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Distributed Attitude Control of Multi-Agent Formations

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Abstract: This paper addresses the problem of how to coordinate the spatial motion of individual agents in a multi-agent system in such a way that the distance between any two agents is maintained at a constant value (but not stabilized) throughout the evolution of the system while simultaneously rotating the formation of agents to a pre-set reference attitude. We propose a distributed control law based on locally available information. The stability of the proposed control law is analyzed using Lyapunov theory in the special case of three agents. The region of attraction is found to encompass the entire state-space, save a single point. A generalization to the case of n agents is performed using the theory of rigid graphs. The geometric underpinnings of these ideas are illustrated by simulation.

Keywords: Multi-agent systems, Formation control, Rigid body kinematics, Graph theory.

1. INTRODUCTION

Cooperating autonomous agents always have the potential to outperform the efforts of any single agent. Such teams of robots are used in a great number of civilian and military applications where the task performed range from routine surveillance to hazardous work in high risk environments such as offshore oil rigs or at remote locations, in outer space for example. The problems thus encountered form an exciting area of study within the fields of robotics and control theory. A fundamental task is formation control, which can roughly be understood as adjusting the relative position or pose between each agent while allowing the group of agents to move as a whole.

In the last decade, formation control of systems with diverse dynamics and various task requirements has been studied using a number of different approaches to control design. Lin, Francis and Maggiore (2005) investigated when a nonholonomic system of agents with two position states and one orientation state can converge to a given geometric formation. Egerstedt and Hu (2001) used a leader-follower approach to design a control law allowing a group of agents to reach a formation asymptotically. Ji and Egerstedt (2007) addressed the connectedness issue in the context of formation control. For an extensive literature review, see Chen and Wang (2005). Problems closely related to formation control are consensus, cooperative target tracking, and path following, see Ren and Beard (2008).

There is a body of research on how to verify and maintain the rigidity of a given structure using tools from graph theory, see for example Whiteley (1997); Yu *et al.* (2007). A theoretical framework for representing split, rejoin and reconfiguration maneuvers for multi-vehicle formations using the theory of rigid graphs is presented by Olfati-Saber and Murray (2002a). They use this framework and the potential function method to solve control problems of formation and reformation in a planar setting (Olfati-Saber and Murray, 2002b). A closely related work is Krick, Broucke and Francis (2008).

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The problem of how to maintain and change the attitude of an already established formation by means of distributed control has generally been given less focus than formation stabilization. There are many possible applications. Picture three persons moving a table together, or three fingers adjusting the orientation of a cup simultaneously. Cooperation in such tasks is both crucial and difficult since any movement caused by one agent will affect the others. If for example the object being carried is rigid, the control should be designed to keep the relative distances between each agent all the time, or else they risk dropping or deforming the object. One way of accomplishing this is by constraining the system of agents to move as a single rigid body, *i.e.* to move in a fixed formation.

In this paper, we address the problem of how to coordinate the spatial motion of each individual agent in a multi-agent formation in such a way that all distances between agents are maintained (not stabilized) at constant values throughout the evolution of the system. The control aim is to rotate the configuration of agents so that a preset reference attitude is reached asymptotically. To do this in a efficient and robust fashion, we require the control to be decentralized and based on locally available information only. A control law based on the geometry of the problem is proposed. The stability of the proposed control is analyzed using Lyapunov theory in the case of three agents, and a generalization to the case of n agents is performed using the theory of rigid graphs.

The remaining sections of this paper are organized as follows: a problem statement is given in § 2; the case of three agents is considered in § 3, while the n agent case is studied in § 4; results from simulations carried out in MATLAB are presented in § 5 and conclusions are drawn in § 6.

2. PROBLEM STATEMENT

The purpose of this paper is to demonstrate that the agents of a multi-agent system can move as a rigid body without relying on any centralized control. We will only do this for a basic type of system on a kinematic level and therefore specify the dynamics of each agent to consist of a single integrator

$$\dot{\mathbf{x}}_i = \mathbf{u}_i, \quad i = 1, 2, \cdots, n, \tag{1}$$

where $\mathbf{x}_i \in \mathbb{R}^3$ is the position of the ith agent in an inertial frame of reference.

Let the Euclidean metric

$$d(\mathbf{x}_i, \mathbf{x}_j) = \|\mathbf{x}_i - \mathbf{x}_j\|$$

denote the distance between the ith and the jth agent. The rigid body motion constraints can then be written as

$$d(\mathbf{x}_i, \mathbf{x}_j) = c_{i,j}, \ \forall i, j = 1, \ldots, n, \tag{2}$$

where $c_{i,j} \in \mathbb{R}^+$ are constants. The position of any agent in an inertial frame of reference can be uniquely determined by the distances between it and three other, non-collinear agents, whose positions are known. Hence, a number of the $n(n-1)/2$ constrains (2) are redundant. In fact, their number can be reduced to $3n - 6$ as will be detailed in § 4.1.

The configuration of agents should be rigid throughout the evolution of the system whereby non-collinearity will be preserved. Therefore, we choose three arbitrary non-collinear agents to act as a kind of reference points. These agents are labeled 1, 2, and 3 for convenience.

The attitude of any rigid body as expressed in an inertial frame of reference can be described using a rotation axis and a rotation angle by Euler's theorem. In our case, a vector that spans the axis is given by the normal of the plane formed by agents 1, 2, and 3. We will not consider the angle of rotation about that axis, *i.e.* we only consider two of the three degrees of freedom that compose the attitude of a rigid body. Let $\mathbf{n} \in \mathbb{R}^3$ be a unit vector representing the reference attitude, and let $P_\mathbf{n} \subset \mathbb{R}^3$ denote the plane with normal $\mathbf{n}$ intersecting the origin. The formation has reached the reference attitude when

$$\frac{(\mathbf{x}_1 - \mathbf{x}_2) \times (\mathbf{x}_2 - \mathbf{x}_3)}{\|(\mathbf{x}_1 - \mathbf{x}_2) \times (\mathbf{x}_2 - \mathbf{x}_3)\|} = \mathbf{n}. \tag{3}$$

Equation (3) determines the attitude of the entire configuration of agents, assuming the rigid body motion constraints are maintained during the evolution of the system.

