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Trajectory Tracking for Nonholonomic Mobile Robots based on Extended Models ⋆

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Abstract: This paper offers a novel approach to circumvent some problems derived from the nonholonomic structure of a two-wheeled mobile robot. It is shown that suitably extending a well-known third-order error model, new kinematics are found that allow direct Lyapunov-based control design as well as straightforward PDC control for its Takagi-Sugeno form. In the latter case, controllability issues are overcome thanks to the proposed design which decouples the uncontrollable mode from the rest of the system. Illustrative examples are provided that show the effectiveness of this technique.

Keywords: Mobile robots, Lyapunov methods, Fuzzy control, Nonholonomic models.

1. INTRODUCTION

The problem of nonholonomic systems control has attracted numerous researchers in the past. A thoroughly studied case with great practical significance is wheeled-mobile robot with kinematic model similar to a unicycle, see Kolmanovsky and McClamroch (1995). The differentially driven mobile robots that are very common in practical applications also have the same kinematic model. Although many researchers coped with the more difficult problem of stabilizing dynamic models for different types of mobile robots, see e.g. Jiang and Nijmeijer (1997) and Pourboghrat and Karlsson (2002), the basic limitations of mobile-robot control still come from their kinematic model, as shown in Brockett (1983), Morin and Samson (2009), and Lizarraga (2004). Kinematic control laws are also very important from the practical point of view, since the wheel-velocity control is often implemented locally on simple, micro-controller-based hardware, while the velocity command comes from high-level hardware that also provides the current control objective.

Traditionally, the problem of mobile-robot control has been approached by point stabilization as in Pourboghrat (2002) or by redefining the problem as a tracking-control one as in Kanayama et al. (1988). We believe that the tracking-control approach is somewhat more appropriate, since the nonholonomic constraints and other control goals (obstacle avoidance, minimum travel time, minimum fuel consumption) are implicitly included in the path-planning procedure, see LaValle (2006), Lepetić et al. (2003), Pozna et al. (2009). It is also easier to extend this approach to more complex schemes such as the control of mobile robot platoons as in Klančar et al. (2009). Many control algorithms were proposed in the path-tracking framework. PID controller is used in Kanayama et al. (1990), Lyapunov-based nonlinear controllers in Oriolo et al. (2002), adaptive controllers in Jiang and Nijmeijer (1997), model-based predictive controllers in Klančar and Škrjanc (2007), fuzzy controllers in Guechi et al. (2010) and Precup and Hellendoorn (2011), etc. Some approaches only guarantee local stability, while others also ensure global stability and global convergence under certain assumptions.

It is very important to find a (kinematic) control law that produces a smooth control signal. If this is not the case, the implementation on the dynamic model becomes impossible. Unfortunately, due to a discontinuity in the orientation error of $\pm 180^\circ$, quite often there is also a discontinuity in the angular-velocity command. This comes from the fact that the classical kinematic model is continuous with respect to orientation (there are no jumps at $\pm\pi$), while in implementation the orientation is often mapped to the $(-\pi, \pi]$ interval. In this paper a novel kinematic model is proposed that overcomes this difficulty, although it is of a higher order. Two control laws that achieve global asymptotic convergence to a predesigned path under some mild conditions are also proposed in this paper.

The problem statement is given in Section 2, the development of the fourth-order error model is given in Section 3. The Lyapunov and the TS control approaches are described in Sections 4 and 5, respectively. The conclusions are stated in Section 6.

2. PROBLEM STATEMENT

Assume a two-wheeled, differentially driven mobile robot as the one depicted in Fig. 1 where (x, y) is the wheel-axis-centre position and θ is the robot orientation. The kinematic motion equations of the mobile robot are equivalent to those of a unicycle. Robots with such an architecture have a nonholonomic constraint of the form:

$$\begin{bmatrix} -\sin\theta(t) & \cos\theta(t) \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = 0 \tag{1}$$

resulting from the assumption that the robot cannot slip in the lateral direction. Only the first-order kinematic model of the system will be treated in this paper:

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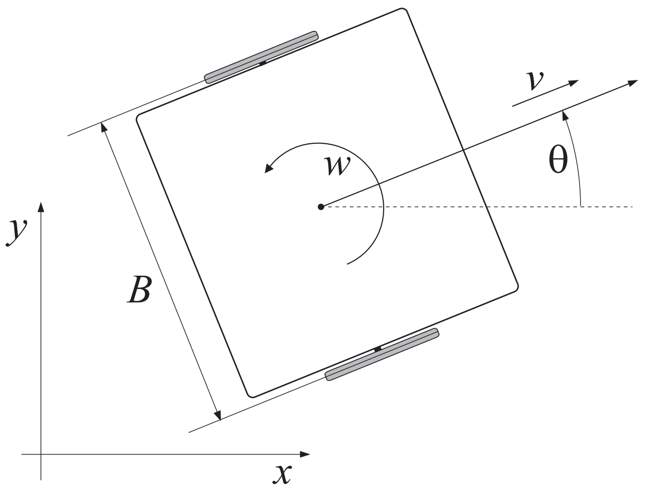

Fig. 1. Two-wheeled, differentially driven mobile robot

$$\dot{q} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} \tag{2}$$

where v and w are the tangential and the angular velocities, respectively, while $q^T(t) = [\, x(t)\ y(t)\ \theta(t) \,]$ is the vector of generalized coordinates. The control design goal is to follow the virtual robot or the reference trajectory, defined by

$$q_r^T(t) = [\, x_r(t)\ y_r(t)\ \theta_r(t) \,] \tag{3}$$

where $q_r(t)$ is a-priori known and smooth. It is very easy to show that the system (2) is flat with flat outputs being x and y, see Fliess et al. (1995). Consequently, (3) can be produced by uniformly continuous control inputs $v_r(t)$ and $w_r(t)$ in the absence of initial conditions, parasitic dynamics and external disturbances. The goal is to design a feedback controller to achieve the tracking and the tracking should be asymptotic under the persistency of excitation (PE) through $v_r(t)$ or $w_r(t)$.

The posture error is not given in the global coordinate system, but rather as an error in the local coordinate system of the robot: e_x gives the error in the direction of driving, e_y gives the error in the lateral direction, and e_θ gives the error in the orientation. The posture error $e = [\, e_x\ e_y\ e_\theta \,]^T$ is determined using the actual posture $q = [\, x\ y\ \theta \,]^T$ and the reference posture $q_r = [\, x_r\ y_r\ \theta_r \,]^T$:

$$\begin{bmatrix} e_x \\ e_y \\ e_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} (q_r - q) \tag{4}$$

From (2) and (4) and assuming that the virtual robot has a kinematic model similar to (2), the posture error model can be written as follows:

$$\begin{bmatrix} \dot{e}_x \\ \dot{e}_y \\ \dot{e}_\theta \end{bmatrix} = \begin{bmatrix} \cos e_\theta & 0 \\ \sin e_\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_r \\ w_r \end{bmatrix} + \begin{bmatrix} -1 & e_y \\ 0 & -e_x \\ 0 & -1 \end{bmatrix} u \tag{5}$$

The transformation (4) is theoretically imposed by the group operation noting that the model (2) is a system on the Lie group SE(2), see Morin and Samson (2009). The approach itself was adopted in Kanayama et al. (1988) where the authors also proposed the PID control for the stabilization of the robot at the reference posture. Later, many authors used the error model (5) for the tracking controller design.

Very often the following control u is used to solve the tracking problem, see Kanayama et al. (1990):

$$u = \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} v_r \cos e_\theta + v_b \\ w_r + w_b \end{bmatrix} \tag{6}$$

Inserting the control (6) into (5), the resulting model is given by:

$$\begin{aligned} \dot{e}_x &= w_r e_y - v_b + e_y w_b \\ \dot{e}_y &= -w_r e_x + v_r \sin e_\theta - e_x w_b \\ \dot{e}_\theta &= -w_b \end{aligned} \tag{7}$$

where $u_b^T = [\, v_b\ w_b \,]$ is the feedback signal to be determined later.

3. FOURTH-ORDER ERROR MODEL OF THE SYSTEM

The problem of using the third-order error model presented in the previous section is that the transformation between the robot posture and the error model is not bijective. This can be observed from the fact that any error-state $[\, 0\ 0\ 2k\pi \,]^T$ $(k \in Z)$ corresponds to the same robot posture. This should be somehow reflected in the kinematic model of the system and also in the error model of the system. This can be achieved by increasing the order of the system to 4. The variable $\theta(t)$ from the original kinematic model (2) is exchanged by two new periodic variables $s(t) = \sin(\theta(t))$ and $c(t) = \cos(\theta(t))$. Their derivatives are:

$$\begin{aligned} \dot{s}(t) &= \cos(\theta(t))\dot{\theta}(t) = c(t)w(t) \\ \dot{c}(t) &= -\sin(\theta(t))\dot{\theta}(t) = -s(t)w(t) \end{aligned} \tag{8}$$

