# ROBUST OPTIMIZATION FOR SCHEDULING UNDER BOUNDED UNCERTAINTY IN PROCESSING TIMES

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*Abstract*

A novel robust optimization methodology is proposed for scheduling under bounded uncertainty in processing times. This approach generates "robust" solutions which are in a sense immune against uncertainty when it is applied to Mixed-Integer Linear Programming (MILP) problems with uncertain coefficients and right-hand-side parameters of the inequality constraints. By introducing a small number of auxiliary variables, a deterministic robust counterpart problem is formulated to determine the optimal solution given the (relative) magnitude of uncertain data and feasibility tolerance. Based on a novel continuous-time short-term scheduling model, uncertainty in processing times of operational tasks is considered. Preliminary computational results are presented to demonstrate the effectiveness of the proposed approach.

*Keywords*

Uncertainty, Scheduling, Chemical Processes, Robust Optimization.

## Introduction

The area of production scheduling has received considerable attention from both the academia and the chemical processing industries over the last few years. Due to lack of accurate process models and variability of process and environmental data, it is of crucial importance to develop systematic methods to address the problem of scheduling under uncertainty, in order to create schedules of high quality.

Most existing work follow the framework of stochastic optimization, in which the uncertainty is modeled, for example, through the use of a number of scenarios based on discretization of continuous probability distribution functions, and the expectation of a certain performance criterion, such as the expected profit, is optimized with respect to the scheduling decision variables. Such methods inevitably enlarge the size of the problem significantly as the number of scenarios increases exponentially with the number of uncertain parameters. This main drawback limits the application of these methods to solve practical problems with a large number of uncertain parameters.

In this work, we propose a novel Robust Optimization approach to address the problem of scheduling under bounded uncertainty in processing times. The underlying framework is based on a Robust Optimization methodology first introduced for Linear Programming (LP) problems by Ben-Tal and Nemirovski (2000) and extended in this work for Mixed-Integer Linear Programming (MILP) problems. The approach produces "robust" solutions which are in a sense immune against uncertainties in both the coefficients and right-hand-side parameters of the inequality constraints. The approach can be applied to address the problem of production scheduling with uncertain processing times, market demands, and/or prices of products and raw materials.

## Problem Statement

The scheduling problem of chemical processes is defined as follows: given (i) production recipes (i.e., the processing times for each task at the suitable units, and the amount of the materials required for the production of each product), (ii) available equipment and the ranges of their capacities, (iii) material storage policy, (iv) production requirement, and (v) time horizon under consideration, determine (i) the optimal sequence of tasks taking place in each unit, (ii) the amount of material being processed at each time in each unit, (iii) the processing time of each task in each unit, so as to optimize a performance criterion, for example, to minimize the makespan or to maximize the overall profit.

The most common sources of uncertainty in the aforementioned scheduling problem are: (i) the processing times of tasks; (ii) the market demands for products; and (iii) the prices of products and/or raw materials.

In this work, we employ the continuous-time formulation proposed in (Ierapetritou and Floudas, 1998a; Ierapetri-

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tou and Floudas, 1998b; Ierapetritou, Hené and Floudas, 1999; Lin and Floudas, 2001) to study the effects of uncertainty on short-term production scheduling of chemical processes. This formulation features the original concept of event points which represents the beginning of a task or utilization of a unit and the resulting mathematical model is an MILP problem.

### Robust Optimization Approach

Consider the following generic MILP problem:

$$\begin{aligned} \underset{x,\,y}{Min/Max}\;\; & c^T x + d^T y \\ s.t.\;\; & Ex + Fy \;=\; e \\ & Ax + By \;\le\; p \\ & \underline{x} \le x \le \bar{x} \\ & y \;=\; 0,1 \end{aligned} \tag{1}$$

Assume that the uncertainty arises in both the coefficients and the right-hand-side parameters of the inequality constraints, namely, $a_{ij}$, $b_{ik}$ and $p_i$. We are concerned about the feasibility of the following constraint.

$$\sum_j a_{ij}x_j + \sum_k b_{ik}y_k \le p_i \tag{2}$$

It can be shown that optimal solutions of MILP problems may become severely infeasible if the nominal data is slightly perturbed, which makes the "nominal" optimal solutions completely meaningless. Our objective is to develop a methodology to generate "reliable" solutions , which is immuned against uncertainty.

