Preprints of the 18th IFAC World Congress
Milano (Italy) August 28 - September 2, 2011

A Method to Teach the Parameterization of All Stabilizing Controllers

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Abstract: A simple and insightful method to teach the Youla-Kučera parameterization of all controllers that stabilize a given plant is presented. The text is intended for first-year graduate students in engineering. The result is derived first using transfer functions. A state-space representation of all stabilizing controllers then evolves from the transfer function result. Thus, the transfer functions and the state-space techniques are presented as connected approaches rather than isolated alternatives.

Keywords: Linear systems, feedback stabilization, Youla-Kučera parameterization, control education.

1. INTRODUCTION

The parameterization of all controllers that stabilize a given plant, often called the Youla-Kučera parameterization, is a fundamental result of control theory. It launched an entirely new area of research and found application, among others, in optimal and robust control.

In its original form, the result was obtained for finite-dimensional, linear time-invariant systems using transfer function methods, see Larin et al. (1971), Kučera (1975), Youla et al. (1976a, b), and Kučera (1979). It was then generalized to cover time-varying systems by e.g. Dale and Smith (1993), infinite dimensional systems by e.g. Desoer et al. (1980), Vidyasagar (1985), Quadrat (2003), and a class of non-linear systems by e.g. Hammer (1985), Paice and Moore (1990), and Anderson (1998). Quite naturally, a state-space representation of all stabilizing controllers was derived for finite dimensional, linear time-invariant systems; see Nett et al. (1984). Modern textbooks on the subject include Antsaklis and Michel (2006) and Colaneri et al. (1997).

This paper presents a simple and insightful method to teach the parameterization result. It is intended for first-year graduate students in engineering. The transfer function result is derived first. A state-space representation of all stabilizing controllers then evolves from the transfer function result. Thus, the transfer functions and the state-space techniques are presented as connected approaches rather than isolated alternatives.

2. TRANSFER FUNCTION APPROACH

Stability is achieved by feedback. Consider systems S_1 and S_2 connected in the feedback configuration shown in Fig. 1. The systems are assumed to be continuous-time, linear and time-invariant. The case of discrete-time systems is analogous. Stability is taken to mean that the states of S_1 and S_2 go to zero from any initial condition as time increases.

Given S_1, the task is to determine all systems S_2 so that the feedback system, S, is stable. Thus S_1 is called the plant and S_2 the stabilizing controller. For any given plant, the set of stabilizing controllers is either empty or infinite. Thus it is convenient to describe the set in parametric form.

Suppose that the plant is described by state-space equations of the form

$$\dot{x} = Ax + Bu, \quad y = Cx + Du \tag{1}$$

or by the transfer function

$$H_1(s) = C(sI - A)^{-1} B + D := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]. \tag{2}$$

The transfer function (2) is a proper rational matrix such that $\hat{y} = H_1 \hat{u}$ holds, where $\hat{y}, \hat{u}$ denotes the one-sided Laplace transforms of y, u. Here and below, the arguments of time functions as well as of their Laplace transforms are dropped, for the sake of brevity, when no ambiguity occurs.

The special notation for the transfer function matrix that is introduced in (2) will be found useful when studying the relations between the state-space and the transfer-function descriptions.

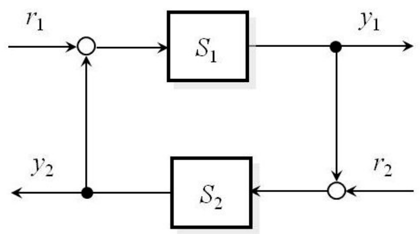

Fig. 1. Feedback system S.

The feedback system S is required to be well posed, that is to say, described by state-space equations of the same form as S_1 and S_2 are. Then, the first observation is that the feedback system S is controllable from inputs r_1 and r_2 and observable from outputs y_1 and y_2 if and only if the constituent systems S_1 and S_2 are both controllable and observable. This can be seen from the eigenvalue criterion. As a consequence, stability of the feedback system can be studied using transfer functions.

