exposure to the radar according to Eqs. (1)-(3), is*

$$R^*(\theta) = R_o\sqrt[3]{\frac{\cos(3\theta - \frac{3\theta_f}{2})}{\cos\frac{3\theta_f}{2}}},\; 0 < \theta \leq \theta_f$$

where θ_f is the angle $\angle AOB$. The length of the optimal trajectory is then given by

$$l^* = \frac{R_o}{\sqrt[3]{\cos\frac{3\theta_f}{2}}} \int_0^{\theta_f} \left[\cos\left(3\theta - \frac{3\theta_f}{2}\right)\right]^{-\frac{2}{3}} d\theta \qquad (16)$$

which is an elliptic integral of the first kind. The cost function evaluates to

$$J^* = \frac{2}{3R_o{}^3} \sin\left(\frac{3\theta_f}{2}\right) \qquad (17)$$

This result holds provided $0 < \theta_f < \frac{\pi}{3}$.

Proof. When $R_f = R_o$ we can write Eq. (5) as

$$\begin{aligned}
\phi &= \text{Arctan}\left(\frac{\sin 3\theta_f}{1 - \cos 3\theta_f}\right) \\
&= \text{Arctan}\left(\frac{2\sin\frac{3\theta_f}{2}\cos\frac{3\theta_f}{2}}{1-(1-2\sin^2\frac{3\theta_f}{2})}\right) \\
&= \text{Arctan}\left(\frac{\cos\frac{3\theta_f}{2}}{\sin\frac{3\theta_f}{2}}\right) \\
&= \text{Arctan}\left(\cot\frac{3\theta_f}{2}\right) \\
&= \text{Arctan}\left(\tan\left[\frac{\pi}{2} - \frac{3\theta_f}{2}\right]\right) \\
&= \frac{\pi}{2} - \frac{3\theta_f}{2} \qquad (18)
\end{aligned}$$

The optimal trajectory $R^*(\theta)$ is then obtained by substituting (18) into (4), whereupon we obtain

$$\begin{aligned}
R^*(\theta) &= R_o \sqrt[3]{\frac{\sin(3\theta + \frac{\pi}{2} - \frac{3\theta_f}{2})}{\sin(\frac{\pi}{2} - \frac{3\theta_f}{2})}} \\
&= R_o \sqrt[3]{\frac{\cos(3\theta - \frac{3\theta_f}{2})}{\cos\frac{3\theta_f}{2}}}
\end{aligned}$$

Similarly, by substituting Eqs. (11) and (18) into the equation for the path length (13) we obtain

$$\begin{aligned}
l^* &= \frac{R_o{}^3}{\sin\left(\frac{\pi}{2} - \frac{3\theta_f}{2}\right)} \int_0^{\theta_f} \frac{1}{R^2}\, d\theta \\
&= \frac{R_o{}^3}{\cos\frac{3\theta_f}{2}} \int_0^{\theta_f} \frac{1}{R^2}\, d\theta
\end{aligned}$$

Substituting the equation for the extremal $R^*(\theta)$ developed for this special case, we obtain

$$\begin{aligned}
l^* &= \frac{R_o{}^3}{\cos\frac{3\theta_f}{2}} \int_0^{\theta_f} \frac{1}{R_o{}^2} \left[\sqrt[3]{\frac{\cos\frac{3\theta_f}{2}}{\cos(3\theta - \frac{3\theta_f}{2})}}\right]^2 d\theta \\
&= \frac{R_o}{\sqrt[3]{\cos\frac{3\theta_f}{2}}} \int_0^{\theta_f} \left[\cos\left(3\theta - \frac{3\theta_f}{2}\right)\right]^{-\frac{2}{3}} d\theta
\end{aligned}$$

Finally, the cost for the optimal trajectory, J^*, is calculated by inserting Eq. (18) into Eq. (15), to obtain

$$\begin{aligned}
J^* &= \frac{1}{3R_o{}^3} \frac{\sin 3\theta_f}{\sin(3\theta_f + \frac{\pi}{2} - \frac{3\theta_f}{2})} \\
&= \frac{1}{3R_o{}^3} \frac{\sin 3\theta_f}{\cos\frac{3\theta_f}{2}} \\
&= \frac{2}{3R_o{}^3} \sin\left(\frac{3\theta_f}{2}\right) \qquad (19)
\end{aligned}$$

□

In the symmetric special case where $R_f = R_o$, the relationship between ϕ and θ_f is linear and is given by Eq. (18). The angle ϕ is evaluated for some interesting θ_f in Table 1. We note that the angle $\phi \to 0$ as $\theta_f \to \frac{\pi}{3}$. The extremal trajectory