The new kinematic model is then obtained:

$$\dot{q} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{s} \\ \dot{c} \end{bmatrix} = \begin{bmatrix} c & 0 \\ s & 0 \\ 0 & c \\ 0 & -s \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} \tag{9}$$

The new error states are defined as:

$$\begin{aligned} e_x &= c(x_r - x) + s(y_r - y) \\ e_y &= -s(x_r - x) + c(y_r - y) \\ e_s &= \sin(\theta_r - \theta) = s_r c - c_r s \\ e_c &= \cos(\theta_r - \theta) = c_r c + s_r s \end{aligned} \tag{10}$$

After derivation of Eq. (10) and some manipulations we obtain the error model of the system:

$$\begin{aligned} \dot{e}_x &= v_r e_c - v + e_y w \\ \dot{e}_y &= v_r e_s - e_x w \\ \dot{e}_s &= w_r e_c - e_c w \\ \dot{e}_c &= -w_r e_s + e_s w \end{aligned} \tag{11}$$

or in the equivalent matrix form

$$\begin{bmatrix} \dot{e}_x \\ \dot{e}_y \\ \dot{e}_s \\ \dot{e}_c \end{bmatrix} = \begin{bmatrix} e_c & 0 \\ e_s & 0 \\ 0 & e_c \\ 0 & -e_s \end{bmatrix} \begin{bmatrix} v_r \\ w_r \end{bmatrix} + \begin{bmatrix} -1 & e_y \\ 0 & -e_x \\ 0 & -e_c \\ 0 & e_s \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} \tag{12}$$

4. LYAPUNOV-BASED CONTROL DESIGN

4.1 Main result

An exponentially stable controller will be developed based on Lyapunov approach. The states e_x, e_y, and e_s should be driven towards 0, while e_c should converge to 1 if we want to achieve perfect tracking. The following Lyapunov function candidate is proposed to achieve this goal:

$$V = \frac{k_x}{2} e_x^2 + \frac{k_y}{2} e_y^2 + \frac{k_s}{2} e_s^2 + \frac{k_c}{2} (e_c - 1)^2 \tag{13}$$

where k_x, k_y, k_s, and k_c are positive constants. Its derivative is:

$$\begin{aligned}\dot{V} = k_x e_x(v_r e_c - v + e_y w) + k_y e_y(v_r e_s - e_x w) + \\ + k_s e_s(w_r e_c - e_c w) + k_c e_c(-w_r e_s + e_s w) - \\ - k_c(-w_r e_s + e_s w)\end{aligned} \quad (14)$$

After simple analysis it is obvious that $k_x = k_y = k$ and $k_s = k_c$ should be used where the latter two constants can be set to 1 without loss of generality. Taking this into account, many terms in Eq. (13) cancel:

$$\dot{V} = k e_x(v_r e_c - v) + e_s(k e_y v_r + w_r - w) \quad (15)$$

Since the values in the parentheses in Eq. (15) should be chosen to make the derivative of the Lyapunov function negative semi-definite, the following control law is proposed with design parameters $\alpha_x > 0$, $\alpha_s > 0$:

$$\begin{aligned} v &= v_r e_c + \alpha_x e_x \\ w &= w_r + k e_y v_r + \alpha_s e_s \end{aligned} \quad (16)$$

By taking into account the control law Eq. (16), the function $\dot{V}$ becomes:

$$\dot{V} = -k\alpha_x e_x^2 - \alpha_s e_s^2 \quad (17)$$

Note that the control law (16) is the same as the one proposed in Kanayama et al. (1990).

Two very well-known lemmas will be used in the proof of a theorem in this section. The first one is Barbălat's lemma and the other one is a derivation of Barbălat's lemma. Both lemmas are taken from Ioannou and Sun (1996) and are given below for the sake of completeness.

Lemma 1. (Barbălat's lemma). If $\lim_{t\to\infty} \int_0^t f(\tau)d\tau$ exists and is finite, and $f(t)$ is a uniformly continuous function, then $\lim_{t\to\infty} f(t) = 0$.

Lemma 2. If $f, \dot{f} \in \mathcal{L}_\infty$ and $f \in \mathcal{L}_p$ for some $p \in [1, \infty)$, then $f(t) \to 0$ as $t \to \infty$.