The transfer function matrix, H, of the feedback system S that relates inputs r_1, r_2 and outputs y_1, y_2 is given by

$$H = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \begin{bmatrix} I & -H_2 \\ -H_1 & I \end{bmatrix}^{-1} = \begin{bmatrix} -H_2 & I \\ I & -H_1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & H_2 \\ H_1 & 0 \end{bmatrix}, \tag{3}$$

where H_1 and H_2 are the transfer function matrices of S_1 and S_2, respectively. This can easily be obtained from the signal relationships in Fig. 1.

Thus, suppose for the present that S_1 and S_2 are jointly controllable and observable. The feedback system S is seen to be well posed if and only if $H(s)$ is proper (that is, analytic at $s = \infty$). Then the feedback system S is stable if and only if $H(s)$ is stable (that is, analytic in Re $s \geq 0$).

2.1 Fractional Descriptions

The set of stabilizing controllers for a given plant may contain controllers whose transfer function is not proper. In order to isolate only those with proper transfer functions, it is convenient to express the transfer function of the plant in the form of a proper and stable matrix fractional description.

Thus, let

$$H_1 = N_1 D_1^{-1} = \widetilde{D}_1^{-1} \widetilde{N}_1, \tag{4}$$

where D_1, N_1 is a pair of right coprime, proper and stable rational matrices, while $\widetilde{D}_1, \widetilde{N}_1$ is a pair of left coprime, proper and stable rational matrices.

Similarly, for the controller, let

$$H_2 = D_2^{-1} N_2 = \widetilde{N}_2 \widetilde{D}_2^{-1}, \tag{5}$$

where D_2, N_2 is a pair of left coprime, proper and stable rational matrices, while $\widetilde{D}_2, \widetilde{N}_2$ is a pair of right coprime, proper and stable rational matrices.

Then the closed-loop system transfer function H is proper and stable if and only if

$$D_2 D_1 - N_2 N_1 = U \tag{6}$$

or

$$\widetilde{D}_1 \widetilde{D}_2 - \widetilde{N}_1 \widetilde{N}_2 = \widetilde{U}, \tag{7}$$

where U and $\widetilde{U}$ are unimodular proper and stable rational matrices, that is, proper and stable rational matrices whose inverses exist and are proper and stable rational.

To prove this result, consider (6) first. It follows that

$$\begin{bmatrix} I & -H_2 \\ -H_1 & I \end{bmatrix} = \begin{bmatrix} D_2^{-1} U & -D_2^{-1} N_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} D_1^{-1} & 0 \\ -N_1 D_1^{-1} & I \end{bmatrix}$$

and

$$\begin{bmatrix} I & -H_2 \\ -H_1 & I \end{bmatrix}^{-1} = \begin{bmatrix} D_1 & 0 \\ N_1 & I \end{bmatrix} \begin{bmatrix} U^{-1} D_2 & U^{-1} N_2 \\ 0 & I \end{bmatrix}.$$

Thus, in view of (3),

$$H = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \begin{bmatrix} I & -H_2 \\ -H_1 & I \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} N_1 \\ D_1 \end{bmatrix} U^{-1} \begin{bmatrix} D_2 & N_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -I & 0 \end{bmatrix}.$$

Since D_1, N_1 are right coprime and D_2, N_2 left coprime, H is seen to be proper and stable if and only if U^{-1} is proper and stable.