*Bounded Uncertainty*

The uncertain data range in the following intervals.

$$\begin{aligned} |\tilde{a}_{ij} - a_{ij}| &\le \epsilon |a_{ij}| \\ |\tilde{b}_{ik} - b_{ik}| &\le \epsilon |b_{ik}| \\ |\tilde{p}_i - p_i| &\le \epsilon |p_i| \end{aligned} \tag{3}$$

where $\tilde{a}_{ij}$, $\tilde{b}_{ik}$ and $\tilde{p}_i$ are the "true" values, $a_{ij}$, $b_{ik}$ and $p_i$ are the nominal values, $\epsilon > 0$ is a given (relative) uncertainty level.

**Property 1:** Given an infeasibility tolerance level ($\delta$), to generate a "reliable" solution which can satisfy the i-th constraint with an error of at most $\delta max[1, |p_i|]$, the following so-called $(\epsilon, \delta)$-Interval Robust Counterpart (IRC$[\epsilon, \delta]$) of the original uncertain MILP problem can be derived. (Lin, Janak and Floudas, 2002)

$$\begin{aligned} \underset{x,\,y,\,u}{Min/Max}\;\; & c^T x + d^T y \\ s.t.\;\; & Ex + Fy = e \\ & Ax + By \le p \\ & \sum_j a_{ij}x_j + \epsilon \sum_{j \in J_i} |a_{ij}| u_j + \sum_{k \notin K_i} b_{ik}y_k + \\ & \sum_{k \in K_i} (b_{ik} + \epsilon |b_{ik}|) y_k \le p_i - \epsilon |p_i| + \delta max[1, |p_i|] \;\; \forall i \\ & -u_j \le x_j \le u_j \;\; \forall j \\ & \underline{x} \le x \le \bar{x} \\ & y_k = 0,1 \;\; \forall k \end{aligned} \tag{4}$$

where $J_i$ and $K_i$ are the set of indices of the x and y variables, respectively, with uncertain coefficients in the i-th inequality constraint.

Note that this remains an MILP problem, but with additional auxiliary variables ($u_j$). It should be pointed out that the aforementioned Robust Optimization methodology circumvents any need for explicit or implicit discretization or sampling of the uncertain data, avoiding undesirable increase of the problem size, and thus renders the potential capability of handling problems with a large number of uncertain parameters.

The parameters involved in the scheduling under uncertainty problem participate in the MILP formulation proposed in (Ierapetritou and Floudas, 1998a; Ierapetritou and Floudas, 1998b; Ierapetritou et al., 1999; Lin and Floudas, 2001) as follows:

- Processing time. The task processing time parameters are linear coefficients of the binary and/or continuous variables in the duration constraint.
- Market demand. The demand data appear as the right-hand-side parameters in the demand constraints.
- Price of product and/or raw material. With simple reformulation, material prices can also be used as linear coefficients of continuous variables in an inequality constraint for the calculation of profit and/or cost.

All of these three types of uncertainty can be addressed effectively based on the aforementioned Robust Optimization framework. An example will be presented in the next section to illustrate how the Robust Optimization technique can be applied for the case of bounded uncertainty in processing times.

### Computational Studies

Consider Example 2 in (Ierapetritou and Floudas, 1998a). The processing recipe and the available units are as shown in Figure 1 and Table 1, respectively. The "nominal" solution of the problem is shown in Figure 2, with an objective value (profit) of 3639. However, this solution can become completely infeasible when there is uncertainty in the processing times of the tasks. If the processing time of each task is increased by 0.1% of its nominal value, the optimal schedule is changed significantly, as shown in Figure 3. In the heater and the separator, the number of tasks as well as processing amounts changes, while in the two reactors, even the task sequences are different. Furthermore, the profit is reduced considerably to 3265.

