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# Hedging Electricity Swing Options in Incomplete Markets ⋆

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**Abstract:** The deregulation of electricity markets renders public utilities vulnerable to the high volatility of electricity spot prices. This price risk is effectively mitigated by swing options, which allow the option holder to buy electric energy from the option writer at a fixed price during a prescribed time period. Unlike many financial derivatives, a swing option cannot be assigned a unique fair value due to market frictions. In this paper we determine the option's no-arbitrage price interval by hedging its payoff stream with basic market securities (such as forward contracts) both from the perspective of the holder and the writer of the option. The end points of the no-arbitrage interval are given by the optimal values of two robust control problems, which we solve in polynomial decision rules via constraint sampling.

Keywords: robust control; uncertain dynamic systems; energy management systems.

## 1. INTRODUCTION

The spot price of electricity is notoriously volatile. Unpredictable demand patterns as well as the limited storability and grid-bound nature of electricity result in frequent price spikes. The associated price risk is absorbed by public utilities which buy energy at uncertain wholesale prices and sell it to end consumers at fixed retail prices. In a regulated market, the government can set prices that allow utility companies to recover their costs. In a liberalised market, however, such compensation is no longer possible.

Utility companies seek to mitigate the resulting price risks by investing in market-traded electricity derivatives such as forwards and options. An electricity forward contract is an obligation to buy electric energy at a prescribed delivery rate over a prescribed delivery period and in return for a predetermined unit price (the forward price). Both the load profile (i.e., the delivery rate in each hour of the delivery period) and the forward price are agreed at the time when the forward is issued. A European call option is the right (but not the obligation) to buy electric energy at a prescribed delivery rate over a prescribed delivery period and at a predetermined unit price (the strike price).

Unfortunately, forwards and European options constitute poor hedging instruments for utility companies. Indeed, forwards neither provide any flexibility in the volume nor in the timing of the energy delivery, both of which are essential for a utility company that is unable to control the energy consumption of its customers. Conversely, using European options to hedge against price spikes results in a costly overprotection, see Jaillet et al. [2004]. As an example, consider a utility company that wishes to hedge against up to five price spikes forecasted for a typical summer month. A perfect hedge is provided by purchasing a European option that is exercisable on each day of the month. However, since the option will be exercised at most on five days, the utility company overpays for the desired protection. Thus, utility companies often hedge spot price risks by purchasing swing options.

A swing option constitutes an agreement to purchase and/or sell electric energy during a fixed period of time and at a predetermined strike price. Like European call options, swing options offer some flexibility both in the timing and the volume of the energy delivery. However, they can only be exercised a limited number of times. For example, a typical swing option may allow its holder to purchase between 500 and 1,000 MWh of electric energy at a unit price of 60 €/MWh during the next month, while not more than 50 MWh may be bought on each day (at constant delivery rate). Swing options are popular hedging instruments in the electricity sector and are used extensively by utility companies, see Carmona and Ludkovski [2010].

The aim of this paper is to determine the monetary value of the right to exercise a swing option, that is, the premium that the option holder has to pay the writer at the time when the option is negotiated. To this end, we assume that the market of basic securities (i.e., cash as well as the forwards and European options on energy) is arbitrage-free. A market is arbitrage-free if there is no self-financing portfolio that transforms a nonpositive initial investment into a nonnegative terminal wealth that is nonzero with positive probability, see Luenberger [1997]. Clearly, the market should remain arbitrage-free when the swing option is added to the existing investment opportunities. It can be shown that the option has a unique no-arbitrage price if and only if there exists a portfolio of basic securities that generates, with certainty, the same cash flow stream as the option. Unfortunately, such a perfect option replication is not possible in real electricity markets due to the spiky behaviour of spot prices, market illiquidity and high transaction costs which preclude frequent adjustments to the replicating portfolio.

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In the language of finance, electricity markets are therefore incomplete.

In this paper we propose a pricing framework for electricity swing options in incomplete markets. Instead of a single objective price, an incomplete market allows for a whole interval of option prices that preserve arbitrage-freeness. We determine this interval by investigating two complementary hedging portfolios of basic securities. We obtain the lower end of the no-arbitrage interval (the holder's price) by computing the maximum amount of money borrowed today that the option holder can repay through exercising the option and trading in the basic securities. Likewise, we obtain the upper end of the no-arbitrage interval (the writer's price) by calculating the minimum amount of money borrowed today that enables the option writer to cover all obligations arising from the option by trading in the basic securities. The option price agreed by the holder and the writer emerges as the result of a negotiation process, and it must lie within the no-arbitrage interval if both contract parties act rationally. The holder's and the writer's price are representable as the optimal values of two robust control problems. These problems do not allow for a direct solution, and we propose an approximate solution scheme that simplifies the underlying information process and solves the resulting problems in polynomial decision rules via constraint sampling. Our approach also has applications to the management of hydroelectric power plants and the valuation of service interruptions of production facilities, see Carmona and Ludkovski [2010].

To the best of our knowledge, all existing pricing schemes for swing options assume either explicitly or implicitly that the underlying market is complete and that swing options can be assigned a unique price. Most approaches rely on stochastic dynamic programming to determine the optimal exercise strategy, see e.g. Jaillet et al. [2004], Haarbrücker and Kuhn [2009] and Thompson [1995]. Recently, the popular Least Squares Monte Carlo technique developed by Longstaff and Schwartz [2001] was employed to estimate the conditional expectations arising in the stochastic dynamic programming iterations, see Ibáñez [2004]. Analytical solutions based on stochastic calculus exploit the connection between swing option pricing and multiple stopping problems, see Carmona and Touzi [2008]. A recent survey of the literature on swing option pricing is provided in Carmona and Ludkovski [2010]. The ramifications of market incompleteness have been studied extensively in the context of financial options theory. This line of research investigates strategies for super- and sub-replicating the option payoffs and evaluates the expected utility generated by the option, see e.g. Hao [2008], Staum [2007] and the references therein. However, all studies of market incompleteness focus on standard financial options and do not easily generalise to the exotic contractual features of swing options and the peculiarities of electricity markets. In addition, emphasis is placed on a theoretical characterisation of option prices, and no numerical techniques are provided to determine an option's no-arbitrage interval.

The structure of this paper is as follows. In §2 we formulate the two control problems that allow us to determine a swing option's no-arbitrage interval. To ensure tractability, we propose several approximation techniques in §3. We illustrate our approach with a numerical example in §4.

