

Improving Moving Horizon Estimation using Parametric Nonlinear Programming Sensitivity

Simen Bjorvand ^{a,1} and Johannes Jäschke ^{a,1}

^a Department of Chemical Engineering, Norwegian University of Science and Technology, Trondheim, Norway

Abstract

The moving horizon estimator (MHE) is a sliding window approximation of a full information least-squares optimization problem. The MHE is related to the full information problem through the arrival cost function, which functions as the prior of the estimation problem. Several methods are described in literature to find arrival cost function like the extended kalman filter method. In this work a new method is proposed to calculate the arrival cost based on parametric nonlinear programming techniques. We propose to approximate the ideal arrival cost using the sensitivity of the optimal solution manifold of the ideal arrival cost. The new method was tested on a quad-tank system and was shown to outperform the EKF approach to update the arrival cost for this case.

Keywords

Nonlinear State Estimation, Optimization, Parametric Nonlinear Programming

Introduction

Moving Horizon Estimation (MHE) is an optimization based technique to estimate the states of a system. MHE is an approximation of the full information Maximum a Posteriori (MAP) batch estimation problem, where only a sliding window of measurements is considered (Rao et al. (2003)). This is to reduce the computational load of solving the optimization problem. The strengths of the MHE are its ability to easily incorporate constraints in the estimation problems as well as its possibility to handle multi-rate measurements. To account for forgotten measurements from the full information problem a term known as the Arrival Cost is introduced to the cost function. From a statistical point of view the arrival cost can be considered a prior of the initial state in the estimation horizon, and the performance of the estimator is dependent on it (Rao et al. (2003)).

Several approaches to calculate the arrival cost for nonlinear systems have been proposed in literature and they can be categorized into two schools. The filtering approach where the prior is based on measurements up to the state in question and the smoothing approach where the prior is based on all available measurements (Elsheikh et al. (2021)). Among the filtering schemes there are the Extended Kalman Filter (EKF) and the smoothed EKF approaches (Robertson et al. (1996)) and the QR-factorization approach (Kühl et al. (2011)). Among the smoothing schemes there are the covariance smoothing scheme (Tenny and Rawlings (2002)), the nonlinear programming sensitivity approach (López-Negrete

and Biegler (2012)) and approximate hessian of MHE cost approach (Fiedler et al. (2020)).

In this work we introduce a novel filtering approach to calculate the arrival cost based on parametric nonlinear programming sensitivities. In particular, we linearize the optimal solution manifold of the ideal arrival cost problem to find a first order approximation of the ideal arrival cost. This is done by considering the arrival cost as a parametric NLP, and to approximate it using fast and efficient sensitivity updates. Our approach is different from previous applications of NLP-sensitivity in MHE, e.g. (Zavala et al. (2007), Das and Jäschke (2017)), who use sensitivity to obtain fast solutions to the MHE optimization problem. Our method should not be confused for (López-Negrete and Biegler (2012)) where the arrival cost is computed using the sensitivity of the optimality conditions of the MHE problem. Here we focus on the arrival cost, and as such, our approach could be combined with the above mentioned approaches for speeding up the solution of the MHE optimization problem. A unique feature of our method is that we can take into account inequality constraints, provided that the active set does not change.

Background

We consider a system described by the following discrete model

$$x_{k+1} = F(u_k, x_k, w_k) \quad (1)$$

$$y_k = h(x_k) + v_k \quad (2)$$

where $x_k \in \mathbb{R}^{n_x}$ denotes the states, $y_k \in \mathbb{R}^{n_y}$ the measurements, $u_k \in \mathbb{R}^{n_u}$ the known input actions, $w_k \in \mathbb{R}^{n_w}$ the uncertain

¹ (E-mail: {simen.bjorvand, johannes.jaschke}@ntnu.no)

process noise and $v_k \in \mathbb{R}^{n_y}$ the measurement noise at time k . $F : \mathbb{R}^{n_u} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_w} \mapsto \mathbb{R}^{n_x}$ represents the discrete process model describing the evolution of the states, $h : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_y}$ represents the measurement model describing the relationship between the measured quantities y_k and the states x_k .