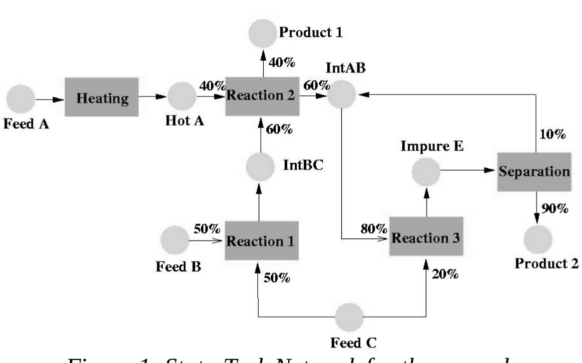


*Figure 1. State-Task Network for the example*

*Table 1. Available units for the example*

| Units | Capacity | Suitability | Processing Time |
|---|---|---|---|
| Heater | 100 | heating | 1.0 |
| Reactor 1 | 50 | reaction1,2,3 | 2.0, 2.0,1.0 |
| Reactor 2 | 80 | reaction1,2,3 | 2.0,2.0,1.0 |
| Separator | 200 | separation | 2.0 |

Based on the continuous-time formulation proposed in (Ierapetritou and Floudas, 1998a), the processing time of a task participates in the so-called duration constraint, as shown below.

$$T^f(i,j,n) - T^s(i,j,n) = \alpha \cdot wv(i,n) \qquad (5)$$

where $wv(i,n)$ is a binary variable, $T^f(i,j,n)$ and $T^s(i,j,n)$ are continuous variables, while $\alpha$ is the fixed processing time.

For simplicity, the indices of the variables are dropped below. We now consider "bounded uncertainty" in $\alpha$.

$$\alpha^L \leq \tilde{\alpha} \leq \alpha^U \qquad (6)$$

The equality constraint (5) has been developed for deterministic problems. If $\alpha$ is uncertain, it makes sense to rewrite it as an inequality constraint, as follows.

$$T^f - T^s \geq \alpha \cdot wv \qquad (7)$$

The IRC[$\epsilon, \delta$] approach (4) is applied to this problem. Because uncertainties only exist for the coefficients of the binary variables, the auxiliary variables ($u_j$) are not needed and only the following additional constraint is added to the original problem.

$$T^f - T^s \geq \alpha^U \cdot wv - \delta \qquad (8)$$

Let us assume that the (relative) uncertainty level ($\epsilon$) is 15%, that is,

$$0.85\alpha \leq \tilde{\alpha} \leq 1.15\alpha \qquad (9)$$

and the infeasibility tolerance level ($\delta$) is 0.10. By solving the IRC[$\epsilon, \delta$] problem, a "robust" schedule is obtained, as shown in Figure 4. The processing time of each task is extended to ensure that the schedule is feasible with the specified uncertainty level and infeasibility tolerance. However, the profit is reduced to 2887.

Figure 5 summarizes the results of the IRC problem with three different levels of uncertainty. Consistent with intuition and other approaches, the results show that the higher the uncertainty level is, the worse objective we can achieve; the larger the feasibility tolerance is, the closer the solution is to the optimal solution without uncertainty. It should be noted that at a given uncertainty level, the objective value of profit as well as the corresponding schedule changes dramatically at discrete points as the infeasibility tolerance increases. This results from the following characteristics of the example problem: the time horizon and the processing times of tasks are both fixed.