The problem we study is the design of a distributed control law; *i.e.* how to specify the input signals (1) in terms of relative information in a decentralized way such that (2) is not violated by the control and (3) is fulfilled asymptotically. It is assumed that all agents know the desired attitude $\mathbf{n}$.

3. A SPECIAL CASE

We begin by analyzing the special case of a system with only three agents. Specifying the position of three non-collinear points on a rigid body uniquely determines the positions of any other point on said body, given the distances between that point and the three reference points. Hence, the extension to the n agents case is easily done once the three agent case has been solved assuming $\mathbf{x}_i - \mathbf{x}_j$ can be sensed by any two agents i and j, as is shown in § 4.1. In § 4.2 we further generalize the results of this section to the case where some agents i and j are unable to sense each other directly.

3.1 Motion Constraints

Let us design the control signals $\mathbf{u}_i$ such that the distances $d(\mathbf{x}_i, \mathbf{x}_j)$ are maintained at constant values throughout the evolution of system (1). Define $\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta} \in \mathbb{R}^3$ by

$$\boldsymbol{\xi} = \mathbf{x}_1 - \mathbf{x}_2, \ \boldsymbol{\eta} = \mathbf{x}_2 - \mathbf{x}_3, \ \boldsymbol{\zeta} = \mathbf{x}_3 - \mathbf{x}_1,$$

whereby the rigid body motion constraints may be rewritten as

$$\boldsymbol{\xi} \cdot \boldsymbol{\xi} = c_{1,2}^2, \ \boldsymbol{\eta} \cdot \boldsymbol{\eta} = c_{2,3}^2, \ \boldsymbol{\zeta} \cdot \boldsymbol{\zeta} = c_{3,1}^2. \tag{4}$$

Note that the variable $\boldsymbol{\zeta}$ is redundant since $\boldsymbol{\zeta} = -(\boldsymbol{\xi} + \boldsymbol{\eta})$. Also note that any function depending only on $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ is a function of relative information.

The constraints (4) are satisfied if the control of system (1) satisfies

$$\begin{aligned} \boldsymbol{\xi} \cdot (\mathbf{u}_1 - \mathbf{u}_2) &= 0, \\ \boldsymbol{\eta} \cdot (\mathbf{u}_2 - \mathbf{u}_3) &= 0, \\ \boldsymbol{\zeta} \cdot (\mathbf{u}_3 - \mathbf{u}_1) &= 0, \end{aligned}$$

i.e. if the vector $\mathbf{u} = [\mathbf{u}_1^\mathsf{T}, \mathbf{u}_2^\mathsf{T}, \mathbf{u}_3^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^9$ of input signals belongs to the null space $\mathcal{N}(M)$ of the matrix $M \in \mathbb{R}^{3\times 9}$ defined by

$$M = \begin{bmatrix} \boldsymbol{\xi}^\mathsf{T} & -\boldsymbol{\xi}^\mathsf{T} & \mathbf{0}^\mathsf{T} \\ \mathbf{0}^\mathsf{T} & \boldsymbol{\eta}^\mathsf{T} & -\boldsymbol{\eta}^\mathsf{T} \\ -\boldsymbol{\zeta}^\mathsf{T} & \mathbf{0}^\mathsf{T} & \boldsymbol{\zeta}^\mathsf{T} \end{bmatrix}. \tag{5}$$

Note that $\mathbf{u} \in \mathcal{N}(M)$ respects the constraints but it does not enforce them, *i.e.* the equilibrium values $c_{i,j}$ of $d(\mathbf{x}_i, \mathbf{x}_j)$ are stable under $\mathbf{u}$ but not asymptotically stable.

A matrix whose columns form a basis for $\mathcal{N}(M)$ is $B(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta}) \in \mathbb{R}^{9\times 6}$ defined by

$$B(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta}) = \begin{bmatrix} I & -S(\boldsymbol{\xi}) + S(\boldsymbol{\zeta}) \\ I & S(\boldsymbol{\xi}) - S(\boldsymbol{\eta}) \\ I & S(\boldsymbol{\eta}) - S(\boldsymbol{\zeta}) \end{bmatrix},$$

where $S(\cdot) \in \mathbb{R}^{3\times 3}$ is the skew-symmetric matrix generated by a vector $\mathbf{a} = [a_1, a_2, a_3]^\mathsf{T} \in \mathbb{R}^3$ through setting

$$S(\mathbf{a}) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}. \tag{6}$$

Note that $S(\mathbf{a})\mathbf{b}$, with $\mathbf{b} \in \mathbb{R}^3$, is an alternative notation for the cross product $\mathbf{a} \times \mathbf{b} \in \mathbb{R}^3$. This implies, among other things, that $S(\mathbf{a})$ is linear and $\mathbf{a} \in \mathcal{N}(S(\mathbf{a}))$. We state two well-known properties of the cross product for future reference:

(i) (*Signed volume*) The factors of the scalar-valued triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ with $\mathbf{c} \in \mathbb{R}^3$ may be reordered as $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$.

(ii) (*bac-cab rule*) The vector-valued triple product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ can be rewritten using dot products as $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$.

