

Robust model predictive control: advantages and disadvantages of tube-based methods^{*}

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Abstract: An important reason for the success of model predictive control is the fact that, for deterministic systems, feedback and open-loop control are equivalent so that ‘optimal’ (but implicit) feedback may be obtained by solving an open-loop optimal control problem at each state encountered. This equivalence is not maintained in the presence of uncertainty complicating the development of robust model predictive control. The advantages and limitations of tube-based model predictive control for dealing with uncertainty of various forms are discussed, firstly in the context of constrained linear systems, and an extension to deal with robustness against unstructured uncertainty is briefly described. Tube-based control requires the determination both of a nominal or reference trajectory and an ancillary controller that constrains deviations of the state of the uncertain systems from the nominal trajectory and is difficult to extend to nonlinear systems. A novel ancillary controller for tube-based control of constrained nonlinear systems with additive disturbances that overcomes some disadvantages of a previous version and is relatively simple to implement is described and assessed; the addition of a terminal equality constraint to the nominal optimal control problem and its removal from the ancillary optimal control problem simplifies the analysis, removes some restrictive assumptions, and enables stronger results to be obtained.

Keywords: Robust Model Predictive Control; Nonlinear Model Predictive Control; Discrete-Time Systems; Tubes; Uncertain Systems.

1. INTRODUCTION

Feedback control is required in most applications and may be achieved, in principle, by dynamic programming as in H_2 and H_∞ optimal control for linear systems. However, dynamic programming is computationally too complex to employ for many control problems. Model predictive control successfully exploits the fact that, for deterministic systems, feedback and open-loop control are equivalent so that optimal (but implicit) feedback may be obtained by solving an open-loop optimal control problem at each state encountered thereby enabling many significant control problems, especially in the process industry, to be solved; control and state constraints and nonlinearity feature in many of these problems. Thus, there exists a fairly mature literature on deterministic model predictive control reviewed in e.g. Mayne et al. (2000).

The addition of uncertainty destroys the equivalence of feedback and open-loop control; there does not exist an open-loop optimal control problem whose solution provides the optimal feedback control at a given state. The powerful theory for deterministic systems no longer applies and ad-hoc solutions have to be sought. One alternative

is to rely on the inherent robustness of deterministic model predictive control, i.e. to solve, as in deterministic model predictive control, the optimal open-loop control problem ignoring the disturbance as in Sokaert et al. (1997). Unfortunately, with this approach, stability relies on Lipschitz continuity of the value function or model predictive control law. As is pointed out in Kellet and Teel (2004), robust stability is obtained if and only if there exists a continuous Lyapunov function; on the other hand, model predictive control employs the value function of the optimal control problem solved online as the Lyapunov function and continuity of the value function can be established in only a few special cases such as linear systems with affine constraints and nonlinear systems with no state or terminal constraints. Alternatively, the optimal open-loop control problem solved online may be modified; merely tightening the constraints does not ensure robust feasibility. Thus, in Michalska and Mayne (1993), a prior predicted control is employed in place of usual predictive control near the tightened constraint boundary and, in Marruedo et al. (2002), modified state constraints in the optimal control problem that become progressively tighter with time are employed. Both methods suffer from the fact that, because of a lack of feedback in the optimal control problem solved online, the spread of the predicted trajec-

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tories may be large, perhaps even rendering the open-loop optimal control problem infeasible.

These difficulties emphasise the desirability of including feedback in the optimal control problem solved online. Ideally, the decision variable at state x for the on-line optimal control problem with horizon N should have the form $\pi(x) = \{u_0, \mu_1(\cdot), \dots, \mu_{N-1}(\cdot)\}$ where, for each i , $\mu_i : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a control law. Such a decision variable is infinite dimensional and therefore impractically complex; indeed, its complexity approaches that of dynamic programming but is mitigated by the fact the domains of the control laws μ_i are restricted to a neighbourhood of the nominal trajectory. The recent literature on robust nonlinear model predictive control often proposes algorithms that have similar complexity since they involve determination of a sequence of control laws or approximations thereto; see, for example Magni et al. (2003) and Limon et al. (2005). Tube-based model predictive control circumvents the complexity problem by replacing $\pi(x)$ by an approximation about a nominal trajectory. When the system is linear, $\mu_i(\cdot)$ has the form $\mu_i(x) = v + K(x - z)$ where x is the current state of the system being controlled, and z and v are the current state and control of the nominal system that generates the reference trajectory. In the nonlinear case, $\mu_i(\cdot)$ necessarily has a more complex form and one purpose of this paper is to present a robust controller for nonlinear systems that has relatively low complexity.

In §2 we summarise the properties of tube-based model predictive control for linear constrained systems, specify some limitations and show how it may be extended to deal with types of uncertainty, such as unmodelled error, not previously handled by this type of robust control. In §3 we analyse the difficulties in extending this method of robust control to constrained non-linear systems and describe a novel method, an extension of a method first described in Mayne and Kerrigan (2007), that overcomes many of these difficulties. We illustrate the performance of the method on a demanding example, discuss its limitations and suggest some desirable extensions. Finally, in §4, we draw some conclusions.