Theorem 3. If the control law (16) is applied to the system where k is a positive constant, α_x and α_s are positive bounded functions, the reference velocities v_r and w_r are bounded, then the tracking errors e_x and e_s converge to 0. The convergence of e_y to 0 is guaranteed provided that at least one of the two conditions is met:

(1) v_r is uniformly continuous and does not go to 0 as $t \to \infty$ while α_s is uniformly continuous,
(2) w_r is uniformly continuous and does not go to 0 as $t \to \infty$ while v_r, α_x, and α_s are uniformly continuous.

Proof:

It follows from (17) that $\dot{V} \leq 0$, and therefore the Lyapunov function is non-increasing and thus has the limit $\lim_{t\to\infty} V(t)$. Consequently, the following can be concluded:

$$e_x, e_y, e_s, e_c \in \mathcal{L}_\infty \quad (18)$$

Based on (18), it follows from (16) that the control signals are bounded, and from (11) that the derivatives of the errors are bounded:

$$v, w, \dot{e}_x, \dot{e}_y, \dot{e}_s, \dot{e}_c \in \mathcal{L}_\infty \quad (19)$$

where we also took into account that v_r, w_r, k, α_x, and α_s are bounded. It follows from Eqs. (18) and (19) that e_x, e_y, e_s, and e_c are uniformly continuous (note that the easiest way to check the uniform continuity of $f(t)$ on $[0, \infty)$ is to see if $f, \dot{f} \in \mathcal{L}_\infty$).

In order to show the asymptotic stability of the system, let us first calculate the following integral:

$$\int_0^\infty \dot{V} dt = V(\infty) - V(0) = -\int_0^\infty k\alpha_x e_x^2 dt - \int_0^\infty \alpha_s e_s^2 dt \quad (20)$$

Since V is a positive definite function, the following inequality holds:

$$\begin{aligned} V(0) \geq \int_0^\infty k\alpha_x e_x^2 dt + \int_0^\infty \alpha_s e_s^2 dt \geq \\ \geq k\underline{\alpha}_x \int_0^\infty e_x^2 dt + \underline{\alpha}_s \int_0^\infty e_s^2 dt \end{aligned} \quad (21)$$

where the lower bounds of functions $\alpha_x(t)$ and $\alpha_s(t)$ are introduced:

$$\begin{aligned} \alpha_x(t) \geq \underline{\alpha}_x > 0 \\ \alpha_s(t) \geq \underline{\alpha}_s > 0 \end{aligned} \quad (22)$$

It follows from (21) that $e_x, e_s \in \mathcal{L}_2$. Applying Lemma 2, the convergence of $e_x(t)$ and $e_s(t)$ to 0 follows immediately. Since e_s and e_c are the sine and the cosine, respectively, of the same argument, e_c^2 converges to 1. Because of $e_s \to 0$, it follows from (11) that $\dot{e}_c \to 0$ and consequently the limit $\lim_{t\to\infty} e_c(t)$ exists and is either 1 or -1. It has been shown that the limit $\lim_{t\to\infty} V(t)$ also exists, and consequently $\lim_{t\to\infty} e_y(t)$ also exists.

Until now we only established the convergence of $e_x(t)$ and $e_s(t)$ to 0, while $e_c(t)$ was shown to converge either to 1 or to -1. To show the convergence of $e_y(t)$ to 0, at least one of the conditions 1 or 2 of Theorem 3 have to be fulfilled. Let us first analyze case 1. Applying Lemma 1 on $\dot{e}_s(t)$ ensures that $\lim_{t\to\infty} \dot{e}_s(t) = 0$ if $\lim_{t\to\infty} e_s(t)$ exists and is finite (which has already been proven) and $\dot{e}_s(t)$ is uniformly continuous. The latter is true (see (11)) if $(w_r - w)e_c$ is uniformly continuous. It has already been shown that e_c is uniformly continuous. The signal $w - w_r$ defined in (16) is uniformly continuous since α_s and v_r are uniformly continuous by the assumption in case 1 of the theorem. The statement $\lim_{t\to\infty} \dot{e}_s(t) = 0$ which is identical to

$$\lim_{t\to\infty} \left(\left(w_r(t) - w(t)\right) e_c(t) \right) = 0 \quad (23)$$

has therefore been proven. Since $e_c(t)$ converges to 1 or to -1, the following can be concluded from (16):

$$\lim_{t\to\infty} (w - w_r) = \lim_{t\to\infty} (k e_y v_r + \alpha_s e_s) = 0 \quad (24)$$

The convergence of e_y to 0 follows from (24):

$$\begin{aligned} k e_y v_r + \alpha_s e_s \to 0, e_s \to 0 \Rightarrow k e_y v_r \to 0 \\ k e_y v_r \to 0, k_y > 0, v_r \nrightarrow 0 \Rightarrow e_y \to 0 \end{aligned} \quad (25)$$

where it was taken into account that v_r does not diminish as $t \to \infty$.