Design

$$\mathbf{u} = B(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{\zeta}) \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} \mathbf{v} + (\boldsymbol{\zeta} - \boldsymbol{\xi}) \times \mathbf{w} \\ \mathbf{v} + (\boldsymbol{\xi} - \boldsymbol{\eta}) \times \mathbf{w} \\ \mathbf{v} + (\boldsymbol{\eta} - \boldsymbol{\zeta}) \times \mathbf{w} \end{bmatrix}, \tag{7}$$

where $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ are new input signals representing the translational and rotational velocities of the formation. Since $\mathbf{u} \in \mathcal{R}(B) = \mathcal{N}(M)$, the constraints (4) are satisfied for all choices of $\mathbf{v}$ and $\mathbf{w}$. Note that $\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 = 3\mathbf{v}$, implying that the centroid $\mathbf{x}_c = (\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3)/3$ moves with translational velocity $\mathbf{v}$. Hence $\mathbf{x}_c$ remains in a fixed position if $\mathbf{v} = \mathbf{0}$, *i.e.* $\mathbf{x}_c$ is then the instantaneous center of zero velocity for the configuration.

3.1.1 Remark. Each of the three agents is capable of implementing its own control law (7) based on local measurements since $\mathbf{u}_i = \mathbf{v} + [(\mathbf{x}_j - \mathbf{x}_i) + (\mathbf{x}_k - \mathbf{x}_i)] \times \mathbf{w}$ (a formula that can easily be obtained from (7)) is symmetric with respect to $\mathbf{x}_j$ and $\mathbf{x}_k$, *i.e.* the same $\mathbf{u}_i$ is obtained no matter how agent i chooses to label itself or j and k.

A straightforward calculation yields $\dot{\boldsymbol{\xi}} = 3\mathbf{w} \times \boldsymbol{\xi}$ and $\dot{\boldsymbol{\eta}} = 3\mathbf{w} \times \boldsymbol{\eta}$. Introduce $\boldsymbol{\omega} \in \mathbb{R}^3$ defined by $\boldsymbol{\omega} = 3\mathbf{w}$ for notational convenience. The dynamics for $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ may now be written as

$$\dot{\boldsymbol{\xi}} = \boldsymbol{\omega} \times \boldsymbol{\xi}, \quad \dot{\boldsymbol{\eta}} = \boldsymbol{\omega} \times \boldsymbol{\eta}. \tag{8}$$

The dynamics (8) conserves distances, as indicated by the following lemma.

3.1.2 Lemma. The norm of any quantity $\mathbf{x} \in \mathbb{R}^3$ governed by the state equation $\dot{\mathbf{x}} = \boldsymbol{\omega} \times \mathbf{x}$ is constant in time.

Proof. Let $X = \frac{1}{2}\|\mathbf{x}\|^2$ and note that $\dot{X} = \mathbf{x} \cdot \dot{\mathbf{x}} = \mathbf{x} \cdot (\boldsymbol{\omega} \times \mathbf{x}) = 0$. □

Note that any vector like the $\mathbf{x}$ of Lemma 3.1.2 is constrained by $\|\mathbf{x}\|$ being constant and hence only has two degrees on freedom. The dynamics (8) therefore has three degrees of freedom (since both $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are governed by the same $\boldsymbol{\omega}$) and the system hence evolves on a set that is equivalent to the rotation group $SO(3)$.

The constraints will be fulfilled for any choice of $\boldsymbol{\omega}$. The problem of how to design $\boldsymbol{\omega}$ such that the agents perform the desired collective behavior (3) will be considered in the next section. Before proceeding, we note that the input signal $\mathbf{v}$ cancels out of the dynamics (8) of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. The freedom provided by $\mathbf{v}$ can be can be used to fix a position $\mathbf{p}$ relative to $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$, through which the rotation axis passes. If $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$ are linearly independent, they span $\mathbb{R}^3$ and any $\mathbf{p} \in \mathbb{R}^3$ can be written as

$$\mathbf{p} = \sum_{i=1}^{3} \lambda_i \mathbf{x}_i$$

with $\lambda_i \in \mathbb{R}$. The rotation axis passes through $\mathbf{p}$ if $\dot{\mathbf{p}} = \mathbf{q}$ for any constant $\mathbf{q} \in \mathbb{R}^3$, *i.e.* if

$$\lambda_1 \mathbf{u}_1 + \lambda_2 \mathbf{u}_2 + \lambda_3 \mathbf{u}_3 = \mathbf{q}. \tag{9}$$

Note that $\mathbf{q}$ is the translational velocity of $\mathbf{p}$ and if for example $\mathbf{q} = \mathbf{0}$, then $\mathbf{p}$ remains in a fixed position during the evolution of the system. Assuming $\lambda_1 + \lambda_2 + \lambda_3 \neq 0$, we may solve (7) and (9) for $\mathbf{v}$ to obtain

$$\mathbf{v} = \frac{\mathbf{q} + [(\lambda_1 - \lambda_2)\boldsymbol{\xi} + (\lambda_2 - \lambda_3)\boldsymbol{\eta} + (\lambda_3 - \lambda_1)\boldsymbol{\zeta}] \times \boldsymbol{\omega}}{\lambda_1 + \lambda_2 + \lambda_3}.$$

The values of λ_i, *i.e.* the value of $\mathbf{p}$ can be chosen arbitrarily. For example, choosing $\lambda_1 = \lambda_2 = \lambda_3 = 1$ gives the centroid of the triangle with corners in $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ a pure translational velocity of $\mathbf{q}/3$. We return to and generalize this result in § 4.2.

3.2 Control Design and Stability Analysis

This section presents a control law $\boldsymbol{\omega}$ based on relative information that asymptotically drive the configuration of agents to the reference attitude.

The current attitude of the formation can be represented by a non-zero vector $\boldsymbol{\psi}$ and a rotation angle about that vector. Here, we do not concern ourself with the rotation angle about $\boldsymbol{\psi}$. Rather, we focus on $\boldsymbol{\psi}$ defined as $\boldsymbol{\psi} = \boldsymbol{\xi} \times \boldsymbol{\eta}$, *i.e.* as a normal to the plane $P_{\boldsymbol{\psi}}$ containing $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ and the origin. The states $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ can be driven to the reference plane by rotating them about the line of intersection between $P_{\mathbf{n}}$ and $P_{\boldsymbol{\psi}}$.