Now consider (7). It follows that

$$\begin{bmatrix} -H_2 & I \\ I & -H_1 \end{bmatrix} = \begin{bmatrix} 0 & I \\ \widetilde{D}_1^{-1} \widetilde{U} & -\widetilde{D}_1^{-1} \widetilde{N}_1 \end{bmatrix} \begin{bmatrix} \widetilde{D}_2^{-1} & 0 \\ -\widetilde{N}_2 \widetilde{D}_2^{-1} & I \end{bmatrix}$$

and

$$\begin{bmatrix} -H_2 & I \\ I & -H_1 \end{bmatrix}^{-1} = \begin{bmatrix} \widetilde{D}_2 & 0 \\ \widetilde{N}_2 & I \end{bmatrix} \begin{bmatrix} \widetilde{U}^{-1} \widetilde{N}_1 & \widetilde{U}^{-1} \widetilde{D}_1 \\ I & 0 \end{bmatrix}.$$

Thus, in view of (3),

$$H = \begin{bmatrix} -H_2 & I \\ I & -H_1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & H_2 \\ H_1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \widetilde{D}_2 \\ \widetilde{N}_2 \end{bmatrix} \widetilde{U}^{-1} \begin{bmatrix} \widetilde{N}_1 & \widetilde{D}_1 \end{bmatrix} + \begin{bmatrix} 0 & -I \\ 0 & 0 \end{bmatrix}.$$

Since $\widetilde{D}_2, \widetilde{N}_2$ are right coprime and $\widetilde{D}_1, \widetilde{N}_1$ left coprime, H is seen to be proper and stable if and only if $\widetilde{U}^{-1}$ is proper and stable. This completes the proof of conditions (6) and (7) for closed-loop stability in terms of fractional descriptions.

2.2 Bézout Identity

The proper and stable rational matrices D_1, N_1 being right coprime, there exist two other proper and stable rational matrices X_1, Y_1 such that

$$X_1 D_1 + Y_1 N_1 = I. \tag{8}$$

The proper and stable rational matrices $\widetilde{D}_1, \widetilde{N}_1$ being left coprime, there exist two other proper and stable rational matrices $\widetilde{X}_1, \widetilde{Y}_1$ such that

$$\widetilde{D}_1 \widetilde{X}_1 + \widetilde{N}_1 \widetilde{Y}_1 = I. \tag{9}$$

It follows from (4) that

$$\widetilde{D}_1 N_1 - \widetilde{N}_1 D_1 = 0. \tag{10}$$

Finally, the matrices X_1, Y_1 and $\widetilde{X}_1, \widetilde{Y}_1$ can be selected in such a way as to satisfy

$$X_1 \widetilde{Y}_1 - Y_1 \widetilde{X}_1 = 0. \tag{11}$$

Indeed, if $X_1 \widetilde{Y}_1' - Y_1 \widetilde{X}_1' = Q \neq 0$ for some pair $\widetilde{X}_1'$, $\widetilde{Y}_1'$ that satisfies (9), then the pair $\widetilde{X}_1 = \widetilde{X}_1' + N_1 Q$, $\widetilde{Y}_1 = \widetilde{Y}_1' - D_1 Q$ will satisfy both (9) and (11). Collecting identities (8), (9), (10), and (11), one obtains the matrix Bézout identity

$$\begin{bmatrix} X_1 & Y_1 \\ -\widetilde{N}_1 & \widetilde{D}_1 \end{bmatrix}\begin{bmatrix} D_1 & -\widetilde{Y}_1 \\ N_1 & \widetilde{X}_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \tag{12}$$

2.3 Parameterization of Stabilizing Controllers

Note that (12) defines one stabilizing controller, namely

$$H_2 = -X_1^{-1}Y_1 = -\widetilde{Y}_1\widetilde{X}_1^{-1}.$$

Any and all controllers having proper transfer function, with the property that the feedback system S is well posed and stable, are now provided in parametric form.