## 2. PROBLEM FORMULATION

We consider a finite planning horizon consisting of $T$ time intervals (periods) $\mathcal{T} = \{1, \ldots, T\}$. Our aim is to find the value of the swing option at the beginning of the first period (i.e., today). We model security prices and dividends as random variables that are defined on a probability space $(\Xi, \mathcal{F}, \mathbb{P})$. We assume that the elements of the sample space $\Xi$ can be represented as $\xi = (\xi_1, \ldots, \xi_{T+1})$, where the subvector $\xi_t \in \mathbb{R}^k$, $t \in \mathcal{T}$, is observed at the beginning of period $t$, while $\xi_{T+1} \in \mathbb{R}^k$ is observed at the end of period $T$. Note that $\xi_1$ is known today and is therefore not random. We assume that $\Xi$ is the smallest closed set that satisfies $\mathbb{P}(\xi \in \Xi) = 1$, and we assume that $\Xi$ is bounded. We denote by $\xi^t = (\xi_1, \ldots, \xi_t) \in \mathbb{R}^{tk}$ the history of observations up to period $t$.

### 2.1 Electricity swing options

An electricity swing option is an agreement to buy and/or sell electric energy at a predetermined strike price $K$ during a fixed delivery period $\{\underline{t}, \ldots, \overline{t}\}$, where $\underline{t}, \overline{t} \in \mathcal{T}$. At the beginning of each period $t \in \{\underline{t}, \ldots, \overline{t}\}$, the option holder can select the amount of energy $e_t$ to be delivered in that period, subject to lower and upper limits $\underline{e}_t$ and $\overline{e}_t$. Moreover, the cumulative amount of energy received during the entire delivery period, $\sum_{t=\underline{t}}^{\overline{t}} e_t$, is subject to lower and upper limits $\underline{c}$ and $\overline{c}$. By setting $\underline{e}_t = \overline{e}_t = 0$ for all $t \in \mathcal{T} \setminus \{\underline{t}, \ldots, \overline{t}\}$, we can assume that $(\underline{t}, \overline{t}) = (1, T)$.

The following comments are in order. One often distinguishes base, peak and off-peak swing options. While base options can be exercised in each period of the delivery period, peak options are only exercisable in peak periods (i.e., from 8am to 8pm on working days). A peak option can be modelled by setting $\underline{e}_t = \overline{e}_t = 0$ for all off-peak periods $t$. Off-peak options, which are only exercisable in off-peak periods, can be modelled accordingly. In practice, the per-period delivery quantity $e_t$ must be chosen prior to period $t$ so that the option writer has time to arrange the delivery. Our assumption that $e_t$ is chosen at the beginning of period $t$ is an acceptable idealisation that facilitates a transparent exposition. Finally, some swing options permit violation of the cumulative energy limits $\underline{c}$ and $\overline{c}$, in which case penalties are imposed for each unit of shortfall or exceedance, respectively.

By our definition, forwards and European options can be regarded as special cases of swing contracts.

### 2.2 Market model

We consider an energy market that consists of a spot exchange and $J$ basic securities indexed by $j = 1, \ldots, J$. We denote the average spot price of electricity over period $t$ by $S_t(\xi_t)$. The price of security $j$ at the beginning of period $t$ is given by $P_t^j(\xi_t)$, while $P_{T+1}^j(\xi_T)$ denotes the price of security $j$ at the end of period $T$. Security $j$ pays off a random dividend $d_t^j(\xi_t)$ at the beginning of period $t$. We assume that $S_t$, $P_t^j$ and $d_t^j$ are continuous functions of the random variables $\xi_t$ observed in period $t$. Security 1 is riskless (e.g. a money market account). For ease of exposition, we assume that the interest rate

is zero, which we model via $P_t^1(\xi_t) = 1$ and $d_t^1(\xi_t) = 0$ for all $\xi \in \Xi$. The other securities represent forwards and options on electricity whose dividends reflect the exercise costs and the revenues generated by selling the delivered electricity on the spot market. We notationally suppress the dependence of $S_t$, $P_t^j$ and $d_t^j$ on $\xi_t$, and we aggregate the prices and dividends to vectors $P_t = (P_t^1, \ldots, P_t^J)$ and $d_t = (d_t^1, \ldots, d_t^J)$.

We assume that the basic market securities are traded in each period $t \in \mathcal{T}$. This is an idealisation since in period $t$ some of the securities may have expired, whereas others may not yet be traded on the market.[1] As will become clear later on, we can add constraints in our control problems to avoid trading in unavailable securities.

We assume that $(i)$ the market is arbitrage-free, $(ii)$ no transaction costs are incurred when trading basic securities, and $(iii)$ the market participants are price-takers, that is, their trades do not affect the market prices. The first assumption is innocent: if the market was not arbitrage-free, then market participants could make infinite profits at no risk. Such arbitrage opportunities would vanish quickly. As for the second assumption, we remark that our models can be extended to accommodate linear transaction costs. The third mild requirement, finally, is standard in the literature.

We now add a swing option to the set of existing securities. Our goal is to determine the interval of swing option prices that preserve arbitrage-freeness of the market. In a complete market, this no-arbitrage interval collapses to a singleton, and the unique no-arbitrage price coincides with the initial value of a portfolio of the basic securities that replicates the swing option's dividend stream. In an incomplete market, however, such a perfect replication is usually not possible, and the no-arbitrage interval is non-degenerate. We obtain the lower end of this interval by computing the maximum loan that the option holder can refinance through exercising the option and trading in the basic securities. This value, which we call the *holder's price*, constitutes the highest price at which all rational market participants would agree to buy the option since there is no risk involved. Any option price below the holder's price would allow the buyer to make arbitrage profits by purchasing swing options and simultaneously trading in the basic securities. Likewise, we obtain the upper end of the no-arbitrage interval by calculating the minimum loan that enables the option writer to cover all obligations arising from the option by trading in the basic securities. This value, which we call the *writer's price*, is the lowest price at which all rational market participants would agree to issue the option since there is no risk involved. Any option price above the writer's price would allow the writer to make arbitrage profits by selling swing options and simultaneously trading in the basic securities. The holder's and the writer's prices are representable as optimal values of robust control problems. The solutions to these problems also reveal arbitrage opportunities that emerge if the option's price falls outside the no-arbitrage interval.

[1] On the European Energy Exchange, for example, trading in a weekly forward starts only five weeks before its delivery period.