The input signal $\boldsymbol{\omega}$ is designed as

$$\boldsymbol{\omega} = \alpha\, \mathbf{l}, \tag{10}$$

where $\alpha \in \mathbb{R}^+$ is a scale factor and $\mathbf{l} \in \mathbb{R}^3$ is a vector that spans the line $L = P_{\mathbf{n}} \cap P_{\boldsymbol{\psi}}$. More specifically, $\mathbf{l} = \boldsymbol{\psi} \times \mathbf{n}$. The resulting dynamics of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are

$$\dot{\boldsymbol{\xi}} = -\alpha\, (\mathbf{n} \cdot \boldsymbol{\xi})\boldsymbol{\psi}, \tag{11}$$

$$\dot{\boldsymbol{\eta}} = -\alpha\, (\mathbf{n} \cdot \boldsymbol{\eta})\boldsymbol{\psi}, \tag{12}$$

as can easily be verified using the *bac-cab* rule for vector triple products.

3.2.1 Remark. The three agents 1, 2, and 3 can calculate the same value of $\boldsymbol{\psi}$ up to sign regardless of how they choose to assign $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, and $\boldsymbol{\zeta}$ since $\boldsymbol{\psi}$ is the signed area of the parallelogram formed by $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$. To remove the sign ambiguity, we suggest a principle allowing the agents to identify agent 1, 2, and 3 amongst themselves. Impose an order by setting $\|\boldsymbol{\xi}\| < \|\boldsymbol{\eta}\| < \|\boldsymbol{\zeta}\|$.* The sense of $\boldsymbol{\xi}$ is determined by requiring that $\boldsymbol{\xi}$ point away from $\boldsymbol{\zeta}$ towards $\boldsymbol{\eta}$ in the triangle formed by $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$. This order is invariant under the evolution of the system since $\|\boldsymbol{\xi}\|$, $\|\boldsymbol{\eta}\|$, and $\|\boldsymbol{\zeta}\|$ are constant. The senses of $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ can then be inferred from $\boldsymbol{\xi} + \boldsymbol{\zeta} + \boldsymbol{\eta} = \mathbf{0}$.

3.2.2 Proposition. If system (8) is controlled by $\boldsymbol{\omega}$ as defined by (10), then $\boldsymbol{\psi}/\|\boldsymbol{\psi}\|$ will converge to $\mathbf{n}$ from any initial point on the 2-sphere S^2 except $\boldsymbol{\psi}(0)/\|\boldsymbol{\psi}(0)\| = -\mathbf{n}$. Furthermore, the rate of convergence is locally exponential.

Proof. The proof is done using Lyapunov theory and consists of three steps. First, we prove that $\boldsymbol{\psi}$ belongs to the 2-sphere. The *bac-cab* rule yields

$$\begin{aligned}
\dot{\boldsymbol{\psi}} &= \dot{\boldsymbol{\xi}} \times \boldsymbol{\eta} + \boldsymbol{\xi} \times \dot{\boldsymbol{\eta}} \\
&= (\boldsymbol{\omega} \times \boldsymbol{\xi}) \times \boldsymbol{\eta} + \boldsymbol{\xi} \times (\boldsymbol{\omega} \times \boldsymbol{\eta}) \\
&= (\boldsymbol{\eta} \cdot \boldsymbol{\omega})\boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \boldsymbol{\eta})\boldsymbol{\omega} + (\boldsymbol{\xi} \cdot \boldsymbol{\eta})\boldsymbol{\omega} - (\boldsymbol{\xi} \cdot \boldsymbol{\omega})\boldsymbol{\eta} \\
&= (\boldsymbol{\eta} \cdot \boldsymbol{\omega})\boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \boldsymbol{\omega})\boldsymbol{\eta} = \boldsymbol{\omega} \times \boldsymbol{\psi},
\end{aligned}$$

whereby $\|\boldsymbol{\psi}\|$ is constant by Lemma 3.1.2.

Second, we introduce $\gamma = \mathbf{n} \cdot \boldsymbol{\psi} = \mathbf{n} \cdot (\boldsymbol{\xi} \times \boldsymbol{\eta})$ to denote the signed volume of $\mathbf{n}$, $\boldsymbol{\xi}$, and $\boldsymbol{\eta}$ and note that

$$\begin{aligned}
\dot{\gamma} &= \mathbf{n} \cdot \dot{\boldsymbol{\psi}} = \mathbf{n} \cdot (\boldsymbol{\omega} \times \boldsymbol{\psi}) = -\boldsymbol{\omega} \cdot (\mathbf{n} \times \boldsymbol{\psi}) \\
&= \boldsymbol{\omega} \cdot \mathbf{l} = \alpha\, \|\mathbf{l}\|^2
\end{aligned} \tag{13}$$

by the properties of scalar triple products.