Theorem. Let $H_1 = N_1 D_1^{-1} = \widetilde{D}_1^{-1}\widetilde{N}_1$ be coprime, proper and stable matrix fractional descriptions. Let X_1, Y_1 and $\widetilde{X}_1, \widetilde{Y}_1$ be proper and stable rational matrices that satisfy the Bézout identity (12). Then all proper H_2 that render the closed-loop feedback system well posed and stable are given by

$$\begin{aligned} H_2 &= -(X_1 + W\widetilde{N}_1)^{-1}(Y_1 - W\widetilde{D}_1) \\ &= -(\widetilde{Y}_1 - D_1 W)(\widetilde{X}_1 + N_1 W)^{-1}, \end{aligned} \tag{13}$$

where the parameter, W, is a proper and stable rational matrix such that $(X_1 + W\widetilde{N}_1)^{-1}$ or $(\widetilde{X}_1 + N_1 W)^{-1}$ exists and is proper.

Proof. All proper stabilizing H_2 are generated by all solutions of equation (6), written in the form

$$(U^{-1}D_2)D_1 + (-U^{-1}N_2)N_1 = I.$$

The solution set can be parameterized as

$$U^{-1}D_2 = X_1 + W\widetilde{N}_1, \quad -U^{-1}N_2 = Y_1 - W\widetilde{D}_1,$$

where W is an arbitrary proper and stable rational matrix. This follows by direct substitution and the use of (8), (10). Thus, in view of (5), all proper stabilizing H_2 are given by

$$H_2 = D_2^{-1}N_2 = (U^{-1}D_2)^{-1}(U^{-1}N_2) = -(X_1 + W\widetilde{N}_1)^{-1}(Y_1 - W\widetilde{D}_1).$$

Alternatively, all proper stabilizing H_2 are generated by all solutions of equation (7), written in the form

$$\widetilde{D}_1(\widetilde{D}_2\widetilde{U}^{-1}) + \widetilde{N}_1(-\widetilde{N}_2\widetilde{U}^{-1}) = I.$$

The solution set can be parameterized as

$$\widetilde{D}_2\widetilde{U}^{-1} = \widetilde{X}_1 + N_1\widetilde{W}, \quad -\widetilde{N}_2\widetilde{U}^{-1} = \widetilde{Y}_1 - D_1\widetilde{W},$$

where $\widetilde{W}$ is an arbitrary proper and stable rational matrix. This follows by direct substitution and the use of (9), (10). Thus, in view of (5), all proper stabilizing H_2 are given by

$$H_2 = \widetilde{N}_2\widetilde{D}_2^{-1} = (\widetilde{N}_2\widetilde{U}^{-1})(\widetilde{D}_2\widetilde{U}^{-1})^{-1} = -(\widetilde{Y}_1 - D_1\widetilde{W})(\widetilde{X}_1 + N_1\widetilde{W})^{-1}.$$

Now (11) implies $W = \widetilde{W}$ and (13) holds. This completes the proof.

3. STATE SPACE APPROACH

The best way of presenting the Youla-Kučera parameterization in state space is to follow the three steps of the transfer function approach and translate each into the state-space parlance. This provides a connected result rather than two isolated alternatives.

For the present, we assume that the plant S_1 and the controller S_2 are jointly controllable and observable. This proviso makes it possible to refer to the transfer function results.

3.1 Fractional Description

Given the state-space description (1) of the plant to be stabilized, we seek appropriate stable systems, each giving rise to a pair of transfer functions that defines the desired fractional description. This can be achieved by applying a stabilizing state feedback and a stabilizing output injection.

To obtain a right fractional description for H_1, consider any stabilizing state feedback $u = Fx + r$ around the plant. The resulting equations

$$\begin{aligned} \dot{x} &= (A + BF)x + Br \\ u &= Fx + r \\ y &= (C + DF)x + Dr \end{aligned}$$

define a stable system with one input, r, and two outputs, u and y. Denote

$$D_1 := \left[\begin{array}{c|c} A + BF & B \\ \hline F & I \end{array}\right]$$

the transfer function between r and u and denote

$$N_1 := \left[\begin{array}{c|c} A + BF & B \\ \hline C + DF & D \end{array}\right]$$

the transfer function between r and y. Then D_1 and N_1 are proper and stable rational matrices and it holds

$$\hat{y} = N_1\hat{r}, \quad \hat{u} = D_1\hat{r}. \tag{14}$$

It follows that

$$\hat{y} = N_1 D_1^{-1}\hat{u} = H_1\hat{u}$$

so that D_1, N_1 is a right matrix fractional description for H_1.