Given the system model, the MHE solves the following optimization problem to estimate the states x at time T

$$\min_{w, v, x} \sum_{k=T-N}^{T-1} w_k^T Q_k^{-1} w_k + \sum_{k=T-N}^T v_k^T R_k^{-1} v_k + \Gamma_{T-N}(x_{T-N}) \quad (3a)$$

$$\text{s.t. } x_{k+1} = F(u_k, x_k, w_k), \quad k = T-N, \dots, T-1 \quad (3b)$$

$$y_k = h(x_k) + v_k, \quad k = T-N, \dots, T \quad (3c)$$

$$x_k \in \mathbb{X}_k, \quad k = T-N, \dots, T \quad (3d)$$

$$w_k \in \mathbb{W}_k, \quad k = T-N, \dots, T-1 \quad (3e)$$

N is the length of the sliding window. The sets \mathbb{X}_k and \mathbb{W}_k denotes inequality constraints on the states and process noise. The matrices Q_k and R_k are tuning matrices for the estimator and if chosen appropriately relate the Moving Horizon estimation problem to the Maximum A Posteriori (MAP) estimation approach. If the process noise in (1) is additive and the process- and measurement noise w_k and v_k are zero mean Gaussian noise, then selecting Q_k and R_k as the covariance matrices of w_k and v_k makes the MHE the MAP estimator. In the rest of this report a filtered estimate for the state at time k will be denoted as \hat{x}_k . The smoothed estimate for the state at time k given $l > k$ measurements will be denoted as $\check{x}_{k|l}$.

$\Gamma_k : \mathbb{R}^{n_x} \mapsto \mathbb{R}$ represents the arrival cost and can be thought of as a prior to the moving horizon estimation problem, as it represents the prior knowledge about the state x_k . The arrival cost also represents the connection between the full information estimation problem and the moving horizon approximation (Rao et al. (2003)). If the ideal arrival cost is used the moving horizon approach is identical to the full information problem. The ideal arrival cost formulation is defined recursively as:

$$\Gamma_{k+1}(x_{k+1}) = \min_{w_k, v_k, x_k} w_k^T Q_k^{-1} w_k + v_k^T R_k^{-1} v_k + \Gamma_k(x_k) \quad (4a)$$

$$\text{s.t. } x_{k+1} = F(u_k, x_k, w_k) \quad (4b)$$

$$y_k = h(x_k) + v_k \quad (4c)$$

$$x_k \in \mathbb{X}_k \quad (4d)$$

$$w_k \in \mathbb{W}_k \quad (4e)$$

For $k = 0, 1 \dots T-N$. $\Gamma_0(\cdot)$ is an explicit function representing the prior knowledge of the states x_k at time $k = 0$. Unfortunately this formulation of the arrival cost is not useful in practice, so an approximation is necessary. If the knowledge of the initial state is sampled from a Gaussian distribution with a known mean \check{x}_0 and covariance Π_0 , then the initial arrival cost becomes $\Gamma_0(x_0) = (\check{x}_0 - x_0)^T \Pi_0^{-1} (\check{x}_0 - x_0)$. For the subsequent states it is common to approximate the arrival cost with the following expression $\Gamma_k(x_k) = (\check{x}_k -$

$x_k)^T \Pi_k^{-1} (\check{x}_k - x_k)$. Several approaches have been proposed in literature to update the parameters \check{x}_k and Π_k . Examples include the Extended Kalman Filter (EKF) approach (Robertson et al. (1996)), the smoothed EKF (Robertson et al. (1996)) or the QR-factorization approach (Kühl et al. (2011)). The commonly used EKF update is found by linearizing the dynamic - and measurement models around the filtered estimate \hat{x}_k , and solving (4) with the linearized models and no inequality constraints. This yields the following update rule:

$$\begin{aligned} \Pi_{k+1} = & A_k(\Pi_k - \Pi_k H_k^T (H_k \Pi_k H_k^T + R_k)^{-1} H_k \Pi_k) A_k^T \\ & + G_k Q_k G_k^T \end{aligned} \quad (5)$$

$$\check{x}_{k+1} = F(u_k, \hat{x}_k, 0) \quad (6)$$

with

$$A_k = \left. \frac{\partial F}{\partial x} \right|_{u_k, \hat{x}_k, 0}, H_k = \left. \frac{\partial h}{\partial x} \right|_{\hat{x}_k}, G_k = \left. \frac{\partial F}{\partial w} \right|_{u_k, \hat{x}_k, 0} \quad (7)$$

coming from the linearized models. A drawback of this method is that the models are linearized around the filtered estimate \hat{x}_k . If the true state x_k is far from the filtered estimate this might lead to a large linearization error and subsequently a bad update of the prior at stage $k+1$. To mitigate this drawback we propose to update the arrival cost as outlined next.