Third, we introduce the Lyapunov function candidate

$$V = \frac{1}{2}\left\| \mathbf{n} - \frac{\boldsymbol{\psi}}{\|\boldsymbol{\psi}\|} \right\|^2 = 1 - \frac{\gamma}{\|\boldsymbol{\psi}\|}$$

with time derivative

$$\dot{V} = -\frac{\dot{\gamma}}{\|\boldsymbol{\psi}\|} = -\frac{\alpha\, \|\mathbf{l}\|^2}{\|\boldsymbol{\psi}\|}$$

since $\|\boldsymbol{\psi}\|$ is constant. Moreover,

$$\begin{aligned}
\|\mathbf{l}\|^2 &= (\boldsymbol{\psi} \times \mathbf{n}) \cdot (\boldsymbol{\psi} \times \mathbf{n}) = -\mathbf{n} \cdot (\boldsymbol{\psi} \times (\boldsymbol{\psi} \times \mathbf{n})) \\
&= -\mathbf{n} \cdot ([\mathbf{n} \cdot \boldsymbol{\psi}]\boldsymbol{\psi} - \|\boldsymbol{\psi}\|^2 \mathbf{n}) = \|\boldsymbol{\psi}\|^2 - \gamma^2 \\
&= (\|\boldsymbol{\psi}\| + \gamma)(\|\boldsymbol{\psi}\| - \gamma) \\
&= (\|\boldsymbol{\psi}\| + \gamma)\|\boldsymbol{\psi}\| V
\end{aligned}$$

so that

$$\dot{V} = -\alpha\, (\|\boldsymbol{\psi}\| + \gamma) V.$$

* This approach can not be applied when the agents do form an isosceles or equilateral triangle. In that case, we may for example assume that each agent possess a unique identifier and that other agents can sense and establish some order among the identifiers.

Note that γ is increasing by (13) since $\alpha \in \mathbb{R}^+$ and $\gamma \geq -\|\boldsymbol{\psi}\|$ by the Cauchy-Schwarz inequality. The inequality only holds with equality if $\boldsymbol{\psi}(0)/\|\boldsymbol{\psi}(0)\| = -\mathbf{n}$ meaning that V is exponentially stable for all $\boldsymbol{\psi}(0)/\|\boldsymbol{\psi}(0)\| \neq -\mathbf{n}$, and $V \to 0$ implies $\boldsymbol{\psi}/\|\boldsymbol{\psi}\| \to \mathbf{n}$. □

3.2.3 Remark. Note that Theorem 3.2.2 implies that $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ will converge to the plane $P_{\mathbf{n}}$ from any initial point since if $\boldsymbol{\psi}(0)/\|\boldsymbol{\psi}(0)\| = -\mathbf{n}$, then $\mathbf{n} \cdot \boldsymbol{\xi} = \mathbf{n} \cdot \boldsymbol{\eta} = 0$. This also shows that $P_{\mathbf{n}}$ is an invariant set of (11) and (12).

4. THE GENERAL CASE

In this section, we extend the results obtained in § 3 for the case of three agents to the general case of n agents. The variables $\mathbf{v}$, $\mathbf{w}$, $\boldsymbol{\omega}$ and $\boldsymbol{\psi}$ are redefined here but have interpretations corresponding to those in the three agent case of § 3.

4.1 Constraint Graph and Rigidity Matrix

Let us start with the constraints. A constraint graph G can be used to represent the rigidity of the configuration of agents. The vertex set $V = \{1, \ldots, n\}$ of G is in one-to-one correspondence with the set of agents and the edge set $E = \{(i, j) \in V \times V\}$ of G contains pairs of vertices corresponding to pairs of agents that should be maintained at a constant distance from each other. The graph G does not necessarily have to be complete since a number of the constraints are redundant in the sense of being implied by the rest. The constraint graph is therefore non-unique in general.

We will use the theory of rigid and flexible frameworks to describe the requirement that the every distance between any two agents should be maintained at a constant level as the configuration of agents moves in $\mathbb{R}^3$. A useful concept is the rigidity matrix (Tay and Whiteley, 1985; Whiteley, 1997). For a configuration of n agents in $\mathbb{R}^3$ with constraint graph G, the rigidity matrix R is the Jacobian matrix formed by the time derivatives of the distance constraints in E

$$(\mathbf{x}_i - \mathbf{x}_j) \cdot (\dot{\mathbf{x}}_i - \dot{\mathbf{x}}_j) = 0, \quad (i, j) \in E. \tag{14}$$

That is $R(G)$ is the $|E| \times 3n$ matrix whose every row is of the form

$$[0 \ldots 0 \; (\mathbf{x}_i - \mathbf{x}_j)^\mathsf{T} \, 0 \ldots 0 \; (\mathbf{x}_j - \mathbf{x}_i)^\mathsf{T} \, 0 \ldots 0]$$

and corresponds to an edge $(i, j) \in E$, where $(\mathbf{x}_i - \mathbf{x}_j)^\mathsf{T} \in \mathbb{R}^3$ is a row vector located in the three columns corresponding to vertex i. The previous M matrix (5) is an example of a rigid matrix representing a triangle. A standard result is that if no set of three or more points are collinear, then the configuration with constraint graph G is rigid if and only if R has rank $3n-6$, which also is the maximum rank of R (Tay and Whiteley, 1985; Whiteley, 1997). Therefore, the number of edges in E should be at least $3n - 6$, and any number greater than $3n - 6$ would imply that some constraints are redundant.

It is clear that rigidity can be obtained by only preserving $3n-6$ distance constraints if there are three non-collinear agents, since the position of one agent can be uniquely determined by the distance from three other non-collinear agents. For example, assume agents 1, 2, and 3 are non-collinear and that their relative distances are fixed. Add the remaining agents one at a time, and assign each agent i to keep its distance with any three non-collinear agents among the previously added $1, \ldots, i-1$ agents. The construction formed in this manner is rigid. The number of constraints is $3 + 3(n-3) = 3n - 6$.

The constraint graph for (2) is complete. But the above analysis suggests that the number of constraints could be reduced to $3n-6$. Assume that redundant constraints are discarded leaving G with $3n - 6$ edges such that the rigidity matrix R has full rank. Let $\mathcal{N}(R)$ denote the kernel or null space of R. By (14), the control $\mathbf{u} = [\mathbf{u}_1^\mathsf{T}, \mathbf{u}_2^\mathsf{T}, \cdots, \mathbf{u}_n^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^{3n}$ should satisfy $\mathbf{u}_i - \mathbf{u}_j \in \mathcal{N}(R)$ for all $(i, j) \in G$ in order to preserve the rigidity during the evolution of the system. A matrix $B \in \mathbb{R}^{3n \times 6}$ whose columns form a basis for $\mathcal{N}(R)$ is given by

$$B = \begin{bmatrix} I & S(\mathbf{x}_1) \\ I & S(\mathbf{x}_2) \\ \vdots & \vdots \\ I & S(\mathbf{x}_m) \end{bmatrix}, \tag{15}$$

where $S(\cdot)$ is defined by (6).