To obtain a left fractional description for H_1, consider the dual situation, namely a state observer for the plant based on any stabilizing output injection Ke, where $e = y - (C\hat{x} + Du)$ is the difference between the actual output and the estimated output. The resulting equations

$$\begin{aligned} \dot{\hat{x}} &= (A - KC)\hat{x} + (B - KD)u + Ky \\ e &= y - (C\hat{x} + Du) \end{aligned}$$

define a stable system with two inputs, u and y, and one output, e. Denote

$$\widetilde{D}_1 := \left[\begin{array}{c|c} A - KC & K \\ \hline -C & I \end{array}\right]$$

the transfer function between y and e and denote

$$\widetilde{N}_1 := \left[\begin{array}{c|c} A - KC & B - KD \\ \hline C & D \end{array}\right]$$

the transfer function between u and $-e$. Then $\widetilde{D}_1$ and $\widetilde{N}_1$ are proper and stable rational matrices and it holds

$$\hat{e} = \widetilde{D}_1 \hat{y} - \widetilde{N}_1 \hat{u} = (\widetilde{D}_1 N_1 - \widetilde{N}_1 D_1)\hat{r} = 0 \tag{15}$$

because e is independent of r. It follows that

$$\widetilde{D}_1^{-1}\widetilde{N}_1 = N_1 D_1^{-1} = H_1$$

so that $\widetilde{D}_1, \widetilde{N}_1$ is a left matrix fractional description for H_1.

3.2 Bézout Identity

The matrix fractional descriptions for the controller can be constructed in a like fashion.

To obtain a right fractional description for H_2, consider a stabilizing state feedback $u = F\hat{x}$ around the observer with output y. The resulting equations

$$\dot{\hat{x}} = (A + BF)\hat{x} + Ke$$
$$u = F\hat{x}$$
$$y = (C + DF)\hat{x} + e$$

define a stable system with one input, e, and two outputs, u and y. Denote

$$\widetilde{X}_1 := \left[\begin{array}{c|c} A + BF & K \\ \hline C + DF & I \end{array}\right] \tag{16}$$

the transfer function between e and y and denote

$$\widetilde{Y}_1 := \left[\begin{array}{c|c} A + BF & K \\ \hline -F & 0 \end{array}\right] \tag{17}$$

the transfer function between e and $-u$. Then $\widetilde{X}_1$ and $\widetilde{Y}_1$ are proper and stable rational matrices and it holds

$$\hat{u} = -\widetilde{Y}_1 \hat{e}, \quad \hat{y} = \widetilde{X}_1 \hat{e}. \tag{18}$$

It follows that

$$\hat{u} = -\widetilde{Y}_1 \widetilde{X}_1^{-1} \hat{y} = H_2 \hat{y}$$

so that $\widetilde{X}_1, \widetilde{Y}_1$ is a right matrix fractional description for H_2.

To obtain a left fractional description for H_2, consider a stabilizing state feedback $u = F\hat{x} + r$ around the observer with output r. The resulting equations

$$\dot{\hat{x}} = (A - KC)\hat{x} + (B - KD)u + Ky$$
$$r = -F\hat{x} + u$$

define a stable system with two inputs, u and y, and one output, r. Denote

$$X_1 := \left[\begin{array}{c|c} A - KC & B - KD \\ \hline -F & I \end{array}\right] \tag{19}$$

the transfer function between u and r and denote

$$Y_1 := \left[\begin{array}{c|c} A - KC & K \\ \hline -F & 0 \end{array}\right] \tag{20}$$