### *2.3 Holder's problem*

The holder's price is given by the optimal value of the following control problem.

$$\begin{array}{ll} \max\limits_{e,x,u} & H = -\langle P_1, x_0 \rangle \\ \text{s.t.} & \underline{c} \le \langle \mathbb{I}, e \rangle \le \overline{c} \\ & \underline{e} \le e \le \overline{e} \\ & \langle P_{T+1}, x_T \rangle \ge 0 \\ & x_t = x_{t-1} + u_t \\ & \langle P_t, u_t \rangle \le e_t\,(S_t - K) + \langle d_t, x_t \rangle \end{array} \Bigg\} \ \forall t \in \mathcal{T}$$

The decision variables are $e_t(\xi^t) \in \mathbb{R}$, $x_t(\xi^t) \in \mathbb{R}^J$ and $u_t(\xi^t) \in \mathbb{R}^J$ for each $t \in \mathcal{T}$ and $\xi \in \Xi$, as well as $x_0 \in \mathbb{R}^J$. We notationally suppress dependence on $\xi$ to avoid clutter. The constraints are understood to hold for all $\xi \in \Xi$ (i.e., with certainty). Here and in the following, $\mathbb{I}$ denotes the vector whose components are all ones.

The variable $e_t$ denotes the amount of energy received in period $t$ from exercising the swing option. An admissible exercise strategy $e = (e_t)_{t\in\mathcal{T}}$ must satisfy the cumulative energy limits $\underline{c}$ and $\overline{c}$ (first constraint) as well as the per-period energy limits $\underline{e} = (\underline{e}_t)_{t\in\mathcal{T}}$ and $\overline{e} = (\overline{e}_t)_{t\in\mathcal{T}}$ (second constraint) specified in the swing option contract. The first two constraints therefore ensure that the exercise strategy is admissible.

The position variable $x_t^j$ denotes the number of basic securities of type $j$ held within period $t$, while the adjustment variable $u_t^j$ denotes the number of securities of type $j$ bought at the beginning of period $t$. For notational simplicity, we define $x_t = (x_t^1, \ldots, x_t^J)$ for $t \in \mathcal{T} \cup \{0\}$ and $u_t = (u_t^1, \ldots, u_t^J)$ for $t \in \mathcal{T}$. The fourth constraint in the holder's problem is a budget constraint which updates the position variables according to the adjustment decisions. The set of admissible portfolio holdings $x_t$ and portfolio adjustments $u_t$ can be restricted further by prohibiting short-sales and portfolio adjustments outside of the market's trading hours. For ease of exposition, we disregard such constraints here.

The option holder attempts to maximise the amount of money that can be borrowed today and repaid with certainty by exercising the swing option. Thus, the holder borrows an amount $-\langle P_1, x_0 \rangle$ of money today by short selling basic securities (note that $-\langle P_1, x_0 \rangle > 0$ only if at least one position variable $x_0^j$ is strictly negative). The fifth constraint in the holder's problem can be viewed as a self-financing condition for the portfolio of the basic securities (determined through the position variable $x_t$) together with the swing option. Exercising the swing option at time $t$ by buying $e_t$ units of energy from the option writer at price $K$ and selling this energy quantity immediately on the spot market at price $S_t$ results in a dividend $e_t(S_t - K)$. Thus, the self-financing condition requires that the costs $\langle P_t, u_t \rangle$ to rebalance the portfolio at the beginning of period $t$ must be recovered with certainty from the dividend $e_t(S_t - K)$ generated by the swing option and the dividends $\langle d_t, x_t \rangle$ received from the basic securities. The third constraint guarantees that the portfolio of basic securities has a nonnegative terminal value with certainty. Together with the self-financing constraint, this ensures that the initial loan can be repaid by only using the dividends received from exercising the swing option.

We can now prove the correctness of the holder's problem.

*Proposition 1.* Let $(e^\star, x^\star, u^\star)$ denote an optimal solution to the holder's problem with objective value $H$.

(1) If the swing option trades at price $V < H$, then the option holder can make arbitrage profits.
(2) If the swing option trades at price $V > H$, no arbitrage can be made by buying the swing option.

**Proof.** Assume first that the option trades at a price $V < H$. Then, buy the swing option at price $V$ and take a loan of amount $H$, which yields an immediate payoff $H - V > 0$. Then, repay the loan by exercising the swing option with strategy $e^\star$ and by managing a portfolio of basic securities according to $(x^\star, u^\star)$. The self-financing constraint and the terminal portfolio constraint ensure that no net payments will arise in the future with certainty. Thus, we have constructed an arbitrage.

Assume now that the swing option trades at a price $V > H$ and there *is* a self-financing portfolio containing the swing option that transforms a nonpositive initial wealth into a nonnegative terminal wealth for all $\xi \in \Xi$. Such a portfolio would correspond to a feasible solution $(e, x, u)$ of the holder's problem with an objective value $V > H$. This contradicts the optimality of $(e^\star, x^\star, u^\star)$. □

The optimal exercise strategy in the holder's problem is sometimes referred to as the 'ruthless' strategy. Some option holders may decide to deviate from a ruthless strategy if they face obligations to deliver energy to third parties.

### *2.4 Writer's problem*

Under the assumption that the swing option holder exercises ruthlessly (see §2.3), the writer's price constitutes the optimal value of the following control problem.

$$\begin{array}{ll}\min\limits_{x,u} & W = \langle P_1, x_0\rangle \\ \text{s.t.} & \langle P_{T+1}, x_T\rangle \geq 0 \\ & \left.\begin{array}{l} x_t = x_{t-1} + u_t \\ \langle P_t, u_t\rangle \leq e_t^\star\,(K - S_t) + \langle d_t, x_t\rangle \end{array}\right\} \forall t \in \mathcal{T}\end{array}$$

The decision variables of this model are $x_t(\xi^t) \in \mathbb{R}^J$ and $u_t(\xi^t) \in \mathbb{R}^J$ for all $t \in \mathcal{T}$ and $\xi \in \Xi$, as well as $x_0 \in \mathbb{R}^J$. We notationally suppress dependence on $\xi$ to avoid clutter, and the constraints are understood to hold for all $\xi \in \Xi$. The portfolio holdings $x$ and portfolio adjustments $u$ have the same interpretation as in §2.3. Unlike the holder, the writer cannot decide on the exercise strategy $e$ of the swing option. Instead, the holder's optimal exercise strategy $e^\star$ is a (uncertainty-affected) parameter in the writer's problem. If the option holder requests $e_t^\star$ units of energy at price $K$, the writer has to provide this energy quantity by buying it on the spot market at price $S_t$. The option writer thus receives a dividend of size $e_t^\star(K - S_t)$ at time $t$ (which is typically negative).