For the second case we again have to guarantee that $\lim_{t\to\infty}(w - w_r) = 0$. This is true if v_r and α_s are uniformly continuous as shown before. Then Barbălat's lemma (Lemma 1) is applied on $\dot{e}_x$ in Eq. (11) after inserting the control law (16):

$$\dot{e}_x = v_r e_c - v + e_y w = -\alpha_x e_x + e_y w_r + k e_y^2 v_r + \alpha_s e_y e_s \quad (26)$$

It has already been shown that e_x, e_y, and e_s are uniformly continuous, while v_r, w_r, α_x, and α_s are uniformly continuous by the assumption of case 2 of the theorem. It has been proven that $\lim_{t\to\infty} \dot{e}_x(t) = 0$. The first term in Eq. (26) goes to 0 as t goes to infinity. The last two terms also converge to 0 due to $\lim_{t\to\infty}(w - w_r) = 0$. Consequently, the product $w_r e_y$ goes to 0. Since w_r is persistently exciting and does not go to 0, e_y has to go to 0.

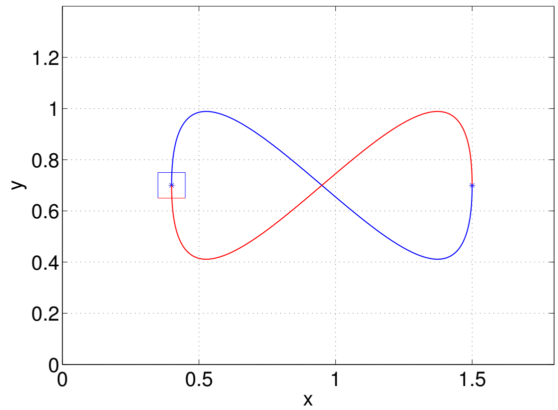

Fig. 2. Smooth reference trajectory for Example 1

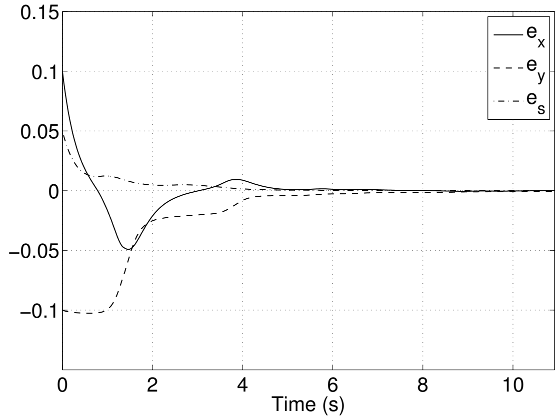

Fig. 3. Error evolution for Example 1

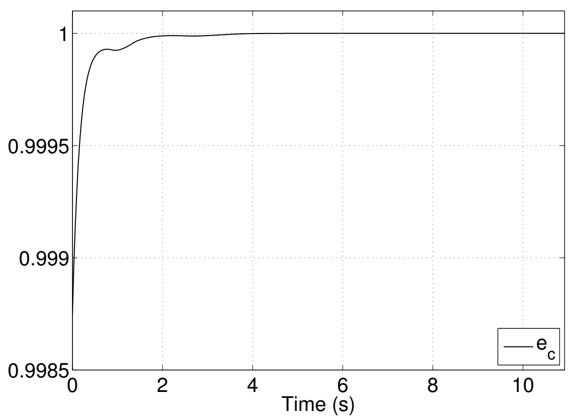

Fig. 4. Evolution of e_c in Example 1

4.2 Example 1

Consider the mobile robot (2) performing trajectory tracking on the path shown in Fig. 2. A direct application of control law (16) with $\alpha_x = 3$, $k = 1$ and $\alpha_s = 3$ allows conditions in Theorem 3 to hold, which guarantees convergence of e_x, e_y, and e_s to 0 as shown in Fig. 3. In other words, Theorem 3 assures trajectory tracking as expected. Fig. 4 shows the time evolution of e_c. As discussed in the theorem's proof above, $e_c \to 1$.