the transfer function between y and r. Then X_1 and Y_1 are proper and stable rational matrices and it holds

$$\hat{r} = Y_1 \hat{y} + X_1 \hat{u} = (Y_1 \widetilde{X}_1 - X_1 \widetilde{Y}_1)\hat{e} = 0 \tag{21}$$

as r is independent of e. It follows that

$$X_1^{-1} Y_1 = \widetilde{Y}_1 \widetilde{X}_1^{-1} = -H_2$$

so that X_1, Y_1 is a left matrix fractional description for H_2.

Collecting equations (14), (15), (18), and (21), one obtains in matrix form

$$\begin{bmatrix} \hat{r} \\ \hat{e} \end{bmatrix} = \begin{bmatrix} X_1 & Y_1 \\ -\widetilde{N}_1 & \widetilde{D}_1 \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{y} \end{bmatrix} \tag{22}$$

and

$$\begin{bmatrix} \hat{u} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} D_1 & -\widetilde{Y}_1 \\ N_1 & \widetilde{X}_1 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{e} \end{bmatrix}. \tag{23}$$

Thus, the Bézout identity (12) holds

$$\begin{bmatrix} X_1 & Y_1 \\ -\widetilde{N}_1 & \widetilde{D}_1 \end{bmatrix} \begin{bmatrix} D_1 & -\widetilde{Y}_1 \\ N_1 & \widetilde{X}_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

and each of the four fractional descriptions involved is actually coprime.

3.3 Parameterization of Stabilizing Controllers

Putting $\hat{r} = 0$ in (22) and (23), it follows that

$$H_2 = -X_1^{-1} Y_1 = -\widetilde{Y}_1 \widetilde{X}_1^{-1} \tag{24}$$

is one stabilizing controller for H_1. Thus it is plausible that involving a parameter in generating $\hat{r}$ one obtains all stabilizing controllers. Accordingly, we put

$$\hat{r} = W\hat{e}. \tag{25}$$

Then (22) implies that

$$0 = \begin{bmatrix} I & -W \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{e} \end{bmatrix} = \begin{bmatrix} I & -W \end{bmatrix} \begin{bmatrix} X_1 & Y_1 \\ -\widetilde{N}_1 & \widetilde{D}_1 \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{y} \end{bmatrix}$$
$$= (X_1 + W\widetilde{N}_1)\hat{u} + (Y_1 - W\widetilde{D}_1)\hat{y}$$

while (23) implies that

$$\begin{bmatrix} \hat{u} \\ \hat{y} \end{bmatrix} = \begin{bmatrix} D_1 & -\widetilde{Y}_1 \\ N_1 & \widetilde{X}_1 \end{bmatrix} \begin{bmatrix} W \\ I \end{bmatrix} \hat{e} = \begin{bmatrix} -(\widetilde{Y}_1 - D_1 W)\hat{e} \\ (\widetilde{X}_1 + N_1 W])\hat{e} \end{bmatrix}.$$

Hence the parameterization formula (13) follows

$$H_2 = -(X_1 + W\widetilde{N}_1)^{-1}(Y_1 - W\widetilde{D}_1) = -(\widetilde{Y}_1 - D_1 W)(\widetilde{X}_1 + N_1 W)^{-1}.$$

This shows that the parameter W introduced in (25) does generate any and all stabilizing controllers.

3.4 State-Space Representation of Stabilizing Controllers

A state-space representation of the stabilizing controller (24), which corresponds to r = 0, is inferred from (16), (17) or (19), (20). It consists of an asymptotic state observer and a stabilizing state feedback from the estimated state. The entire set of stabilizing controllers is then generated by putting $r = We$ as follows,

$$\dot{\hat{x}} = A\hat{x} + Bu + K[y - (C\hat{x} + Du)]$$

$$u = F\hat{x} + W(q)[y - (C\hat{x} + Du)],$$

where the parameter W is an arbitrary proper and stable rational matrix in the differential operator $q := d/dt$ and is such that the feedback system is well posed. A state-space representation of all stabilizing controllers is shown in Fig. 2.