The writer's problem differs from the holder's problem only in the objective function and the self-financing constraint. Instead of maximising the loan that can be refinanced, the writer minimises the loan that has to be taken today to cover the obligations arising from the swing option. The last constraint in the writer's problem stipulates that the costs $\langle P_t, u_t\rangle$ to rebalance the portfolio at the beginning of period $t$ are recovered from the dividends $e_t^\star(K - S_t)$ arising from the swing option and the dividends $\langle d_t, x_t\rangle$ received from the basic securities. The dividends of the swing option cannot be influenced by the writer.

If the writer is unsure whether the holder exercises ruthlessly, he may want to hedge against *any* exercise strategy of the holder. In that case, the last constraint of the writer's problem must hold for any $e^\star$ within the set

$$\left\{e \,:\, \underline{c} \leq \langle \mathbb{I}, e\rangle \leq \overline{c},\ \underline{e} \leq e \leq \overline{e}\right\}.$$

In the next section we will propose a technique to solve both versions of the writer's problem.

The following proposition establishes the correctness of the writer's problem. Its proof follows the lines of Proposition 1 and can therefore be omitted.

*Proposition 2.* Let $(x^\star, u^\star)$ denote an optimal solution to the writer's problem with objective value $W$.

(1) If the swing option trades at price $V > W$, then the option writer can make arbitrage profits.
(2) If the swing option trades at price $V < W$, no arbitrage can be made by selling the swing option.

We now show that the holder's and the writer's price indeed form an interval of no-arbitrage prices.

*Proposition 3.* The optimal values $H$ and $W$ of the holder's and writer's problems satisfy $H \leq W$. Equality holds if the market of basic securities is complete.

**Proof.** We first show that $W - H \geq 0$. To this end, we transform the holder's problem into a minimisation problem and determine $W - H$ by solving the holder's and the writer's problem simultaneously:

$$\begin{array}{ll}\min\limits_{e^{\mathrm{h}},x^{\mathrm{h}},u^{\mathrm{h}},x^{\mathrm{w}},u^{\mathrm{w}}} & \left\langle P_1, x_0^{\mathrm{h}} + x_0^{\mathrm{w}}\right\rangle \\ \text{s.t.} & \underline{c} \leq \left\langle \mathbb{I}, e^{\mathrm{h}}\right\rangle \leq \overline{c} \\ & \underline{e} \leq e^{\mathrm{h}} \leq \overline{e} \\ & \left\langle P_{T+1}, x_T^{\mathrm{h}}\right\rangle \geq 0 \\ & \left\langle P_{T+1}, x_T^{\mathrm{w}}\right\rangle \geq 0 \\ & \left.\begin{array}{l} x_t^{\mathrm{h}} = x_{t-1}^{\mathrm{h}} + u_t^{\mathrm{h}} \\ x_t^{\mathrm{w}} = x_{t-1}^{\mathrm{w}} + u_t^{\mathrm{w}} \\ \left\langle P_t, u_t^{\mathrm{h}}\right\rangle \leq e_t^{\mathrm{h}}\,(S_t - K) + \left\langle d_t, x_t^{\mathrm{h}}\right\rangle \\ \left\langle P_t, u_t^{\mathrm{w}}\right\rangle \leq e_t^\star\,(K - S_t) + \left\langle d_t, x_t^{\mathrm{w}}\right\rangle \end{array}\right\} \forall t \in \mathcal{T}\end{array}$$

In this problem, $(e^{\mathrm{h}}, x^{\mathrm{h}}, u^{\mathrm{h}})$ and $(x^{\mathrm{w}}, u^{\mathrm{w}})$ represent the decisions of the holder and the writer, respectively. We obtain a lower bound on the optimal value of this problem (and hence a lower bound on $W - H$) by aggregating (adding) the pairs of associated constraints and substituting $x = x_t^{\mathrm{h}} + x_t^{\mathrm{w}}$ as well as $u = u_t^{\mathrm{h}} + u_t^{\mathrm{w}}$:

$$\begin{array}{ll}\min\limits_{e^{\mathrm{h}},x,u} & \langle P_1, x_0\rangle \\ \text{s.t.} & \underline{c} \leq \left\langle \mathbb{I}, e^{\mathrm{h}}\right\rangle \leq \overline{c} \\ & \underline{e} \leq e^{\mathrm{h}} \leq \overline{e} \\ & \langle P_{T+1}, x_T\rangle \geq 0 \\ & \left.\begin{array}{l} x_t = x_{t-1} + u_t \\ \langle P_t, u_t\rangle \leq \left(e_t^{\mathrm{h}} - e_t^\star\right)(S_t - K) + \langle d_t, x_t\rangle \end{array}\right\} \forall t \in \mathcal{T}\end{array}$$

Since the option holder exercises ruthlessly, $e^{\mathrm{h}} = e^\star$ and the problem simplifies to

$$\begin{array}{ll}\min\limits_{x,u} & \langle P_1, x_0\rangle \\ \text{s.t.} & \langle P_{T+1}, x_T\rangle \geq 0 \\ & \left.\begin{array}{l} x_t = x_{t-1} + u_t \\ \langle P_t, u_t\rangle \leq \langle d_t, x_t\rangle \end{array}\right\} \forall t \in \mathcal{T}\end{array} \tag{1}$$

This problem determines the minimum initial value of a self-financing portfolio with nonnegative terminal value. Since the market of basic securities is arbitrage-free, the optimal value of this problem must be nonnegative. We therefore conclude that $W - H$ is nonnegative as well.

We now show that $H = W$ if the market of basic securities is complete. To this end, let $e^\star$ denote an optimal exercise strategy of the holder. We assume in the following that $e^\star$ is unique, but our arguments extend to the case of multiple optimal exercise strategies. Since the market is complete, there is a trading strategy $(x^\star, u^\star)$ that replicates the dividend stream generated by purchasing and exercising the swing option. Since all dividends are zero after period $T$, the terminal value $\langle P_{T+1}, x_T^\star \rangle$ of the replicating portfolio must be zero as well. By construction, the dividends $[\langle d_t, x_t^\star \rangle - \langle P_t, u_t^\star \rangle]_{t\in\mathcal{T}}$ of the trading strategy $(x^\star, u^\star)$ match the dividends $[e_t^\star(S_t - K)]_{t\in\mathcal{T}}$ of the swing option. Hence, the solution $(e^\star, x^\star, u^\star)$ is *feasible* in the holder's problem. We now show that $(e^\star, x^\star, u^\star)$ is indeed *optimal* in the holder's problem. Assume to the contrary that there is an alternative trading strategy $(\overline{x}, \overline{u})$ that leads to a higher objective value, that is, $\langle P_1, \overline{x}_0 \rangle < \langle P_1, x_0^\star \rangle$. In this case, the trading strategy $(\overline{x} - x^\star, \overline{u} - u^\star)$ would be feasible in (1) with a strictly negative objective value, indicating that the market of basic securities is not arbitrage-free. Since this contradicts our assumptions, $(e^\star, x^\star, u^\star)$ must be optimal in the holder's problem. By construction, the trading strategy $(-x^\star, -u^\star)$ is feasible in the writer's problem and yields an objective value of $H$, indicating that the writer's price does not exceed the holder's price. Equality of $H$ and $W$ now follows from the first part of the proof. □