Arrival cost update by NLP sensitivity

From problem (4) we note that the ideal arrival cost is a parametric nonlinear program with respect to x_{k+1} . We propose to use parametric nonlinear programming (PNLP) sensitivities to approximate the arrival cost as an alternative to the EKF.

NLP sensitivity concepts

Equation (4) can be written in a general form as a parametric NLP

$$\begin{aligned} \min_{\chi} \quad & J(\chi, p) \\ \text{s.t.} \quad & c(\chi, p) = 0 \quad (\lambda) \\ & g(\chi, p) \leq 0 \quad (\mu) \end{aligned} \quad (8)$$

where $\chi \in \mathbb{R}^{n_\chi}$ are the decision variables and $p \in \mathbb{R}^{n_p}$ are the parameters. $\lambda \in \mathbb{R}^{n_c}$ are the Lagrange multipliers related to the equality constraints $c(\cdot)$ and $\mu \in \mathbb{R}^{n_g}$ are the Lagrange multipliers related to the inequality constraints $g(\cdot)$. We assume $f(\cdot)$, $c(\cdot)$ and $g(\cdot)$ are twice differential in x and p . Let $s^*(p_0) = [\chi^{*T}(p_0), \lambda^{*T}(p_0), \mu^{*T}(p_0)]^T$ denote a optimal primal-dual solution satisfying the first order KKT conditions for the parameter being p_0

$$\begin{aligned}
& \nabla_{\chi} J(\chi^*, p_0) + \nabla_{\chi} c(\chi^*, p_0) \lambda^* + \nabla_{\chi} g(\chi^*, p_0) \mu^* = 0 \\
& c(\chi^*, p_0) = 0 \\
& g(\chi^*, p_0) \leq 0 \\
& \mu^* \geq 0 \\
& g(\chi^*, p_0)^T \mu^* = 0
\end{aligned} \tag{9}$$

For the KKT conditions to be a condition of optimality a constraint qualification is necessary, e.g. the linear independence constraint qualification (LICQ):

Definition: LICQ holds for some $\chi^* \in \mathbb{R}^{n_{\chi}}$ if $\nabla c_i(\chi^*, p_0)$ for $i \in \{1, 2 \dots n_c\}$ and $\nabla g_i(\chi^*, p_0)$ for $i \in \{i | g_i(\chi^*, p_0) = 0\}$ are linearly independent.

If the second order sufficient Condition (SOSC) holds at a KKT point $s^*(p_0)$, then it is a local minimizer.

Definition: Second-Order Sufficient Condition (SOSC) holds for $s^*(p_0)$ if $d^T \nabla_{\chi}^2 \mathcal{L}(z^*(p_0)) d > 0$ for all $d \neq 0$ such that $\nabla_{\chi} c(\chi^*, p_0) d = 0$ and $\nabla_{\chi} g_i(\chi^*, p_0) d = 0$ for $i \in \{j | g_j(\chi^*, p_0) = 0, \mu_j > 0\}$.

where $\mathcal{L}(\chi, \lambda, \mu, p) = J(\chi, p) + c(\chi, p)^T \lambda + g(\chi, p)^T \mu$ is the Lagrangian of (8).

Definition: Strict Complementarity holds for a primal-dual solution $s^*(p_0)$ if $\mu^*(p_0) - g(\chi^*(p_0), p_0) > 0$

With these definitions the following result from (Fiacco (1983)) can be established

Theorem 1: (NLP sensitivity.) If $f(\cdot)$, $c(\cdot)$ and $g(\cdot)$ are at least twice differentiable in χ and p in a neighbourhood of $\chi^*(p_0)$ and p_0 , LICQ and SOSC hold for $s^*(p_0)$, and strict complementary holds then,

- $s^*(p_0)$ is a unique local solution.
- For a p in a neighborhood of p_0 there exist a unique differentiable function $s(p) = [\chi(p)^T, \lambda(p)^T, \mu(p)^T]^T$ corresponding to a unique local minima.
- For a p in a neighborhood of p_0 the set of active inequality constraints remains unchanged.