Let $\mathbf{x}_{i,j} = \mathbf{x}_i - \mathbf{x}_j \in \mathbb{R}^3$ denote the relative information. Since the elements of the rigidity matrix R contain only relative information $\mathbf{x}_{i,j}$, the basis for the null space of R can be represented using relative information only. The expression for such a basis depends on the form of R, *i.e.* of G. Let $A \in \mathbb{R}^{3m \times 6}$ denote a matrix whose (i) first three columns form a block matrix with 3×3 identity blocks, (ii) columns form a basis for $\mathcal{N}(R)$, and (iii) elements are all expressed in terms of relative information. Since the columns of B form a basis for $\mathcal{N}(R)$ for all constraint graphs G, there exists a matrix $T \in \mathbb{R}^{6 \times 6}$ that transform B into A, *i.e.* $A = BT$. The matrix T has full rank and is of the form

$$T = \begin{bmatrix} I & \cdot \\ 0 & \cdot \end{bmatrix}. \tag{16}$$

4.1.1 Example. Let us look at an example. Agents 1, 2, and 3 preserve the distance constraints with each other, and all other agents keep their distance to agents 1, 2, and 3. Figure 1 provides an illustration for the case of $n = 6$. The constraint graph G has $3n - 6$ edges, and the rigidity matrix $R \in \mathbb{R}^{(3n-6) \times 3n}$ has the form of

$$R = \begin{bmatrix} M & 0 & 0 & \cdots & 0 \\ M_4 & X_4 & 0 & \cdots & 0 \\ M_5 & 0 & X_5 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_n & 0 & 0 & \cdots & X_n \end{bmatrix},$$

where $M \in \mathbb{R}^{3 \times 9}$ is defined by (5) and

$$M_i = \begin{bmatrix} \mathbf{x}_{1,i}^\mathsf{T} & \mathbf{0}^\mathsf{T} & \mathbf{0}^\mathsf{T} \\ \mathbf{0}^\mathsf{T} & \mathbf{x}_{2,i}^\mathsf{T} & \mathbf{0}^\mathsf{T} \\ \mathbf{0}^\mathsf{T} & \mathbf{0}^\mathsf{T} & \mathbf{x}_{3,i}^\mathsf{T} \end{bmatrix}, \quad X_i = \begin{bmatrix} \mathbf{x}_{i,1}^\mathsf{T} \\ \mathbf{x}_{i,2}^\mathsf{T} \\ \mathbf{x}_{i,3}^\mathsf{T} \end{bmatrix},$$

where $M_i \in \mathbb{R}^{3 \times 9}$ and $X_i \in \mathbb{R}^{3 \times 3}$.

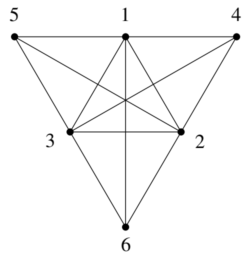

Figure 1. A rigid constraint graph for a system of six agents.

The columns of B in (15) is a basis for $\mathcal{N}(R)$. Another matrix whose columns form a basis for $\mathcal{N}(R)$ and whose elements only contain relative information is

$$A = \begin{bmatrix} I & S(\mathbf{x}_{2,1} + \mathbf{x}_{3,1}) \\ I & S(\mathbf{x}_{1,2} + \mathbf{x}_{3,2}) \\ I & S(\mathbf{x}_{1,3} + \mathbf{x}_{2,3}) \\ I & S(\mathbf{x}_{1,4} + \mathbf{x}_{2,4} + \mathbf{x}_{3,4}) \\ I & S(\mathbf{x}_{1,5} + \mathbf{x}_{2,5} + \mathbf{x}_{3,5}) \\ \vdots & \vdots \\ I & S(\mathbf{x}_{1,n} + \mathbf{x}_{2,n} + \mathbf{x}_{3,n}) \end{bmatrix},$$

where $S(\cdot)$ is defined by (6). The relation $A = BT$, where

$$T = \begin{bmatrix} I & S(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3) \\ 0 & -3I \end{bmatrix}$$

can easily be verified.

4.2 Control Design

In this section we will discuss the design of a control that makes a set of vectors $\{\mathbf{x}_{ij} \in \mathbb{R}^3 \,|\, i, j \in J\} \subset P_{\boldsymbol{\psi}}$, where J is an index set and $P_{\boldsymbol{\psi}}$ is a plane intersecting the origin ($\boldsymbol{\psi}$ will be defined later), converge to the reference plane $P_{\mathbf{n}}$. The plane $P_{\boldsymbol{\psi}}$ represents part of the attitude of the rigid structure formed by the configuration of n agents.

Design the control $\mathbf{u} = [\mathbf{u}_1^\mathsf{T}, \mathbf{u}_2^\mathsf{T}, \cdots, \mathbf{u}_n^\mathsf{T}]^\mathsf{T}$ as

$$\mathbf{u} = A \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix} = B\,T \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix}. \tag{17}$$

where $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ are new input signals. In fact, $\mathbf{v}$ represents the translational velocity of the formation and $\mathbf{w}$ is related to its angular velocity. Note that $\mathbf{u}$ is expressed in terms of A, allowing agent i to calculate its own input signal

$$\mathbf{u}_i = \begin{bmatrix} I & S(\mathbf{x}_i) \end{bmatrix} T \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix}, \quad i \geq 4.$$

To see this the reader may compare $\mathbf{u}_i$ in Example 4.1.1 with $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ in the context of Remark 3.1.1.