## 3. TRACTABLE REFORMULATION

Unfortunately, without suitable approximations, the problems $(\mathcal{H})$ and $(\mathcal{W})$ are computationally intractable for three reasons: $(i)$ they involve a large number of time periods, $(ii)$ they optimise over function spaces and therefore exhibit infinitely many decision variables, and $(iii)$ their constraints hold for all $\xi \in \Xi$, that is, the models involve infinitely many constraints.

To gain tractability, we successively apply three approximations to $(\mathcal{H})$ and $(\mathcal{W})$: $(i)$ we reduce the number of time periods via aggregation, $(ii)$ we approximate the functional decisions via polynomials to obtain finitely many decision variables, and $(iii)$ we apply constraint sampling to obtain finitely many constraints. We discuss the three approximations in the context of problem $(\mathcal{H})$. Our results immediately extend to problem $(\mathcal{W})$.

### *3.1 Aggregation of time periods*

Typical swing options have delivery periods of up to one year and are exercisable in intervals of 15 min up to one day. Thus, the planning horizon $\mathcal{T}$ may cover thousands of periods. This renders problem $(\mathcal{H})$ intractable since the decision variables $e_t$, $x_t$ and $u_t$ are multivariate functions of the observation history $\xi^t$.

To reduce the problem's complexity, we propose to aggregate the time periods in $\mathcal{T}$, which we henceforth call *micro periods*, to fewer *macro periods*. Formally, we denote the set of macro periods by $\mathcal{M} = \{1, \ldots, M\}$, and we assume that there is a strictly increasing function $\tau : \mathcal{M} \cup \{M+1\} \mapsto \mathcal{T} \cup \{T+1\}$ such that $\tau(m)$ represents the first micro period within the $m^{\text{th}}$ macro period. We set $\tau(1) = 1$ and $\tau(M+1) = T+1$, so that the $m^{\text{th}}$ macro period consists of the micro periods $\tau(m), \ldots, \tau(m+1)-1$. With a slight abuse of notation, we also define the reduced observation histories $\xi^m = (\xi_{\tau(1)}, \ldots, \xi_{\tau(m)})$, $m \in \mathcal{M}$, consisting only of the risk factors observed at the beginning of each macro period. As before, $\xi^t = (\xi_1, \ldots, \xi_t)$ represents the full observation history, and we use the indices $t$ and $m$ to indicate whether a vector has components for each micro period $(t)$ or each macro period $(m)$.

By studying the optimality conditions of problem $(\mathcal{H})$, one can show that the optimal exercise strategy $e^\star$ is of bang-bang type, that is,

$$e_t^\star(\xi^t) = \begin{cases} \overline{e}_t & \text{if } S_t(\xi_t) \geq q_t(\xi^t), \\ \underline{e}_t & \text{otherwise,} \end{cases}$$

for some unknown function $q_t(\xi^t)$ which typically changes slowly with $t$ and $\xi^t$. This function can conveniently be interpreted as an exercise threshold. Whenever the spot price exceeds (falls short of) $q_t$, the swing option is exercised at maximum (minimum) delivery rate. This structural knowledge about the optimal exercise strategy motivates us to consider a finite number of *candidate exercise strategies* $e_t^l$, $l = 1, \ldots, L$, defined through

$$e_t^l(\xi_t) = \begin{cases} \overline{e}_t & \text{if } S_t(\xi_t) \geq q_l, \\ \underline{e}_t & \text{otherwise,} \end{cases}$$

for some prescribed constant exercise thresholds $q_l$, $l = 1, \ldots, L$. We assume that $q_1 = \infty$ and $q_L = 0$, that is, the first (last) candidate exercise strategy always exercises the swing option at minimum (maximum) delivery rate.

The basic idea to simplify problem $(\mathcal{H})$ is the following. Instead of choosing an individual exercise decision $e_t(\xi^t)$ at the beginning of each micro period, we choose one of the finitely many candidate exercise strategies at the beginning of each macro period. We achieve this by assigning each candidate exercise strategy $e_t^l$ a weight $\lambda_m^l(\xi^m) \geq 0$, $\sum_{l=1}^L \lambda_m^l(\xi^m) = 1$, that is held fixed in each macro period. We then obtain the exercise strategy

$$e_t(\xi^t) = \sum_{l=1}^{L} \lambda_m^l(\xi^m) e_t^l(\xi_t) \text{ for } t = \tau(m), \ldots, \tau(m+1) - 1. \tag{2}$$

The weight vectors $\lambda_m(\xi^m) = \left[\lambda_m^l(\xi^m)\right]_l$ represent the new decision variables that replace the original variables $e_t$. In our numerical examples we will be able to choose $ML \ll T$, which results in a substantial complexity reduction.

In order to reformulate problem $(\mathcal{H})$ in terms of the new decision variables, we define

$$e_m^l(\xi) = \sum_{t=\tau(m)}^{\tau(m+1)-1} e_t^l(\xi_t)$$

as the cumulative energy consumption of the candidate exercise strategy $e_t^l$ within macro period $m$ and

$$\delta_m^l(\xi) = \sum_{t=\tau(m)}^{\tau(m+1)-1} e_t^l(\xi_t) \left[S_t(\xi_t) - K\right]$$

as the aggregate dividend earned by exercising the swing option according to $e_t^l$ within macro period $m$. We also define the vectors $e_m(\xi) = \left[e_m^l(\xi)\right]_l$ and $\delta_m(\xi) = \left[\delta_m^l(\xi)\right]_l$.