2. CONSTRAINED LINEAR SYSTEMS

The system to be controlled is described

$$x^+ = f(x, u) + w \quad (1)$$

where, in this section, $f(x, u) \triangleq Ax + Bu$ and (A, B) is assumed to be stabilizable. The additive disturbance w lies in the compact, convex set $\mathbb{W}$ that contains the origin. The system is subject to the state and control constraints $x \in \mathbb{X}$ and $u \in \mathbb{U}$ where $\mathbb{X}$ is closed and $\mathbb{U}$ is compact. The control objective is to steer an initial state $x_0 \in \mathbb{X}$ to a neighbourhood of an equilibrium target state x_e satisfying $f(x_e, v_e) = x_e$ for some $v_e \in \mathbb{U}$; the neighbourhood of x_e should be as small as possible. The *nominal* system is

$$z^+ = f(z, v) \quad (2)$$

where, here, $f(z, v) = Az + Bv$. The nominal cost that is minimized by the model predictive controller is

$$\bar{V}_N(z, \mathbf{v}; x_e) \triangleq \sum_{i=0}^{N-1} \ell(z(i) - x_e, v(i) - v_e) + V_f(z(N), x_e) \quad (3)$$

where $\mathbf{v} \triangleq \{v(0), v(1), \dots, v(N-1)\}$ and $z(i)$ is the associated solution of (2) with initial state z . The terminal cost $V_f(z, x_e)$ and terminal constraint set $\mathbb{Z}_f(x_e)$ satisfy the usual stability conditions. The nominal optimal control problem $\bar{\mathbb{P}}_N(z; x_e)$ is minimization of $\bar{V}_N(z, \mathbf{v}; x_e)$ with respect to $\mathbf{v}$ subject to the state, control and terminal constraints $z \in \mathbb{Z}$, $v \in \mathbb{V}$ and $z(N) \in \mathbb{Z}_f(x_e)$ respectively; an appropriate choice for $\mathbb{Z}$ and $\mathbb{V}$ is presented below.

2.1 Prototype tube-based controller

Given (x_0, x_e) , the prototype tube-based controller controls the composite system with state (x, z) as follows. At each state (x, z) , conventional model predictive control is used to control the nominal system yielding the implicit feedback control $v = \bar{\kappa}_N(z)$ with the property that x_e is asymptotically or exponentially stable for the controlled system $z^+ = f(z, \bar{\kappa}_N(z))$. The control u applied to the uncertain system is $u = \kappa_N(x, z) \triangleq \bar{\kappa}_N(z) + K(x - z)$ where K is chosen so that $A_K \triangleq A + BK$ is stable and suitably attenuates the disturbance. It has been shown that, if $x(0) \in z(0) + S_K$, this controller generates trajectories $\{x(i), z(i)\}$ satisfying $z(i) \rightarrow x_e$, $x(i) \in z(i) + S_K$ and $u(i) \in v(i) + K S_K$ where $S_K \triangleq \sum_{i=0}^{\infty} A_K^i \mathbb{W}$ is the minimal robust invariant set for $e^+ = A_K e + w$, $w \in \mathbb{W}$ and is compact; see Kolmanovsky and Gilbert (1998). It is then possible to choose state and control constraint sets, $\mathbb{Z}$ and $\mathbb{V}$ respectively, such that, if the nominal system $z^+ = f(z, \bar{\kappa}_N(z))$ satisfies these constraints, the controlled uncertain system $x^+ = f(x, \kappa_N(x, z)) + w$ satisfies the original state and control constraints $\mathbb{X}$ and $\mathbb{U}$. An important observation is that $\mathbb{Z}$ and $\mathbb{V}$ may be determined without computing S_K ; see Raković et al. (2005) and Rawlings and Mayne (2009). It is possible to tune K to reduce the effect of the disturbance w .

2.2 Improved tube-based controller

A simple modification permits improved performance and rapid adjustment to changes in the target state; at state (x, z) of the composite system, the solution to two nominal optimal control problems, $\bar{\mathbb{P}}_N(x; x'_e)$ and $\bar{\mathbb{P}}_N(z; x'_e)$ where x'_e is the (possibly new) target state and to reset z to x if the value function for $\bar{\mathbb{P}}_N(\cdot)$ satisfies $\bar{V}_N^0(x; x'_e) < \bar{V}_N^0(z; x'_e)$; in this case, the control u applied to the uncertain system is $u = \bar{\kappa}_N(x)$ rather than $u = \bar{\kappa}_N(z) + K(x - z)$. Thus the nominal trajectory responds to the 'environment'.

A further possibility is modification of the nominal optimal control problem $\bar{\mathbb{P}}_N(z; x_e)$ to permit minimization with respect to $(z, \mathbf{v})$ subject to the additional constraint $x \in z + S_K$ on z ; see Mayne et al. (2005). This method of control has theoretical advantages but implementation is more difficult because the set S_K , or an invariant approximation, has to be computed.