Introduce a new input signal $\boldsymbol{\omega} \in \mathbb{R}^3$ as

$$\boldsymbol{\omega} = -\begin{bmatrix} 0 & I \end{bmatrix} T \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix}.$$

Note that due to the form and non-singularity of T (see equation (16)), $\boldsymbol{\omega}$ is uniquely determined by $\mathbf{w}$ and vice versa. Then

$$\begin{aligned} \dot{\mathbf{x}}_{i,j} &= \mathbf{u}_i - \mathbf{u}_j \\ &= \begin{bmatrix} I & S(\mathbf{x}_i) \end{bmatrix} T \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix} - \begin{bmatrix} I & S(\mathbf{x}_j) \end{bmatrix} T \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix} \\ &= \begin{bmatrix} 0 & S(\mathbf{x}_i - \mathbf{x}_j) \end{bmatrix} T \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix} \\ &= -S(\mathbf{x}_i - \mathbf{x}_j)\boldsymbol{\omega} = \boldsymbol{\omega} \times \mathbf{x}_{i,j}, \end{aligned}$$

wherefore the distances between agent i and j are preserved by Lemma 3.1.2.

It remains to design $\boldsymbol{\omega}$ so that the desired orientation is achieved asymptotically. Note that each agent should be able to calculate $\boldsymbol{\omega}$ from its own relative information, and that all agents should arrive at the same result.

4.2.1 Theorem. If each agent has three non-collinear neighbors on $P_{\boldsymbol{\psi}}$ (agents positioned on $P_{\boldsymbol{\psi}}$ need only two neighbors), then an $\boldsymbol{\omega}$ that rotates $P_{\boldsymbol{\psi}}$ into $P_{\mathbf{n}}$ can be designed based on the relative information.

Proof. Design the control $\mathbf{u}$ as done in equation (17). It remains to calculate $\boldsymbol{\omega}$. Suppose agent i has three non-collinear neighbors on $P_{\boldsymbol{\psi}}$, whereby $P_{\boldsymbol{\psi}}$ is uniquely determined. Agent i can then calculate a normal to $P_{\boldsymbol{\psi}}$. Let $\boldsymbol{\psi}$ be such a normal satisfying $\|\boldsymbol{\psi}\| = 1$ and $\gamma = \mathbf{n} \cdot \boldsymbol{\psi} > 0$ (if $\gamma = 0$, then apply $\mathbf{u}$ with some arbitrary but pre-set $\boldsymbol{\omega}$ for a pre-set period of time). All agents calculate the same γ due to the dot product being invariant under coordinate changes. Morever, γ is increasing by the proof of Proposition 3.2.2 wherefore the sign of γ does not change during the evolution of the system. Hence, all agents calculate the same $\boldsymbol{\psi}$. If we set

$$\boldsymbol{\omega} = \alpha\, \boldsymbol{\psi} \times \mathbf{n},$$

then $\boldsymbol{\omega}$ is (i) identical for all i, (ii) based on relative information and (iii) equal to a positive constant times the three agent case input signal given by equation (10). Proposition 3.2.2 then guarantees convergence. □

4.2.2 Remark. Note that by requiring $\gamma > 0$ we impose a limit on the feasible reference attitudes, making Theorem 4.2.1 a more limited result than Proposition 3.2.2 in this respect. This limitation could be removed by assuming the agents to have some additional communication capabilities.

We also present an alternative approach to control design.

4.2.3 Proposition. Assume G is a tree. Consider a system of n agents with constraint graph G. If the control satisfies

$$\mathbf{u}_i - \mathbf{u}_j = \boldsymbol{\omega} \times \mathbf{x}_{i,j} \quad \forall\, (i, j) \in E, \tag{18}$$

then all distance constraints (2) are preserved throughout the evolution of the system.

Proof. For any $i, j \in \{1, 2, \ldots, n\}$, there exists a path $i, i_1, i_2, \cdots, i_m, j$ such that each edge belongs to G, namely

$$\{(i, i_1), (i_k, i_{k+1}), (i_m, j) \in E \,|\, k = 1, \cdots, n\} \subseteq E.$$

Therefore,

$$\begin{aligned} \dot{\mathbf{x}}_{i,j} &= \mathbf{u}_i - \mathbf{u}_j \\ &= \mathbf{u}_i - \mathbf{u}_{i_1} + \mathbf{u}_{i_1} - \mathbf{u}_{i_2} + \ldots + \mathbf{u}_{i_m} - \mathbf{u}_j \\ &= \boldsymbol{\omega} \times \mathbf{x}_{i,i_1} + \boldsymbol{\omega} \times \mathbf{x}_{i_1,i_2} + \ldots + \boldsymbol{\omega} \times \mathbf{x}_{i_m,j} \\ &= \boldsymbol{\omega} \times (\mathbf{x}_{i,i_1} + \mathbf{x}_{i_1,i_2} + \ldots + \mathbf{x}_{i_m,j}) \\ &= \boldsymbol{\omega} \times \mathbf{x}_{i,j} \end{aligned}$$

is feasible and all distance constraints (2) are fulfilled throughout the evolution of the system by Lemma 3.1.2. □

A tree composes a minimal set of edges connecting all vertices. In the following, we therefore assume that G is a connected graph and that $\boldsymbol{u}$ is given by (17), thereby satisfying Proposition 4.2.3.