Table 1. Interesting Values of ϕ for the Special Case $R_f = R_o = 1$

θ_f	ϕ	$\sin\phi$
0^{o+}	90^o	1
10^o	$\text{Arctan}\left(\frac{1}{2-\sqrt{3}}\right)$	$\frac{1}{2\sqrt{2-\sqrt{3}}}$
15^o	$\text{Arctan}\left(\frac{1}{\sqrt{2}-1}\right)$	$\frac{1}{2\sqrt{1-\frac{1}{\sqrt{2}}}}$
20^o	60^o	$\frac{\sqrt{3}}{2}$
30^o	45^o	$\frac{1}{\sqrt{2}}$
45^o	$\text{Arctan}\left(\frac{1}{\sqrt{2}+1}\right)$	$\frac{1}{2\sqrt{1+\frac{1}{\sqrt{2}}}}$
60^o	0^o	0

(4) is shown in Fig. 3 for the case where $R_o = R_f = 1$ and $\theta_f = 45^o$.

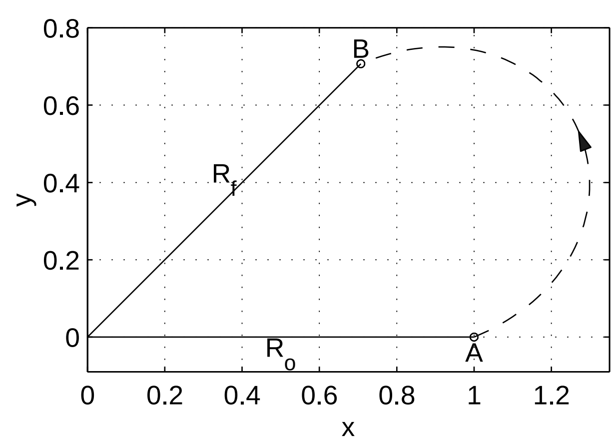

Fig. 3. Extremal $R^*(\theta)$ for $R_o = R_f = 1$, $\theta_f = 45^o$

Remark 1. *$R(\theta) \equiv const$, viz., the trajectory of a circular arc, satisfies the Euler equation of the Calculus of Variations. Hence for $R_f = R_o$, $R(\theta) = R_o$ is a candidate solution. However, the cost associated with this solution, J_c, is*

$$
\begin{aligned}
J_c &= \int_0^{\theta_f} \frac{\sqrt{\dot{R}^2+R^2}}{R^4}\, d\theta \\
&= \int_0^{\theta_f} \frac{R_o}{{R_o}^4}\, d\theta \\
&= \frac{1}{{R_o}^3}\theta_f
\end{aligned}
$$

Considering the optimal cost for the trajectory when $R_f = R_o$ given by (19), we see that $J_c > J^*$ for all admissible R_o, R_f, and θ_f.

Proof.

$$
\begin{aligned}
& \theta_f > \sin\theta_f\,, \quad \text{for all } \theta_f > 0 \\
\Rightarrow \quad & \frac{3}{2}\,\theta_f > \sin\frac{3}{2}\,\theta_f \\
\Rightarrow \quad & \theta_f > \frac{2}{3}\sin\frac{3}{2}\theta_f \\
\Rightarrow \quad & \frac{1}{{R_o}^3}\theta_f > \frac{2}{3{R_o}^3}\sin\frac{3}{2}\theta_f \\
\Rightarrow \quad & J_c > J^*
\end{aligned}
$$

Thus a circular arc trajectory is not optimal under the condition $R_f = R_o$. □

An additional argument can be made concerning the nature of the function that minimizes the reflected RF energy along the flight path: For example, one could argue that by minimizing the distance traveled between points A and B, one would lower the amount of time exposed to the radar and thus lower the cost function. Clearly, time of exposure is not the only factor of concern, as proximity to the radar is also a factor. Alternatively, it might be suggested that the aircraft should travel as far away from the radar as quickly as possible, to minimize the energy received at the radar. Of course, the aircraft must eventually reach point B. Thus, the shape of the extremal shown in Fig. 3 represents the tradeoff between minimizing the exposure time (path length) and the RF power received by the radar over time.

The extremal arc no longer exists as the angle $\theta_f \to \theta_C \equiv \frac{\pi}{3}$, a critical angle. In other words, the path length becomes infinite at this critical angular separation of the segments *OA* and *OB*. Beyond this critical angle, there does not exist a finite length path that minimizes our cost function. That is, the aforementioned tradeoff breaks down and it is advantageous for the aircraft to travel away from the radar to infinity. Thus for $\theta_f \geq \frac{\pi}{3}$, a path length constraint must be included to render the optimization problem well posed.

3. CONCLUSION

The problem of determining the flight path connecting the point of departure and the point of arrival, such that the exposure of an aircraft to illumination by a tracking radar located at the origin is minimized, has been solved using the Calculus of Variations. A closed form solution was obtained. The optimum was shown to exist iff the angle θ_f included between the radials from the radar to the points of departure and arrival is less than 60^o. The analytic solution obtained provided valuable insight into the problem. Expressions for the path length and optimal cost are determined. A solution of a representative path planning problem is provided.

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