Proof: See (Fiacco (1976)).

With this result the sensitivity of $s^*(p_0)$ with respect to p can be found by solving the following linear system of equations

$$\begin{aligned}
& \begin{bmatrix} \nabla_{\chi}^2 \mathcal{L}(s^*(p_0)) & \nabla_{\chi} c(\chi^*, p_0) & \nabla_{\chi} g_+(\chi^*, p_0) \\ \nabla_{\chi} c^T(\chi^*, p_0) & 0 & 0 \\ \nabla_{\chi} g_+^T(\chi^*, p_0) & 0 & 0 \end{bmatrix} (\nabla_p s^*)^T \\
& = - \begin{bmatrix} \nabla_p^2 \mathcal{L}(s^*(p_0)) \\ \nabla_p c(\chi^*, p_0) \\ \nabla_p g_+(\chi^*, p_0) \end{bmatrix}
\end{aligned} \tag{10}$$

where $\mathcal{L}(s^*(p_0)) = J(\chi^*(p_0), p_0) + c^T(\chi^*(p_0), p_0) \lambda^*(p_0) + g^T(\chi^*(p_0), p_0) \mu^*(p_0)$ and $g_+ = [g_i]$ for $i \in \{i | g_i(\chi^*(p_0), p_0) = 0, \mu_i^*(p_0) > 0\}$.

As a result from Theorem 1, for a small Δp the optimal primal-dual solution $s^*(p_0 + \Delta p)$ can be approximated as $s^*(p_0) + \nabla_p s^* \Delta p$, using a first order Taylor expansion.

Arrival cost update through NLP sensitivity

The ideal Arrival Cost (4) is a parametric nonlinear program with the states at time $k + 1$ as the parameter $p = x_{k+1}$, and the process noise, measurement noise and states at time k as decision variables $\chi = [w_k^T \ v_k^T \ x_k^T]^T$. If problem (4) is solved for some point $p_0 = \bar{x}_{k+1}$, then the sensitivity of the optimal primal solution with respect to the parameter can be found for this point by (10), $S(p_0 = \bar{x}_{k+1}) = \nabla_p \chi^*(p_0 = \bar{x}_{k+1})^T = [S_w^T \ S_v^T \ S_x^T]^T$. Using these sensitivities the optimal primal solution can be approximated as an explicit expression of the parameter:

$$\begin{aligned}
w_k^*(x_{k+1}) & \approx w_k^*(\bar{x}_{k+1}) + S_w(x_{k+1} - \bar{x}_{k+1}) \\
v_k^*(x_{k+1}) & \approx v_k^*(\bar{x}_{k+1}) + S_v(x_{k+1} - \bar{x}_{k+1}) \\
x_k^*(x_{k+1}) & \approx x_k^*(\bar{x}_{k+1}) + S_x(x_{k+1} - \bar{x}_{k+1})
\end{aligned} \tag{11}$$

We propose to approximate the ideal arrival cost by inserting (11) into the cost function of the ideal Arrival Cost (4a) and obtain the following expression:

$$\begin{aligned}
& \Gamma_{k+1}(x_{k+1}) = \\
& [w_k^*(\bar{x}_{k+1}) + S_w(x_{k+1} - \bar{x}_{k+1})]^T Q_k^{-1} [w_k^*(\bar{x}_{k+1}) + S_w \\
& (x_{k+1} - \bar{x}_{k+1})] + [v_k^*(\bar{x}_{k+1}) + S_v(x_{k+1} - \bar{x}_{k+1})]^T R_k^{-1} \\
& [v_k^*(\bar{x}_{k+1}) + S_v(x_{k+1} - \bar{x}_{k+1})] + [\check{x}_k - x_k^*(\bar{x}_{k+1}) - S_x \\
& (x_{k+1} - \bar{x}_{k+1})]^T \Pi_k^{-1} [\check{x}_k - x_k^*(\bar{x}_{k+1}) - S_x(x_{k+1} - \bar{x}_{k+1})]
\end{aligned} \tag{12}$$