2.3 Output tube-based controller

When the state is not accessible it has to be estimated; state estimation error introduces a new kind of uncertainty. However, the state estimate $\hat{x}$, which depends on the available output y , obeys a difference equation similar

to (1). The controllers described in §2.1 and §2.2 may then be employed to control $\hat{x}$ using, of course, the same control for the uncertain system; the state $\hat{x}(i)$ then lies in the set $z(i) + S_K$ where $z(i)$ is the state of the nominal system corresponding to the state estimator and S_K is the associated minimal robust positively invariant set. Since, under reasonable assumptions, $x(i) \in \hat{x}(i) + E$ for some compact computable set E , it is possible to choose the constraints $\mathbb{Z}$ and $\mathbb{V}$ for the nominal optimal control problem to satisfy $\mathbb{Z} + S_K + E \subset \mathbb{X}$ and $\mathbb{V} + KS_K \subset \mathbb{U}$ thereby ensuring that the controlled uncertain system satisfies the original state and control constraints $x \in \mathbb{X}$ and $u \in \mathbb{U}$; see Mayne et al. (2006).

2.4 Offset-free tube-based control

In one important area, that of offset-free control, the tube-based methodology has a disadvantage. Offset-free control ensures that a specified output $r = Hy$ of a deterministic system converges to a set point $\bar{r}$ with zero offset even though the system has an unknown constant input d . This requires the controller/estimator system to include an ‘integrator’ which is achieved by assuming a model with state $\phi \triangleq (x, d)$ of the form:

$$x^+ = Ax + Bu + B_d d + w_x \quad (4)$$

$$d^+ = d + w_d \quad (5)$$

$$y = C_x x + C_d d + v, \quad r = Hy. \quad (6)$$

or, more concisely

$$\phi^+ = \tilde{A}\phi + \tilde{B}u + w \quad (7)$$

$$y = \tilde{C}\phi + v, \quad r = Hy \quad (8)$$

where $\phi \triangleq (x, d)$, $w \triangleq (w_x, w_d)$ and

$$\tilde{A} = \begin{bmatrix} A & B_d \\ 0 & I \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \tilde{C} = [C \quad C_d]. \quad (9)$$

It can be shown that the estimated state $\hat{\phi} \triangleq (\hat{x}, \hat{d})$ satisfies a difference equation of the form

$$\hat{\phi}^+ = \tilde{A}\hat{\phi} + \tilde{B}u + \delta \quad (10)$$

where δ takes values in a compact set; the estimator state $\hat{d}$ satisfies $\hat{d}^+ = \hat{d} + \delta_d$. The associated nominal system is

$$z^+ = \tilde{A}z + \tilde{B}v. \quad (11)$$

Two difficulties, currently unresolved, are apparent. The model (4)–(6), while appropriate for the design of a time-invariant state estimator, is not suitable for control design since the evolution of d is unbounded. Secondly, the pair $(\tilde{A}, \tilde{B})$ is not stabilizable (the subsystem $z_d^+ = z_d$ is obviously uncontrollable) so that the methodology of §2.3 cannot be directly employed.

2.5 Robustness against unmodelled dynamics

This topic has received very little attention in the predictive control literature a notable exception being Løvaas et al. (2008) where relevant literature is referenced. We describe briefly an approach currently being considered. The standard model for robustness against unstructured uncertainty consists of a plant (operator) P with two inputs u and v and two outputs y and z satisfying $u = \kappa y$ and $v = \Delta z$ where κ is the controller and Δ the unmodelled dynamics. For simplicity, we consider the case when

$y = x$ (accessible state). It is usual to require that the operator Δ lies in the class $\mathcal{D}$ of systems with bounded ℓ_2 gain. We suggest, that for constrained control, that the class $\mathcal{D}$ of systems with bounded ℓ_∞ gain is more appropriate. Hence, we assume that each $\Delta \in \mathcal{D}$ satisfies

$$\|\mathbf{v}_t\|_\infty \leq \|\Delta \mathbf{z}_t\|_\infty \leq c_\delta \|\mathbf{z}_t\|_\infty \quad (12)$$

where, here, for each integer t $\mathbf{v}_t \triangleq \{\dots, \nu(t-2), \nu(t-1), \nu(t)\}$, $\|\cdot\|_\infty$ denotes the ℓ_∞ norm and c_δ bounds the induced ℓ_∞ gain of any $\Delta \in \mathcal{D}$. Standard tube-based methodology may then be employed to achieve robustness against unmodelled dynamics with bounded ℓ_∞ gain by removing Δ from the system description, regarding ν as an extra disturbance input with bound $|\nu| \leq c$, and imposing an extra output constraint $|z| \leq c/c_\delta$ (the constraint $|z| \leq c/c_\delta$ ensures $\|\mathbf{v}_t\|_\infty \leq c$ so that $|\nu(t)| \leq c$ for all t); c is a design parameter and $|\cdot|$ denotes the usual Euclidean norm.

3. CONSTRAINED NONLINEAR SYSTEMS

The system to be controlled is described by (1) and the associated nominal system by (2) where, now, $f(\cdot)$ is nonlinear; we assume $f(\cdot)$ and its partial derivatives are Lipschitz continuous. As before, the control objective is to steer any initial state x_0 to a small neighbourhood of the target state x_e .