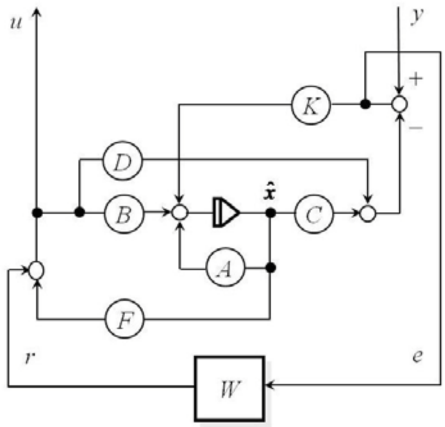

Fig. 2. A state-space representation of all stabilizing controllers.

Thus, every stabilizing controller can be viewed as a combination of an asymptotic state observer and a stabilizing feedback from the estimated state, plus an additional system that depends on the parameter W. The order of any stabilizing controller is the order of the plant plus the order of a state-space realization of W. Thus the set of stabilizing controllers contains controllers of arbitrarily high order.

The initial assumption of controllability and observability for the plant and the controller can be relaxed. If the plant is uncontrollable and/or unobservable, then the closed-loop system has an uncontrollable and/or unobservable eigenvalue. Such an eigenvalue is invariant under state feedback and/or output injection. Thus, for all such eigenvalues to be stable, the plant must be stabilizable and detectable. This is a necessary and sufficient condition for the stabilization of a plant given by (1) using the feedback configuration of Fig. 1. It is to be noted that a stabilizing controller can also turn uncontrollable and/or unobservable provided it is jointly stabilizable and detectable.

4. EXAMPLE

Simple examples are most illustrative. Consider an integrator plant described by the equations $\dot{x} = u,\ y = x,$ which correspond to (1) with $A = 0$, $B = 1$, $C = 1$, and $D = 0$. The transfer function of the plant is $H_1(s) = 1/s$.

Determine all stabilizing controllers with proper transfer function H_2 using the transfer-function approach. Proper and stable fractional descriptions of H_1 are provided, for example, by

$$D_1(s) = \widetilde{D}_1(s) = \frac{s}{s+1}, \quad N_1(s) = \widetilde{N}_1(s) = \frac{1}{s+1}.$$

The Bézout identity is completed by solving the equations

$$X_1 D_1 + Y_1 N_1 = \widetilde{D}_1 \widetilde{X}_1 + \widetilde{N}_1 \widetilde{Y}_1 = 1$$

to obtain

$$X_1 = \widetilde{X}_1 = 1,\ Y_1 = \widetilde{Y}_1 = 1.$$

Thus,

$$H_2(s) = -\left(1 - \frac{s}{s+1}\overline{W}\right)\left(1 + \frac{1}{s+1}\overline{W}\right)^{-1},$$

where $\overline{W}$ is an arbitrary proper and stable rational function. Note that the indicated inverse exists and is proper for any $\overline{W}$. For example, $\overline{W} = 0$ implies $H_2 = -1$.

Following the state-space approach, we first construct an asymptotic state observer

$$\dot{\hat{x}} = u + K(y - \hat{x}) \tag{26}$$

with an output injection gain K such that $A - KC = -K$ is stable, hence $K > 0$. We then apply a stabilizing state feedback $u = F\hat{x} + r$ with a gain F such that $A + BF = F$ is stable, hence $F < 0$. We complete the design of all stabilizing controllers that have a proper transfer function by adding a term that depends on the parameter, W, as follows

$$u = F\hat{x} + W(q)(y - \hat{x}), \tag{27}$$

where W is an arbitrary proper and stable rational function in the differential operator q. Note that the closed-loop system is well posed for any W. For example, $W = 0$ implies

$$H_2(s) = \left[\begin{array}{c|c} F-K & K \\ \hline F & 0 \end{array}\right].$$

The stabilizing controllers described by (26) – (27) are shown in Fig. 3.