A tree has $n - 1$ edges, so the number of equations in (18) is $n-1$. Hence there is still some degree of freedom left to specify in $\mathbf{u}$. As done in § 3.1, in the three agents case, we can use this degree of freedom to fix a location $\mathbf{p}$ through which the rotation axis passes by setting

$$\mathbf{p} = \sum_{i=1}^{n} \lambda_i \mathbf{x}_i, \quad \dot{\mathbf{p}} = \sum_{i=1}^{n} \lambda_i \mathbf{u}_i = \mathbf{q}$$

for any constant $\mathbf{q} \in \mathbb{R}^3$ representing the translational velocity of $\mathbf{p}$ if $\sum_{i=1}^{n} \lambda_i \neq 0$. This degree of freedom should be used to find a location for rotation that can be sensed by every agent.

4.2.4 Example. Let G be a star shaped tree with agent 1 as the center (agent 1 may very well be an virtual agent corresponding to a location that every other agent can sense as mentioned above). An illustration for the case of $n = 6$ is provided in Figure 2. According to Proposition 4.2.3, if the control satisfies

$$\mathbf{u}_i - \mathbf{u}_1 = \boldsymbol{\omega} \times \mathbf{x}_{i,1}, \quad \forall i = 1, \cdots, n, \tag{19}$$

then the distances between agents will be maintained at constant levels. Set $\mathbf{u}_1 = 0$ then $\mathbf{u}_i = \boldsymbol{\omega} \times \mathbf{x}_{i,1}$, which is a simpler control structure than that of (17). The design of $\boldsymbol{\omega}$ can be done as in Proposition 4.2.1.

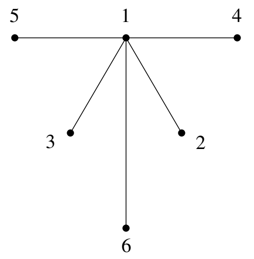

Figure 2. A spanning tree for G in Example 1.

5. SIMULATION RESULTS

The behavior of the system is demonstrated by simulation in MATLAB. The differential equations (11)–(12) for a system of six coplanar agents with $\mathbf{u}_i$ as in Theorem 4.2.1 are solved for $\mathbf{x}_i$ as functions of time using the built-in function `ode45`.

Figure 3 a, b and c display the results from a simulation of the system given in Example 4.1.1 using the technique of setting $\mathbf{u}_1 = 0$. All three images show the same trajectories, but from different viewpoints. We set the number of agents to six, requiring the agents to be able to sense each other as indicated by the graph in Figure 1 and designing the control law (19) from the rigidity graph in Figure 2. Agents 1, 2 and 3 form an equilateral triangle so we have to assume that all agents are able to distinguish between them, see Remark 3.2.1. We can disregard the limitation discussed in Remark 4.2.2 since all agents calculate $\boldsymbol{\omega}$ using their relative position with regard to the same three agents on $P_{\boldsymbol{\psi}}$. The squares (□) in Figure 3 a, b, c and d correspond to initial values while the circles (○) denote final values. The formation can be visualized by drawing lines between squares and circles in a 'connect the dots' fashion.

Figure 3 a and b show that the trajectory of each agent is confined to a plane. The planes of different agents are parallel with each other and with $P_{\mathbf{l}}$, the plane through the origin whose normal is $\mathbf{l}$. This is natural since $\mathbf{l}$ spans the rotation axis of each agent, and since $\mathbf{l}$ is constant in direction which is easily shown,

$$\begin{aligned}\dot{\mathbf{l}} &= \dot{\boldsymbol{\psi}} \times \mathbf{n} = (\boldsymbol{\omega} \times \boldsymbol{\psi}) \times \mathbf{n} \\ &= (\mathbf{n} \cdot \boldsymbol{\omega})\boldsymbol{\psi} - (\mathbf{n} \cdot \boldsymbol{\psi})\boldsymbol{\omega} = -\alpha\gamma\mathbf{l}.\end{aligned}$$

Figure 3 c gives a three-dimensional view of the agents' trajectories. The initial and final attitude of the configuration of agents are clearly distinguishable. Figure 3 d illustrates that the trajectory of the normal is a geodesic on the 2-sphere.

6. CONCLUSIONS

The proposed control law rotates a formation of n agents in $\mathbb{R}^3$ allowing it to reach a pre-set reference attitude asymptotically. The reference is limited to two of the three degrees of freedom that the attitude of a rigid body has. If, for example, the formation should consist of a convex polytope, then the reference attitude limited to requiring that one of the two-dimensional faces is parallel to a reference plane. A in-depth study of the three agent case makes way for a straightforward generalization to the n agents case, assuming the communication graph has a spanning three and that any agent is able to sense at least three other agents on the reference plane. Lyapunov based stability analysis shows that a formation of three agents will converge to the reference plane from any initial condition on the 2-sphere except for an unstable equilibrium point. Simulations in MATLAB illustrates the geometric underpinnings of these ideas.

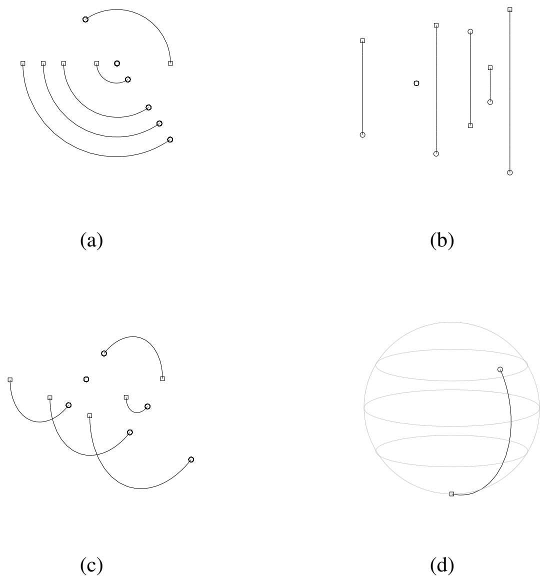

Figure 3. Trajectories of six agents as seen from different viewpoints, a–c. Trajectory of the normal on the 2-sphere, d. Squares (□) denote initial positions, while circles (○) denote final positions.

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