3.1 Prototype tube-based controller

The prototype controller, as in the linear case, has two components: a reference trajectory, obtained by solving a nominal optimal control problem $\mathbb{P}_N(x_0, x_e)$ and an ancillary controller that maintains trajectories of the uncertain system (1) in a small neighbourhood of the reference trajectory.

Reference trajectory: The nominal optimal control problem $\mathbb{P}_N(x_0, x_e)$ with feasible set $\mathcal{X}(x_e)$ is defined as in §2 with cost $V_N(z, \mathbf{v}; x_e)$ and tightened state and control constraints $z \in \mathbb{Z}$ and $v \in \mathbb{V}$ except that, for reasons given below, $V_f(z, x_e) \equiv 0$ and $Z_f(x_e) = \{x_e\}$. Let $\mathbf{v}^0(x_0; x_e)$ denote the solution of the nominal control problem with initial state $z = x_0$ and let $\mathbf{z}^0(x_0; x_e)$ denote the associated state trajectory. Clearly, $z^0(0; x_0, x_e) = x_0$ and $z^0(N; x_0, x_e) = x_e$; in the absence of the disturbance w , the control sequence $\mathbf{v}^0(x_0; x_e)$ steers x_0 to x_e . The reference state and control trajectories are the infinite sequences $\mathbf{z}^*(x_0, x_e)$ and $\mathbf{v}^*(x_0, x_e)$, respectively, defined by

$$\mathbf{z}^*(x_0, x_e) \triangleq \{\mathbf{z}^0(x_0, x_e), x_e, x_e, \dots\} \quad (13)$$

$$\mathbf{v}^*(x_0, x_e) \triangleq \{\mathbf{v}^0(x_0, x_e), v_e, v_e, \dots\}. \quad (14)$$

The fact that, here, the reference trajectories are constant for times greater than the horizon N enables stronger properties of the controlled system to be established than is the case in Mayne and Kerrigan (2007) and is the main reason for considering the reference trajectory explicitly as a function of time rather than implicitly, as in the linear case, as the solution of $z^+ = f(z, \bar{\kappa}_N(z))$.

Ancillary controller: The crucial difference between the linear and nonlinear case is that in the former, due to linearity, it is possible to choose easily an ancillary controller $u(x, i) = v^*(i; x_0, x_e) + K(x - z^*(i; x_0, x_e))$ that

steers any initial state (x, i) of the *nominal* system to the reference trajectory and, hence, to x_e , exponentially fast and thereby maintains the state of the *uncertain* system in the neighbourhood of the reference trajectory. This is not possible in the nonlinear case; what is required is a nonlinear feedback controller that renders the reference trajectory asymptotically or exponentially stable for the controlled *nominal* system; determination of control laws or policies is usually prohibitively difficult.

This obstacle is easily overcome if we recall that the success of model predictive control is due to the fact that it replaces determination of a (feedback) control law by determination of the solution to an open-loop optimal control problem at each state encountered. Hence, instead of determining a-priori an ancillary control *law*, we determine an *implicit* control law via an ancillary model predictive controller that computes a control *action* at each state encountered. The ancillary model predictive controller solves, at each state x of the uncertain system that is encountered, an open-loop optimal control problem in which the system is the nominal system and the cost is a measure of the deviation from the reference trajectories $z^*(x_0, x_e)$ and $v^*(x_0, x_e)$. Compared with the approach in Mayne and Kerrigan (2007), the ancillary optimal control problem used here has no terminal constraint permitting stronger stability results to be obtained. The ancillary optimal control problem is therefore time-varying so we use t to denote current time. The optimal control problem $\mathbb{P}_N(x, t; x_0, x_e)$ solved by the ancillary controller at (x, t) (i.e. at state x , time t) is minimization of the cost $V_N(x, t, \mathbf{u}; x_0, x_e)$ defined by:

$$V_N(x, t, \mathbf{u}; x_0, x_e) \triangleq \sum_{i=0}^{N-1} \ell(x(i) - z^*(t+i; x_0, x_e), u(i) - v^*(t+i; x_0, x_e)) + \beta V_f(x(N); x_e) \quad (15)$$

where, for each i , $x(i) \triangleq \bar{\phi}(i; x, \mathbf{u})$ the solution of $x^+ = f(x, u)$ if the initial state is x at time 0 (or the solution at time $t+i$ if the initial state is x at time t) and the control sequence is $\mathbf{u} \triangleq \{u(0), u(1), \dots, u(N-1)\}$. Problem $\mathbb{P}_N(x, t; x_0, x_e)$ is defined by

$$\mathbb{P}_N(x, t; x_0, x_e) : \quad V_N^0(x, t; x_0, x_e) = \min_{\mathbf{u}} \{V_N(x, t, \mathbf{u}; x_0, x_e) \mid \mathbf{u} \in \mathbb{U}^N\} \quad (16)$$