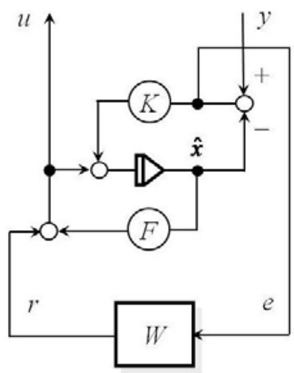

Fig. 3. The set of all controllers that stabilize an integrator.

Any stabilizing controller in the set has order at least one. The McMillan degree of its transfer function may be lower, however, in case the controller is uncontrollable and/or unobservable. For example, the parameter

$$W(q) = -\frac{q+K-F+KF}{q+1}$$

yields the controller

$$H_2(s) = \left[\begin{array}{cc|c} F+K-1 & 1 & K-1 \\ -1+(K-F+KF) & -1 & 1-(K-F+KF) \\ \hline F+1 & 1 & -1 \end{array}\right] = -1$$

that is uncontrollable and unobservable, of McMillan degree zero. It corresponds to the controller effected by $\overline{W} = 0$.

On the other hand, if it is of interest to obtain the controller that corresponds to $W = 0$ in (25) by employing transfer function techniques, put

$$D_1(s) = \frac{s}{s-F}, \; N_1(s) = \frac{1}{s-F}$$

and

$$\widetilde{D}_1(s) = \frac{s}{s+K}, \; \widetilde{N}_1(s) = \frac{1}{s+K}.$$

Complete the Bézout identity while enforcing strictly proper Y_1 and $\widetilde{Y}_1$. In this case,

$$X_1(s) = \frac{s+(K-F)}{s+K}, \; Y_1(s) = \frac{-KF}{s+K}$$

and

$$\widetilde{X}_1(s) = \frac{s+(K-F)}{s-F}, \; \widetilde{Y}_1(s) = \frac{-KF}{s-F}.$$

Then

$$H_2(s) = -\left(\frac{s+(K-F)}{s+K}\right)^{-1}\left(\frac{-KF}{s+K}\right) = -\left(\frac{-KF}{s-F}\right)\left(\frac{s+(K-F)}{s-F}\right)^{-1}$$
$$= \frac{KF}{s+(K-F)}.$$

5. CONCLUSIONS

The transfer function approach to determining all controllers with proper transfer function H_2 that stabilize a given plant H_1 consists of three steps: (i) determine coprime, proper and stable fractional descriptions for H_1, (ii) solve a Bézout equation to obtain one stabilizing controller, and (iii) form the transfer functions H_2 of all stabilizing controllers using a parameter that is a proper and stable rational matrix.

The state space approach to determining all controllers S_2 with proper transfer function that stabilize a given plant S_1 consists of two steps: (i) obtain one stabilizing controller by constructing an asymptotic state observer for S_1 and applying a stabilizing state feedback from the estimated state, and (ii) form all stabilizing controllers S_2 by connecting to the initial controller a parameter that is a system with proper and stable rational transfer function.

Thus, in the state space approach, there is no need to construct proper and stable fractional descriptions for the plant, or to solve the Bézout equation. The observer-based stabilizing controller is given directly by stabilizing gains F and K. A particular selection of F and K then corresponds to a choice of a particular proper and stable fractional description used in the transfer-function approach.

The transfer function result is the result of reference and it is amenable to generalizations for time-varying, infinite-dimensional, and nonlinear systems. The state space result is simple and useful in the design of control systems.

ACKNOWLEDGMENT

Supported by the Ministry of Education of the Czech Republic, Project 1M0567 Center for Applied Cybernetics.

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