As for the trading strategy, we only allow portfolio adjustments $u_t(\xi^t)$ in the first micro period of each macro period, that is, for micro periods $t = \tau(m)$ for some $m \in \mathcal{M}$. All other adjustment variables $u_t(\xi^t)$ are set to zero. This implies that $x_t(\xi^t) = x_{t-1}(\xi^{t-1})$ for all $\tau(m) < t < \tau(m+1)$, $m \in \mathcal{M}$ and $\xi \in \Xi$, that is, the position variables $x_t(\xi^t)$ do not change within the macro periods. With these simplifications, we can replace the original position and adjustment variables with new decision variables $x_m(\xi^m)$ and $u_m(\xi^m)$, respectively, where $x_m(\xi^m)$ represents the portfolio positions during macro period $m$ and $u_m(\xi^m)$ the portfolio adjustments at the beginning of macro period $m$. Finally, we define the aggregate dividend earned by holding the $j^{\text{th}}$ basic security in macro period $m$ through

$$d_m^j(\xi) = \sum_{t=\tau(m)}^{\tau(m+1)-1} d_t^j(\xi_t),$$

and we set $d_m(\xi) = \left[d_m^j(\xi)\right]_j$.

We now formulate the *aggregated holder's problem* $(\mathcal{AH})$:

$$\begin{array}{ll} \max\limits_{\lambda,x,u} & -\langle P_1, x_0\rangle \\ \text{s.t.} & \langle P_{T+1}, x_M\rangle \geq 0 \\ & \underline{c} \leq \sum\limits_{m\in\mathcal{M}} \langle e_m, \lambda_m\rangle \leq \overline{c} \\ & \left.\begin{array}{l} \lambda_m \geq 0,\ \langle \mathbb{I}, \lambda_m\rangle = 1 \\ x_m = x_{m-1} + u_m \\ \left\langle P_{\tau(m)}, u_m\right\rangle \leq \langle \delta_m, \lambda_m\rangle + \langle d_m, x_m\rangle \end{array}\right\} \forall m \in \mathcal{M} \end{array}$$

The decision variables of this problem are $\lambda_m(\xi^m) \in \mathbb{R}^L$, $x_m(\xi^m) \in \mathbb{R}^J$ and $u_m(\xi^m) \in \mathbb{R}^J$, $m \in \mathcal{M}$, as well as $x_0 \in \mathbb{R}^J$. We notationally suppress dependence on $\xi$ to avoid clutter. The constraints are understood to hold for all $\xi \in \Xi$. Note that all decision variables are chosen once per macro period, and they only depend on the reduced observation histories $\xi^m$.

The objective function as well as the first, second and fourth constraint have the same meaning as in problem $(\mathcal{H})$. Note that the per-period energy constraints are implicitly satisfied as the exercise decisions (2) represent convex combinations of candidate strategies satisfying these limits. The third constraint in $(\mathcal{AH})$ ensures that $\lambda_m$ is a vector of convex weights for all $m \in \mathcal{M}$. The last constraint stipulates that the costs $\left\langle P_{\tau(m)}, u_m\right\rangle$ to rebalance the portfolio of basic securities at the beginning of macro period $m$ must be recovered with certainty from the aggregate dividend $\langle \delta_m, \lambda_m\rangle$ generated by the swing option and the aggregate dividends $\langle d_m, x_m\rangle$ received from the basic securities.

We can simplify problem $(\mathcal{W})$ in a similar way to obtain the aggregated writer's problem $(\mathcal{AW})$. We now show that $(\mathcal{AH})$ and $(\mathcal{AW})$ represent conservative approximations for the original problems $(\mathcal{H})$ and $(\mathcal{W})$, respectively.

*Proposition 4.* The optimal values $H$ and $W$ of the problems $(\mathcal{H})$ and $(\mathcal{W})$, as well as the optimal values $AH$ and $AW$ of the problems $(\mathcal{AH})$ and $(\mathcal{AW})$, satisfy

$$AH \leq H \leq W \leq AW.$$

The aggregated problems $(\mathcal{AH})$ and $(\mathcal{AW})$ are feasible iff the original problems $(\mathcal{H})$ and $(\mathcal{W})$ are feasible.

**Proof.** We can decompose any feasible solution $(\lambda, x, u)$ in problem $(\mathcal{AH})$ into a feasible solution $(e', x', u')$ in problem $(\mathcal{H})$ so that $x_0' = x_0$. This may require trading in the riskless security so that $(e', x', u')$ satisfies the self-financing constraint in $(\mathcal{H})$ for all $t \in \mathcal{T}$. Since $(\lambda, x, u)$ and $(e', x', u')$ attain the same objective values in their respective problems, we have $AH \leq H$. A similar argument shows that $W \leq AW$ for the problems $(\mathcal{W})$ and $(\mathcal{AW})$. Finally, the inequality $H \leq W$ is proven in Proposition 3.

We now show that $(\mathcal{W})$ and $(\mathcal{AW})$ are always feasible. To this end, consider problem $(\mathcal{W})$. Compactness of the support $\Xi$ and continuity of the spot price $S_t$ imply that the cumulative obligations arising from the swing option, $\sum_{t\in\mathcal{T}} e_t^\star(S_t - K)$, are bounded above by a constant $O$. We thus obtain a feasible solution $(x, u)$ to problem $(\mathcal{W})$ if we take a large loan today and use this loan to cover all obligations arising from the swing option. A similar argument shows that $(\mathcal{AW})$ is always feasible as well.

The arguments from the previous paragraph extend to problem $(\mathcal{H})$. Hence, $(\mathcal{H})$ is feasible iff there is a feasible exercise strategy $e$ for the swing option, that is, if

$$\left[\langle \mathbb{I}, \underline{e}\rangle, \langle \mathbb{I}, \overline{e}\rangle\right] \cap [\underline{c}, \overline{c}] \neq \emptyset.$$

Assume that these two intervals indeed have a nonempty intersection, and let $c$ be contained in that intersection. Then $c = \gamma \langle \mathbb{I}, \underline{e}\rangle + (1-\gamma)\langle \mathbb{I}, \overline{e}\rangle$ for some $\gamma \in [0, 1]$. Choose $\lambda_m^1 = \gamma$, $\lambda_m^L = 1 - \gamma$ and $\lambda_m^l = 0$ for $l \notin \{1, L\}$ for each macro period $m \in \mathcal{M}$. This choice of $\lambda$ satisfies the second constraint in $(\mathcal{AH})$, and we can marry $\lambda$ with a trading strategy $(x, u)$ so that $(\lambda, x, u)$ is feasible in $(\mathcal{AH})$. □