The minimizer of $\mathbb{P}_N(x, t; x_0, x_e)$ is $\mathbf{u}^0(x, t; x_0, x_e)$ and the control $\kappa_N(x, t; x_0, x_e)$ applied to the system at (x, t) is the first element of this N-sequence:

$$\kappa_N(x, t; x_0, x_e) = u^0(0; x, t, x_0, x_e). \quad (17)$$

The corresponding state sequence is $\mathbf{x}^0(x, t; x_0, x_e)$. An important feature of $\mathbb{P}_N(x, t; x_0, x_e)$ is that it has no state constraints. If $x = z^*(t; x_0, x_e)$, then $V_N^0(x, t; x_0, x_e) = 0$ and the optimal trajectory coincides with the reference trajectory. The sampling period and stage cost $\ell(\cdot)$ may differ from that for the reference trajectory.

3.2 Analysis

Let $\mathbb{M}$ denote the compact set of initial and target states such that $\mathbb{P}_N(x_0, x_e)$ is feasible for each $(x_0, x_e) \in \mathbb{M}$. We assume that $f(\cdot)$ and $f_{(x,u)}(\cdot)$ are Lipschitz continuous and that $\ell(x, u) \triangleq |x|_Q^2 + |u|_R^2$ where Q and R are

positive definite. As shown in (Rawlings and Mayne, 2009, page 137), for each admissible x_e there exists a pair $(V_f(\cdot; x_e), \mathbb{X}_f(x_e))$ that satisfies the usual stability axiom: $\min_u \{V_f(f(x, u); x_e) + \ell(x - x_e, u - v_e)\} \mid u \in \mathbb{U}, f(x, u) \in \mathbb{X}_f(x_e)\} \leq V_f(x; x_e)$ for all $x \in \mathbb{X}_f(x_e)$; $V_f(x; x_e) = x'Q_f(x_e)x$ where $Q_f(x_e)$ is positive definite.

The first problem we have to address is that $\mathbb{P}_N(x, t; x_0, x_e)$ has no terminal constraint. Let

$$\mathbb{Z}_{\bar{V}}(t; x_0, x_e) \triangleq \{(x, \mathbf{u}) \in \mathbb{R}^n \times \mathbb{U}^N \mid V_N(x, t, \mathbf{u}; x_0, x_e) \leq \bar{V}, \bar{\phi}(N; x, \mathbf{u}) \in \mathbb{X}_f(x_e)\} \quad (18)$$

$$\mathbb{X}_{\bar{V}}(t; x_0, x_e) \triangleq \{x \in \mathbb{R}^n \mid \exists (x, \mathbf{u}) \in \mathbb{Z}_{\bar{V}}(t; x_0, x_e)\}. \quad (19)$$

Proposition 1. (Stewart et al. (2010)). For all $\bar{V} > 0$, all $(x_0, x_e) \in \mathbb{M}$, there exists a $\beta > 0$ such that, for all $x \in \mathbb{X}_{\bar{V}}(t; x_0, x_e)$, all $t \in \mathbb{I}_{\geq 0} \triangleq \{0, 1, 2, \dots\}$, the terminal state $x^0(N; x, t, x_0, x_e)$ lies in $\mathbb{X}_f(x_e) \subseteq \mathbb{X}_{\bar{V}}(x_0, x_e)$.

It follows that the terminal constraint is *automatically satisfied* by solutions of $\mathbb{P}_N(x, t; x_0, x_e)$ if $x \in \mathbb{X}_{\bar{V}}(x_0, x_e)$ and $\beta \geq \beta$. We assume in the sequel that $\beta \geq \max\{\beta, 1\}$.

Proposition 2. For all $(x_0, x_e) \in \mathbb{M}$, there exist constants $c_2 > c_1 > 0$ such that $V_N^0(\cdot; x_0, x_e)$ satisfies:

$$c_1 |x - z^*(t; x_0, x_e)|^2 \leq V_N^0(x, t; x_0, x_e) \quad \forall x \in \mathbb{R}^n$$

$$V_N^0(x, t; x_0, x_e) \leq c_2 |x - z^*(t; x_0, x_e)|^2 \quad \forall x \in \mathbb{X}_{\bar{V}}^\dagger(t; x_0, x_e)$$

$$V_N^0(x^+, t^+; x_0, x_e) \leq \gamma V_N^0(x, t; x_0, x_e) \quad \forall x \in \mathbb{X}_{\bar{V}}(t; x_0, x_e)$$

for all $t \in \mathbb{I}_{\geq 0}$ where $\gamma \triangleq 1 - c_1/c_2 \in (0, 1)$, $x^+ \triangleq f(x, \kappa_N(x, t; x_0, x_e))$, $t^+ \triangleq t + 1$ and $\mathbb{X}_{\bar{V}}^\dagger(t; x_0, x_e) \triangleq \mathbb{X}_{\bar{V}}(t; x_0, x_e) \cup \mathbb{W}$.