5. TAKAGI-SUGENO FUZZY CONTROL DESIGN

In this section the Takagi-Sugeno (TS) model of the mobile robot kinematic model (11) or (12) will be developed. Firstly, the control signals will be separated to the the "feedforward" term and the feedback term to be determined by the TS control law:

$$\begin{aligned} v &= v_r e_c + v_b \\ w &= w_r + w_b \end{aligned} \tag{27}$$

where "feedforward" is not entirely suitable since the linear velocity command depends on the state e_c. But on the other hand, if the control law is chosen properly, e_c should converge to 1, thus meaning that $v_r e_c$ would converge to a feedforward term v_r. Inserting Eq. (27) into Eq. (11) we obtain:

$$\begin{aligned} \dot{e}_x &= -v_b + e_y w_r + e_y w_b \\ \dot{e}_y &= v_r e_s - e_x w_r - e_x w_b \\ \dot{e}_s &= -e_c w_b \\ \dot{e}_c &= e_s w_b \end{aligned} \tag{28}$$

or in the equivalent matrix form

$$\begin{bmatrix} \dot{e}_x \\ \dot{e}_y \\ \dot{e}_s \\ \dot{e}_c \end{bmatrix} = \begin{bmatrix} 0 & w_r & 0 & 0 \\ -w_r & 0 & v_r & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_x \\ e_y \\ e_s \\ e_c \end{bmatrix} + \begin{bmatrix} -1 & e_y \\ 0 & -e_x \\ 0 & -e_c \\ 0 & e_s \end{bmatrix} \begin{bmatrix} v_b \\ w_b \end{bmatrix} \tag{29}$$

The TS model of (29) can be constructed by the sector non-linearity approach if v_r, w_r, e_x, e_y, e_s, and e_c are chosen as the antecedent variables with a-priori known upper and lower bounds. It must be kept in mind that a TS model is an exact representation of the original model in a compact set of the state variables. A natural way to control such a system is to use parallel distributed compensation (PDC) as shown in Tanaka and Wang (2001). It is very easy to determine that the PDC design on the model (29) would result in an infeasible system of linear matrix inequalities. This is due to the fact that the system is not controllable in the linear sense. This problem can be circumvented if the state space description is split into two systems:

$$\dot{e} = \begin{bmatrix} \dot{e}_x \\ \dot{e}_y \\ \dot{e}_s \end{bmatrix} = \begin{bmatrix} 0 & w_r & 0 \\ -w_r & 0 & v_r \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_x \\ e_y \\ e_s \end{bmatrix} + \begin{bmatrix} -1 & e_y \\ 0 & -e_x \\ 0 & -e_c \end{bmatrix} \begin{bmatrix} v_b \\ w_b \end{bmatrix} \tag{30}$$

$$\dot{e}_c = e_s w_b \tag{31}$$

First, the TS model of the system (30) will be developed and then the PDC control will be designed for this system. Eq. (31) represents the uncontrollable mode of the system, and it will be important to check if e_c converges to 1 as desired.

5.1 Takagi-Sugeno model of the system

The TS model is represented through the following polytopic form:

$$\dot{e}(t) = \sum_{i=1}^{r} h_i\left(z(t)\right)\left(A_i e(t) + B_i u(t)\right) \tag{32}$$

In order to construct the TS model the sector nonlinearity approach will be used. This means that the nonlinearities have to be taken from the nonlinear model and used in the premise (or antecedent) vector $z(t)$. The antecedent vector has 5 elements in this case:

$$z(t) = [\, w_r(t) \ v_r(t) \ e_y(t) \ e_x(t) \ e_c(t) \,]^T \tag{33}$$

The system (30) is controllable in the vicinity of the point $[\, e_x \ e_y \ e_s \,]^T = [\, 0 \ 0 \ 0 \,]^T$ if e_c does not approach 0 and either v_r or w_r do not approach 0. These are actually the conditions for the feasibility of LMIs for the determination of control gains as it will be shown later. The lower bounds $\underline{z}_j$ and the upper bounds $\overline{z}_j$ ($j = 1, 2, 3, 4, 5$) of the elements of the antecedent vector are needed for the construction of the PDC control. The bounds on v_r and w_r are obtained from the actual reference trajectory, while the bounds on the tracking error are selected on the basis of any a priori knowledge available.