### *3.2 Decision rule approximation*

Although the number of time periods has been reduced in $(\mathcal{AH})$ and $(\mathcal{AW})$, both problems still involve infinitely many decision variables and constraints. Following the approach developed by Bertsimas and Caramanis [2007] and Vayanos et al. [2011], we restrict the functional decisions $\lambda_m$, $x_m$ and $u_m$ to be polynomials in $\xi^m$ of finite degree $d \in \mathbb{N}_0$. Hence, we replace the portfolio holdings $x_m$ in macro period $m \in \mathcal{M}$ with a polynomial decision rule of the form

$$x_m(\xi^m) = \sum_{\substack{\alpha\in\mathbb{N}_0^{mk}: \\ \langle \mathbb{I}, \alpha\rangle \leq d}} p_\alpha (\xi^m)^\alpha,$$

where $(\xi^m)^\alpha = \prod_{i=1}^{mk} (\xi_i^m)^{\alpha_i}$, and we introduce similar decision rules for the portfolio adjustments $u_m$ and weights $\lambda_m$ of the candidate exercise strategies. The resulting approximate problems $(\mathcal{PAH})$ and $(\mathcal{PAW})$ are linear in the polynomial coefficients, which constitute the new decision variables that replace $\lambda_m$, $x_m$ and $u_m$.

We now show that $(\mathcal{PAH})$ and $(\mathcal{PAW})$ are conservative approximations for $(\mathcal{AH})$ and $(\mathcal{AW})$, respectively.

*Proposition 5.* Assume given the optimal values

(1) $H$ and $W$ of the problems $(\mathcal{H})$ and $(\mathcal{W})$,
(2) $AH$ and $AW$ of the problems $(\mathcal{AH})$ and $(\mathcal{AW})$, and
(3) $PAH$ and $PAW$ of the problems $(\mathcal{PAH})$ and $(\mathcal{PAW})$.

These values satisfy the chain of inequalities

$$PAH \leq AH \leq H \leq W \leq AW \leq PAW.$$

The problems $(\mathcal{PAH})$ and $(\mathcal{PAW})$ are feasible iff the problems $(\mathcal{H})$ and $(\mathcal{W})$ are feasible.

**Proof.** The chain of inequalities follows from Proposition 4 and the fact that any feasible solution to $(\mathcal{PAH})$ and $(\mathcal{PAW})$ is also feasible in $(\mathcal{AH})$ and $(\mathcal{AW})$, respectively. The proof of the second statement parallels the proof of Proposition 4 and can therefore be omitted. □

### *3.3 Constraint sampling*

Although $(\mathcal{PAH})$ and $(\mathcal{PAW})$ involve finitely many decision variables, they remain intractable since their constraints are parameterised by $\xi \in \Xi$, while $\Xi$ typically has infinite cardinality. We obtain tractable approximations for $(\mathcal{PAH})$ and $(\mathcal{PAW})$ if we replace $\Xi$ with finite subsets $\Xi^N \subset \Xi$ of cardinality $N$. We denote these approximate problems by $(\mathcal{PAH}^N)$ and $(\mathcal{PAW}^N)$.

The following theorem shows that randomly sampled subsets $\Xi^N$ of moderate cardinality $N$ are sufficient to obtain good approximations. We state this result for problem $(\mathcal{PAH})$, but it immediately extends to $(\mathcal{PAW})$.

*Theorem 6.* (Campi and Garatti [2008]). Suppose that $(\mathcal{PAH})$ is feasible and accommodates $n$ decision variables. For a prespecified violation probability $\epsilon \in (0,1)$ and confidence $\beta \in (0,1)$, define

$$N(\epsilon,\beta) = \min\left\{ N \in \mathbb{N} : \sum_{i=0}^{n-1} \binom{N}{i} \epsilon^i (1-\epsilon)^{N-i} \leq \beta \right\}.$$

Then, the probability mass of all $\xi \in \Xi$ whose associated constraints are violated by an optimal solution of $(\mathcal{PAH}^N)$, $N \geq N(\epsilon,\beta)$, does not exceed $\epsilon$ with confidence $1-\beta$.

The parameter $\epsilon$ describes the probability that an optimal solution to $(\mathcal{PAH}^N)$ violates a constraint of $(\mathcal{PAH})$ for a randomly chosen sample $\xi \in \Xi$. A constraint violation implies that the terminal wealth is negative, the per-period or cumulative energy constraints are violated, or that the portfolio containing the swing option and the basic securities is not self-financing. The option holder thus faces a risk when she implements an optimal solution to $(\mathcal{PAH}^N)$, and this risk can be controlled by choosing an appropriate value for $\epsilon$.

Since an optimal solution to $(\mathcal{PAH}^N)$ depends on the random sample $\Xi^N \subset \Xi$, it is itself a random variable. The violation probability $\epsilon$ is satisfied by the optimal solutions corresponding to some random samples $\Xi^N \subset \Xi$, but it may be violated by others. The parameter $\beta$ quantifies the confidence that the violation probability $\epsilon$ is satisfied by an optimal solution to $(\mathcal{PAH}^N)$. Since the sample size $N$ only grows logarithmically as $\beta$ goes to zero, one can assign a negligibly small quantity to $\beta$ (e.g. $10^{-10}$).

If we fix the degree $d$ of the polynomial decision rules as well as the violation probability $\epsilon$ and confidence $\beta$, then the required sample size $N(\epsilon,\beta)$ is bounded from above by a polynomial in the parameters $k$, $J$ and $M$. Thus, the approximate problems $(\mathcal{PAH}^N)$ and $(\mathcal{PAW}^N)$ are linear programs that involve polynomially many variables and constraints and can thus be solved efficiently.

## 4. NUMERICAL RESULTS

We consider a planning horizon of five weeks subdivided into periods of one hour (i.e., $T = 958$). We hedge an hourly exercisable peak swing option with a delivery period of one month starting today (i.e., at the beginning of period 1). The per-period energy limits are $\underline{e}_t = 0$ MWh, $t \in \mathcal{T}$, $\overline{e}_t = 1$ MWh for peak hours and $\overline{e}_t = 0$ MWh for off-peak hours. The option's strike price is $K = 30$ €/MWh. The cumulative energy limits are $\underline{c} = 0$ MWh and $\overline{c} = 72$ MWh, and exceedance of the upper energy limit $\overline{c}$ is penalised by a cost of 10 €/MWh.

We hedge the swing option with a money market account and all the electricity forwards that would be traded on the European Energy Exchange [2] during the planning horizon. In particular, we consider weekly forwards for five weeks and monthly forwards for two months. In both cases, we consider base and peak contracts. In total, the market thus contains $J = 15$ basic securities. The spot price of electricity and the prices as well as dividends of the basic securities are modelled with the approach described by Haarbrücker and Kuhn [2009]. We remark, however, that any other price model can be used instead.