Since $\mathbb{P}_N(x, t; x_0, x_e)$, unlike the situation in Mayne and Kerrigan (2007), has no state constraints, the value function $V_N^0(\cdot)$ has an important property:

Proposition 3. The value function $V_N^0(\cdot)$ is uniformly Lipschitz continuous in x , i.e. for all $(x_0, x_e) \in \mathbb{M}$ there exists a constant c_3 such that $|V_N^0(x + w, t; x_0, x_e) - V_N^0(x, t; x_0, x_e)| \leq c_3 |w|$ for all $w \in \mathbb{W}$, all $(x, t, x_0, x_e) \in \mathbb{X} \times \mathbb{I}_{\geq 0} \times \mathbb{M}$.

A crucial property of the tube-based control system can now be established. For all $d > 0$, let $\mathbb{S}_d(t; x_0, x_e) \triangleq \{x \mid V_N^0(x, t; x_0, x_e) \leq d\}$ denote a level set of $V_N^0(\cdot)$.

Proposition 4. There exists a $d^* \triangleq c_3 |\mathbb{W}| / (1 - \gamma)$ such that, for all $d \in [d^*, \bar{V}]$, $x \in \mathbb{S}_d(t; x_0, x_e)$ implies $x^+ \in \mathbb{S}_d(t^+; x_0, x_e)$, $x^+ \triangleq f(x, \kappa_N(x, t; x_0, x_e)) + w$, for all $w \in \mathbb{W}$ with $|\mathbb{W}| \triangleq \max_w \{|w| \mid w \in \mathbb{W}\}$.

The set $\mathbb{S}_d(t; x_0, x_e)$ is the analog of the set $z^* + S_K$ in the linear case. We assume in the sequel that $d \geq d^*$. Proposition 4 ensures that all state trajectories of the uncertain system commencing at (x, t) lying in the state tube $\mathbb{T}_x(x_0, x_e) \triangleq \{\mathbb{S}_d(t; x_0, x_e) \mid t \in \mathbb{I}_{\geq 0}\}$ remain in this tube and the corresponding control trajectories lie in the control tube $\mathbb{T}_u(x_0, x_e) \triangleq \{\kappa_N(\mathbb{S}_d(t; x_0, x_e), t; x_0, x_e) \mid t \in \mathbb{I}_{\geq 0}\}$ and, therefore, remain, respectively, in bounded neighbourhoods of the reference trajectories $\mathbf{z}^*(x_0, x_e)$ and $\mathbf{v}^*(x_0, x_e)$. Since the reference trajectories are constant for $t \geq N$, all state trajectories of the uncertain system lie in $\mathbb{S}_d(N; x_0, x_e)$ for all $t \geq N$.

Because the state and control tubes are bounded neighbourhoods of the reference trajectories we can, analogously

with the linear case, make the following additional assumption that is reasonable if $\mathbb{W}$ is sufficiently small:

Assumption 1. There exist non-empty sets $\mathbb{Z}$ and $\mathbb{V}$ such that, for all $(x_0, x_e) \in \mathbb{M}$, the state and control tubes $\mathbb{T}_x(x_0, x_e)$ and $\mathbb{T}_u(x_0, x_e)$ lie inside the original constraint sets $\mathbb{X}$ and $\mathbb{U}$ respectively.

Because $z^*(t; x_0, x_e) = x_e$ for all $t \geq N$, this result can be strengthened. The value function $V_N^0(\cdot)$ and associated control law $\kappa_N(\cdot)$ are time invariant for all $t \geq N$. Hence the value function $V_N^0(\cdot)$, the associated control law $\kappa_N(\cdot)$ and associated level set $\mathbb{S}_d(\cdot)$ are all independent of both t and x_0 for $t \geq N$ and may be replaced by

$$\begin{aligned} V_N^{x_e}(x) &\triangleq V_N^0(x, N; x_0, x_e), \quad \mathbb{S}_d^{x_e} \triangleq \{x \mid V_N^{x_e}(x) \leq d\} \\ \kappa_N^{x_e}(x) &\triangleq \kappa_N(x, N, x_0, x_e). \end{aligned} \quad (20)$$

We can now state our main result:

Theorem 1. (i) For each $(x_0, x_e) \in \mathbb{M}$, every solution $\{x(t) \mid t \in \mathbb{I}_{\geq 0}\}$ of the controlled uncertain system $x^+ = f(x, \kappa_N(x, t; x_0, x_e)) + w$, with initial state $x(0) = x_0$, lies in the state tube $\mathbb{T}_x(x_0, x_e)$ while the control sequence $\{u(t) = \kappa_N(x, t; x_0, x_e) \mid t \in \mathbb{I}_{\geq 0}\}$ lies in the control tube $\mathbb{T}_u(x_0, x_e)$ so that the original state and control constraints $x(t) \in \mathbb{X}$ and $u(t) \in \mathbb{U}$ are satisfied for all $t \in \mathbb{I}_{\geq 0}$. The state $x(t)$ converges, in finite time N , to the set $\mathbb{S}_d^{x_e}$ and remains in this set thereafter.