The matrices A_z and B_z are:

$$A_z = \begin{bmatrix} 0 & z_1 & 0 \\ -z_1 & 0 & z_2 \\ 0 & 0 & 0 \end{bmatrix} \quad B_z = \begin{bmatrix} -1 & z_3 \\ 0 & -z_4 \\ 0 & -z_5 \end{bmatrix} \tag{34}$$

The number of fuzzy rules is $r = 2^5 = 32$. The matrices of the linear submodels are:

$$A_i = \begin{bmatrix} 0 & \varepsilon_i^1 & 0 \\ -\varepsilon_i^1 & 0 & \varepsilon_i^2 \\ 0 & 0 & 0 \end{bmatrix} \quad B_i = \begin{bmatrix} -1 & \varepsilon_i^3 \\ 0 & -\varepsilon_i^4 \\ 0 & -\varepsilon_i^5 \end{bmatrix} \quad i = 1 \ldots 2^5 \tag{35}$$

where

$$\varepsilon_i^j = \underline{z}_j + i_j \left(\overline{z}_j - \underline{z}_j \right), \ \ i = 1 \ldots 2^5, \ \ j = 1 \ldots 5$$

and i_j corresponds to the j-th bit of the binary representation of $(i-1)$, i.e., they represent all the vertices of the hypercube numbered using binary enumeration. Finally the membership functions $h_i(z)$ in (32) need to be defined:

$$h_i(z) = \prod_{j=1}^{5} w_{i_j}^j(z_j) \ \ i = 1, 2, \ldots 32$$

$$w_1^j(z_j) = \frac{z_j - \underline{z}_j}{\overline{z}_j - \underline{z}_j}, \ \ w_0^j(z_j) = 1 - w_1^j(z_j) \ \ j = 1 \ldots 5$$

5.2 PDC-based tracking control of a mobile robot

In order to stabilize the TS fuzzy model (32), a PDC (Parallel Distributed Compensation) control law is used:

$$u_b(t) = -\sum_{i=1}^{r} h_i\left(z(t)\right) F_i e(t) = -F_z e(t) \tag{36}$$

Several results concerning PDC controller design for TS models are available in the literature. The problem is often solved within the LMI framework.

In this work the following optimization problem is solved, which finds the maximum decay rate of the system under given input constraints; see Tanaka and Wang (2001) and Guechi et al. (2010):

$$\begin{array}{ll} \underset{X, M_1 \ldots M_r}{\text{maximize}}\, \gamma \text{ subject to} & \\ \begin{bmatrix} 1 & e(0)^T \\ e(0) & X \end{bmatrix} \geq 0 & \\ \begin{bmatrix} X & M_i^T \\ M_i & \mu^2 I \end{bmatrix} \geq 0, & i = 1, 2, \ldots r \\ \Upsilon_{ii} < 0, & i = 1, 2, \ldots r \\ \frac{2}{r-1}\Upsilon_{ii} + \Upsilon_{ij} + \Upsilon_{ji} < 0, & i, j = 1, 2, \ldots r, i \neq j \end{array} \tag{37}$$

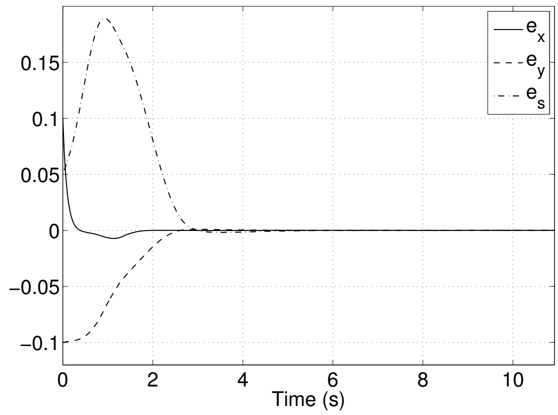

Fig. 5. Error evolution of Example 2

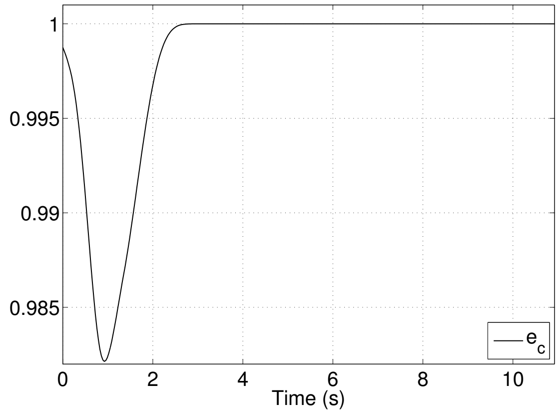

Fig. 6. Evolution of e_c in Example 2

with $\Upsilon_{ij} = XA_i^T + A_iX - M_j^T B_i^T - B_iM_j + 2\gamma X$, $\gamma > 0$, $|u(t)| \leq \mu$, $\mu > 0$, and $e(0) = [\, e_x(0) \ e_y(0) \ e_s(0) \,]^T$.