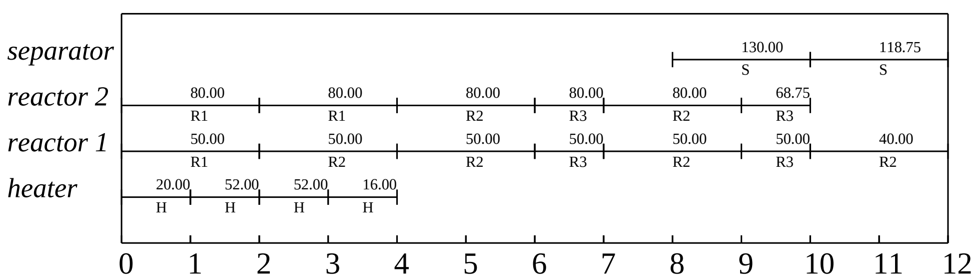


*Figure 2. Optimal solution with nominal processing times*

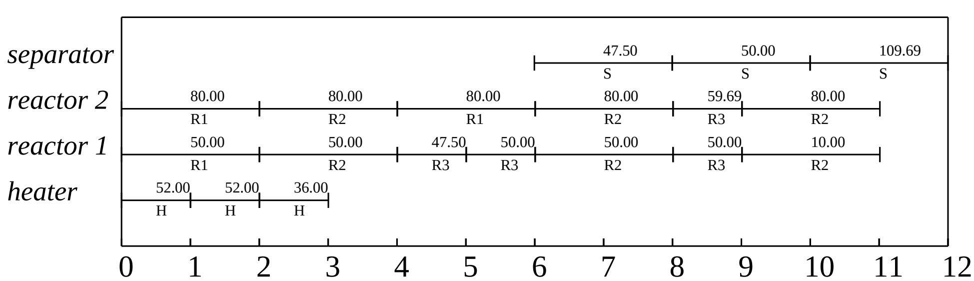


*Figure 3. Optimal solution with processing times increased by 0.1%*

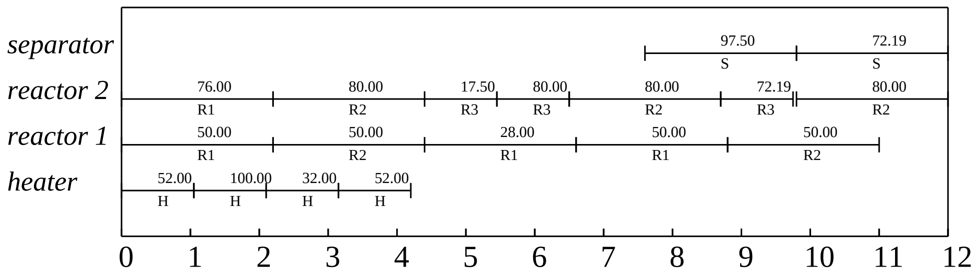


*Figure 4. Robust solution of the example ($\epsilon = 15\%, \delta = 0.10$)*

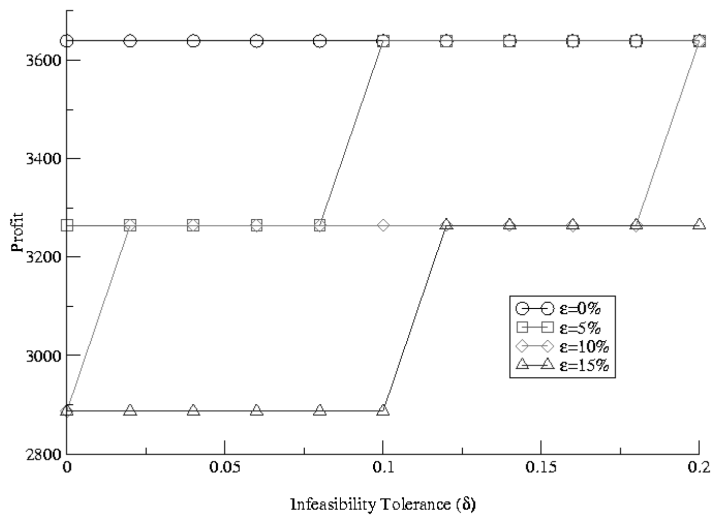


*Figure 5. Effects of uncertainty on profit*

## Conclusions

In this work, we propose a new approach to address the problem of scheduling under bounded uncertainty in processing times based on a Robust Optimization methodology. When applied to Mixed-Integer Linear Programming (MILP) problems, this approach produces “robust” solutions which are in a sense immune against uncertainties in both the coefficients and right-hand-side parameters of the inequality constraints. The approach can be extended to address uncertainty in market demands and prices of products and/or raw materials (Lin et al., 2002). Our preliminary computational results show that this approach provides an effective way to study scheduling problems under uncertainty, generating helpful insights on the tradeoffs between conflicting objectives.

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