(ii) Let $c_3^e \leq c_3$ denote the Lipschitz constant of $V_N^{x_e}(\cdot)$ on $\mathbb{S}_d^{x_e}$. If $c_3^e < c_3$, there exists a $d^\dagger < d^* < d$ such that every solution of $x^+ = f(x, \kappa_N^{x_e}(x)) + w$, $w \in \mathbb{W}$ with initial state in $\mathbb{S}_d^{x_e}$ converges robustly to $\mathbb{S}_{d^\dagger}^{x_e} \subset \mathbb{S}_d^{x_e}$ in finite time and remains in this set thereafter.

Thus solutions of the controlled uncertain system converge robustly to $\mathbb{S}_{d^\dagger}^{x_e}$ in finite time.

3.3 Illustrative example

The example is a stirred tank reactor described by

$$\dot{x}_1 = (1/\theta)(1 - x_1) - kx_1e^{-(M/x_2)}$$

$$\dot{x}_2 = (1/\theta)(x_f - x_2) + kx_1e^{-(M/x_2)} - \alpha u(x_2 - x_c) + w$$

The model parameters are $\theta = 20$, $k = 300$, $M = 5$, $x_f = 0.3947$, $x_c = 0.3816$ and $\alpha = 0.117$. The state, control and disturbance constraint sets are

$$\mathbb{X} = \{x \in \mathbb{R}^2 \mid x_1 \in [0, 1], x_2 \in [0, 1]\},$$

$$\mathbb{U} = [0, 2], \quad \mathbb{W} = [-0.001, 0.001].$$

The controller is required to steer the system from a locally stable steady state $x(0) = (0.9831, 0.3918)$ to, or close to, a locally unstable steady state $z_e = (0.2632, 0.6519)$. The control problem is demanding because the linearised system is stable at the initial state but unstable at the final state. The controller uses $N = 140$ and a prediction horizon $T = 420s$ corresponding to a sampling period Δ of 3 seconds. The control v_e satisfies $z_e = f(z_e, v_e)$, i.e. (z_e, v_e) is an equilibrium pair. The constraint sets for the central path optimal control problem are $\mathbb{Z} = \mathbb{X}$, $\mathbb{V} = \{v \in \mathbb{R}^m \mid v \in [0.12, 1.99]\}$ and $Z_f = \{z_e\}$; state constraints are not activated so there is no need to tighten $\mathbb{X}$ but increasing the lower limit for v from 0 to 0.12 provides leeway for the ancillary controller to counteract the disturbance but reduces the speed of response. The disturbance $w(t) = A \sin(\omega t)$ where A and ω are uniformly distributed random variables lying, respectively, in

the sets $[0, 0.001]$ and $[0, 1]$. For comparison, a standard model predictive controller is determined using the same stage cost $\ell(\cdot)$, the same terminal constraint, the same sampling period and the same horizon as used for the central path controller. The ancillary controller may be tuned by choosing its stage cost to reject more effectively external disturbances. Figure 1 compares the standard and tube-based model predictive controllers for the case when the nominal and standard controllers employ the stage cost $\ell(x, u) \triangleq 0.5|x|^2 + 5|u|^2$ whereas the ancillary control is obtained using the more aggressive stage cost $\ell_a(x, u) \triangleq 50|x|^2 + 0.5|u|^2$. The spread of trajectories is considerably less with the tube-based controller both during the transient and steady state phases. The tube-based controller is somewhat slower because the lower limit of 0.12 for v restricts the speed of the central path controller; in the absence of disturbances a lower limit of 0 may be used. The ancillary controller may also be tuned by

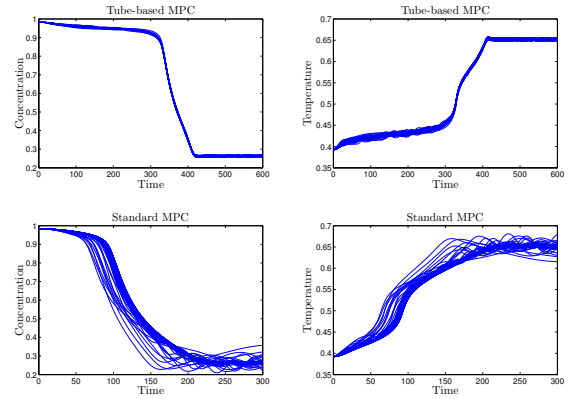


Fig. 1. Top: Tube-based MPC; Bottom: Standard MPC

increasing its sampling frequency enabling it to decrease the effect of disturbances with higher frequencies. Figure 2 shows the variation of concentration with time t when the disturbance is $w(t) = 0.002 \sin(0.4t)$, the sampling period for the reference trajectories is $\Delta = 12s$ and the sampling period for the ancillary controller is 12s, 8s and 4s; the prediction horizon is 480s. For the reference trajectory, the stage cost is $\ell(x, u) \triangleq 0.5|x|^2 + 5|u|^2$ whereas for the ancillary controller the stage cost is $\ell_a(x, u) \triangleq 50|x|^2 + 0.5|u|^2$. The effectiveness of the ancillary controller in reducing the disturbance is increased as its sampling period is reduced from 12s to 4s.