The solutions of the above generalized eigenvalue problem give the control gains:

$$F_i = M_i X^{-1}, \quad i = 1, 2, \ldots r. \tag{38}$$

The Lyapunov function is given by

$$V(e(t)) = e^T(t) P e(t), \tag{39}$$

with $P = X^{-1}$.

The necessary condition for the solution of the above generalized eigenvalue problem is that e_c should be positive, meaning that the orientation error must be kept within $\pm 90°$. If this is satisfied, it can be shown that e_c does converge to 1 since e_s and w_b in (31) are "in average" of the same sign and consequently $\dot{e}_c$ is "in average" positive.

5.3 Example 2

Consider a compact set of the premise variables in model (31) as described in the previous section, for instance, $w_r(t) \in [-2.5481, 2.5481]$, $v_r(t) \in [0.0141, 0.7494]$ (from the trajectory described in Fig. 2), $e_x(t) \in [-0.15, 0.15]$, $e_y(t) \in [-0.15, 0.15]$ and $e_c(t) \in [0.9, 1.1]$ (as for the possible error intervals). Moreover, assume initial conditions $e(0) = [\, e_x(0) \ e_y(0) \ e_s(0) \,]^T = [\, 0.1 \ -0.1 \ 0.05 \,]^T$ and control input bound of $|u(t)| \leq \mu$, $\mu = 5$.

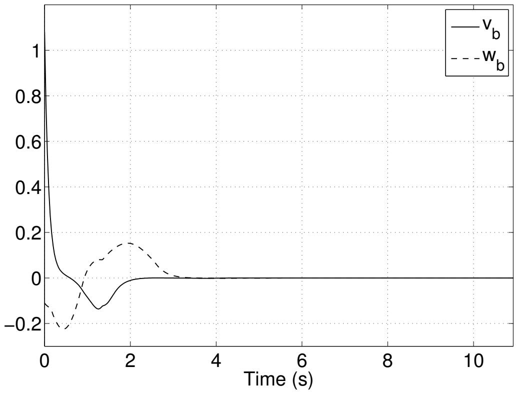

Fig. 7. Control law signals in Example 2

Applying the PDC control design described by linear matrix inequalities (37) to TS model (32) under the bounds described above, the following common matrix $P > 0$ for the quadratic Lyapunov function (39) has been found:

$$P = \begin{bmatrix} 26.2966 & 0.0030 & 0.0009 \\ 0.0030 & 78.5366 & 19.3415 \\ 0.0009 & 19.3415 & 6.5017 \end{bmatrix}. \quad (40)$$

Gains $M_i, i = 1, 2, \ldots 32$, have been also found and are omitted for brevity. The maximum decay rate found feasible under the previous conditions was $\gamma = 0.06$.

Fig 5 shows the time evolution of the error signals e_x, e_y, and e_s. As expected, trajectory tracking is performed since convergence of the decoupled TS model (30) to $(0, 0, 0)$ as well as convergence of the uncontrollable mode e_c to 1 (see Fig 6), is guaranteed by the PDC control design. In Fig. 7 the control law is shown to hold the prescribed bound $|u(t)| \leq \mu$. Recall this control law correspond to $u(t) = [\, v_b \; w_b \,]^T$.

6. CONCLUSION

In this paper a novel kinematic model is proposed where the transformation between the robot posture and the system state is bijective. Two control approaches are proposed to solve the tracking problem. One approach is based on the Takagi-Sugeno fuzzy model where a parallel distributed compensation control is used. The alternative approach is to use Lyapunov stability analysis to construct a nonlinear controller that achieves asymptotic stability if reference velocities satisfy the condition of persistent excitation.

Although the proposed fourth-order model is not controllable in the linear sense and the associated PDC control design is not feasible, the Lyapunov-based control design easily results in a controller that achieves asymptotic stability under the usual demands of persistently exciting reference velocities. When the uncontrollable mode is separated from the controllable states, the PDC control is obtained for the controllable part but the state e_c is restricted to the interval $(0, 1]$ (the orientation error must be within $\pm 90°$) while in the Lyapunov-based design is restricted to $(-1, 1]$ (the orientation error must be within $\pm 180°$).

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