We partition the planning horizon into $M$ macro periods of (approximately) equal length, see §3.1. Moreover, we consider convex combinations of candidate exercise strategies $e^l$, $l = 1,\ldots,L$, whose exercise thresholds $q_l$ are selected uniformly between the strike price $K$ and the maximum of the spot price over the planning horizon, see §3.2. Note that exercise thresholds below the strike price $K$ are always suboptimal because the swing option holder is not obligated to purchase electric energy (since $\underline{e} = 0$ and $\underline{c} = 0$). We choose the number $N$ of random samples $\xi \in \Xi$ such that a given violation probability $\epsilon$ is satisfied at the confidence level $\beta = 0.01\%$, see §3.3.

In the following, we denote by $\mathcal{H}$ the holder's problem under ruthless exercise, by $\mathcal{W}$ the writer's problem under ruthless exercise, and by $\mathcal{RW}$ the writer's problem under worst-case exercise, that is, under the assumption that the holder chooses the exercise strategy that maximises the cumulative obligations $\sum_{t\in\mathcal{T}} e_t^\star(K - S_t)$ of the writer.

We first investigate the impact of the approximation parameters $M$, $L$, $d$ and $\epsilon$ on the holder's and writer's price of the swing option. As Fig. 1(a) shows, the no-arbitrage interval shrinks considerably when the number of macro periods $M$ increases, indicating that dynamic hedging of the swing option is essential. The no-arbitrage interval saturates when about 7 macro periods are used. The chart also shows that one obtains severely suboptimal no-arbitrage intervals if the holder or the writer hedges the swing option statically, that is, if the hedging portfolio can only be adjusted in the first macro period. Fig. 1(b) shows that a small number of candidate exercise strategies ($L \geq 4$) suffices for a good approximation of the optimal exercise strategy. Fig. 1(c) illustrates the holder's and writer's price as the degree $d$ of the polynomial decision rules is increased. The chart shows that the price gap reduces significantly when moving from constant to linear decision rules, while a further increase in $d$ does not have a noticeable effect. Finally, Fig. 1(d) illustrates the

[2] European Energy Exchange: `www.eex.com`

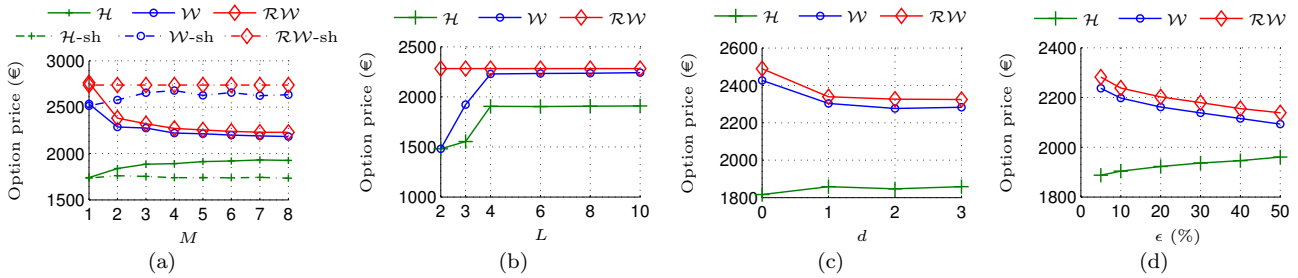



Fig. 1. No-arbitrage interval as a function of $M$, $L$, $d$ and $\epsilon$ (from left to right). The suffix "-sh" indicates static hedging. The basic parameter setting is $M = 5$, $L = 5$, $d = 1$ and $\epsilon = 5\%$.

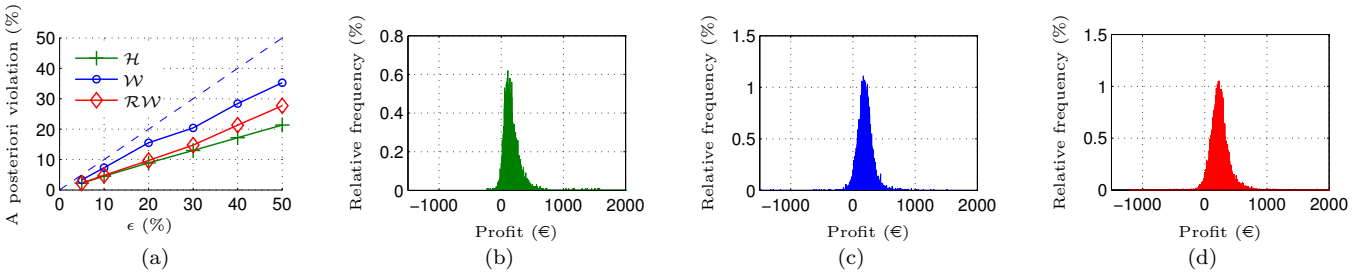



Fig. 2. A posteriori violation (left chart) and empirical profit/loss distributions for the holder under ruthless exercise, the writer under ruthless exercise, and the writer under worst-case exercise (second to fourth graph, respectively). The basic parameter setting is $M = 5$, $L = 5$, $d = 1$ and $\epsilon = 5\%$.

impact of the violation probability $\epsilon$ on the option's no-arbitrage interval. With larger values of $\epsilon$, the holder and the writer accept to take on part of the risk that results from the inability to perfectly replicate the option's payoff streams. This causes the holder's and writer's price to converge as $\epsilon$ increases.

We now assess the risk faced by the holder and the writer of the swing option. To this end, we draw 5000 independent samples $\xi \in \Xi$. We define the a posteriori violation probability of the holder and the writer as the percentage of those 5000 samples that violate any one of the constraints of $(\mathcal{PAH})$ and $(\mathcal{PAW})$, respectively. The a posteriori violation probabilities for different levels of (a priori) violation probabilities $\epsilon$ are shown in Fig. 2(a). As expected from Theorem 6 and our choice of $\beta$, the a posteriori violation never exceeds $\epsilon$ (dashed line).

The other charts in Fig. 2 visualise the empirical profit/loss distributions. The charts show that the expected profits of both the holder and the writer are positive. The distributions have a positive skewness, indicating that the tail on the right side (profit) is longer than on the left side (loss).

We finally remark that apart from the instances considered in Fig. 1(a), all problems were solved within 20 min on a 2.66GHz Intel Core i7-920 machine running CPLEX 12.2. The problems in Fig. 1(a) were solved within 0.04 sec ($M = 1$) and 55 min ($M = 8$).

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