4. CONCLUSION

Tube based controllers address many problems in robust control of constrained linear systems; the reference trajectory responds to target changes and the ancillary controller to disturbances. There are unresolved problems in offset free model predictive control and robustness against unmodelled dynamics. The robust nonlinear controller can be employed for constrained linear systems, an advantage being maintenance of model predictive control once the target state is reached by the reference trajectory. The reference trajectory may be seen as a transformation of the original optimal control problem permitting employment of a simpler form of model predictive control

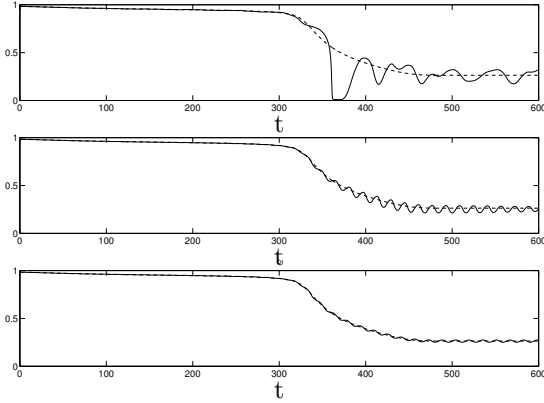


Fig. 2. Concentration versus time with sampling period for ancillary controller of 12s (top), 8s (middle) and 4s (bottom)

that ignores state constraints. For constrained nonlinear systems, the novel extension of tube based control has the advantage, compared with nominal model predictive control, of constraining the spread of trajectories of the controlled uncertain system; this is especially important if the open-loop system is unstable. Trajectories of the uncertain systems are bounded by a level set of a value function that varies in a complex way with (x, t) making estimation of suitable constraint sets for determination of the reference trajectories more difficult than in the linear case.

Appendix A. OUTLINE OF PROOFS

Proposition 1: This result follows from Proposition 1 in Stewart et al. (2010).

Proposition 2: From Proposition 1, the terminal state $x^0(N; x, t, x_0, x_e)$ lies in $X_f(x_e)$. Since $(V_f(\cdot; x_e), \mathbb{X}_f(x_e))$ satisfies the usual stability axiom, the lower bound for $V_N^0(x, t; x_0, x_e)$ and the upper bound for $V_N^0(x^+, t^+; x_0, x_e)$ follow by the usual arguments as does the upper bound for $V_N^0(x, t; x_0, x_e)$ in the set $\mathbb{X}_f$; this bound can be extended to $\mathbb{X}_V^+(t; x_0, x_e)$ as shown in Proposition 2 in *Notes on recent MPC literature* (webpage for Rawlings and Mayne (2009), url: www.nobhillpublishing.com/mpc/).

Proposition 3: By virtue of our assumptions on $f(\cdot)$, $\ell(\cdot)$ and $V_f(\cdot; x_e)$, it follows that $V_N(\cdot; x_0, x_e)$ is Lipschitz continuous on $\mathbb{X} \times \mathbb{U}^N \times \mathbb{I}_{0:N} \times \mathbb{M}$ with Lipschitz constant c_3 . Proposition 3 then follows from Theorem C.29 in Rawlings and Mayne (2009).

Proposition 4: From Propositions 2 and 3, $V_N^0(x^+, t^+; x_0, x_e) \leq \gamma V_N^0(x, t; x_0, x_e) + c_3|\mathbb{W}|$, $x^+ = f(x, \kappa_N(x, t; x_0, x_e)) + w$, $w \in \mathbb{W}$ for all $(x, t; x_0, x_e)$ such that $x \in \mathbb{X}_V^+(t; x_0, x_e)$, $t \in \mathbb{I}_{0:N}$ and $(x_0, x_e) \in \mathbb{M}$. Hence $V_N^0(x, t; x_0, x_e) \leq d$ implies $V_N^0(x^+, t^+; x_0, x_e) \leq d$ if $d \geq d^* \triangleq c_3|\mathbb{W}|/(1 - \gamma)$.

Theorem 1: (i) is a direct consequence of Proposition 4 and the fact that the set $\mathbb{S}_d(t; x_0, x_e)$ has the constant value $\mathbb{S}_d^{x_e}$ for all $t \geq N$. (ii) $V_N^{x_e}(x^+) \leq \gamma V_N^{x_e}(x) + c_3^e|\mathbb{W}|$ for all $x^+ \in F(x) \triangleq f(x, \kappa_N^e(x), \mathbb{W})$ if $x \in \mathbb{S}_d^{x_e}$. Thus, for all $x^+ \in F(x)$, $V_N^{x_e}(x^+) \leq V_N^{x_e}(x) - \varepsilon$ if $x \in \mathbb{S}_d^{x_e} \setminus \mathbb{S}_{d^*}^{x_e}$ and

$V_N^{x_e}(x^+) \leq d^*$ if $x \in \mathbb{S}_{d^*}^{x_e}$ where $d^* \triangleq [c_3^e|\mathbb{W}| + \varepsilon]/(1 - \gamma)$; $d^* < d^*$ if $\varepsilon > 0$ is sufficiently small.

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