This expression can be rearranged, and upon removing all terms that are not a function of the decision variable x_{k+1} we get,

$$\Gamma_{k+1}(x_{k+1}) = x_{k+1}^T \Pi_{k+1}^{-1} x_{k+1} - 2 \check{x}_{k+1}^T \Pi_{k+1}^{-1} x_{k+1} \tag{13}$$

where

$$\Pi_{k+1}^{-1} = S_w^T Q_k^{-1} S_w + S_v^T R_k^{-1} S_v + S_x^T \Pi_k^{-1} S_x \tag{14}$$

and

$$\begin{aligned}
\check{x}_{k+1} & = \bar{x}_{k+1} - \Pi_{k+1} [S_w^T Q_k^{-1} w_k^*(\bar{x}_{k+1}) \\
& + S_v^T R_k^{-1} v_k^*(\bar{x}_{k+1}) + S_x^T \Pi_k^{-1} (x_k^*(\bar{x}_{k+1}) - \check{x}_k)]
\end{aligned} \tag{15}$$

These two expressions can be used to update the prior. Our method differs from the EKF and QR-factorization approaches in how inequalities are handled in the update. In the EKF and QR-factorization methods the inequalities are not considered. In our proposed method the sensitivity of the solution manifold of the ideal arrival cost with respect to the states at time $k + 1$ is used. This means the strongly active

inequality constraints given by the chosen point \bar{x}_{k+1} are included in the update of the priors. If the the correct set of active inequality equations is selected, this can be expected to yield a better approximation of the full information problem.

Case Study

To test the parametric NLP-sensitivity approach for updating the arrival cost, a simulation study was conducted. A quad-tank (Raff et al. (2006)) case is considered. The quad tank system consists of four interconnected tanks as can be seen in figure 1. The dynamical model describing the quad tank is given as

$$\begin{aligned} \frac{dx_1}{dt} &= -\frac{a_1}{A_1} \sqrt{2g(x_1 + w_3)} + \frac{a_3}{A_1} \sqrt{2g(x_3 + w_5)} + \frac{\gamma_1 + w_1}{A_1} u_1 \\ \frac{dx_2}{dt} &= -\frac{a_2}{A_2} \sqrt{2g(x_2 + w_4)} + \frac{a_4}{A_2} \sqrt{2g(x_4 + w_6)} + \frac{\gamma_2 + w_2}{A_2} u_2 \\ \frac{dx_3}{dt} &= -\frac{a_3}{A_3} \sqrt{2g(x_3 + w_5)} + \frac{1 - (\gamma_2 + w_2)}{A_3} u_2 \\ \frac{dx_4}{dt} &= -\frac{a_4}{A_4} \sqrt{2g(x_4 + w_6)} + \frac{1 - (\gamma_1 + w_1)}{A_4} u_1 \end{aligned} \quad (16)$$

The states $x = [x_1, x_2, x_3, x_4]^T$ represent the liquid level in the the tanks. The inputs $u = [u_1, u_2]^T$ are the pump flow rates. The process noise $w = [w_1, w_2, w_3, w_4, w_5, w_6]^T$ represents two types of uncertainty/disturbances to the system. w_1 and w_2 represents uncertainty in the flow rate split fractions γ_1 and γ_2 . w_3, w_4, w_5 and w_6 represents uncertainty in the pressure drop over the outlet valves of the tanks.

For this case study it is assumed that the liquid level in the two bottom tanks are measured, ie.

$$\begin{aligned} y_1 &= x_1 + v_1 \\ y_2 &= x_2 + v_2 \end{aligned} \quad (17)$$

where $v = [v_1, v_2]^T$ is the measurement noise. For the simulations the measurement and process noise are randomly generated. w_3, w_4, w_5, w_6, v_1 and v_2 are all drawn from a standard normal distribution, while w_1 and w_2 are uniformly drawn to be realized as the discrete values $\{-0.15, 0, 0.15\}$.

Moving Horizon Estimator setup

A moving horizon of 30 seconds was used, with a sampling time of 10 seconds. Orthogonal collocation was used to discretize the dynamic model, using third order Gauss-Radau polynomials per finite element. The initial arrival cost was selected as $\Pi_0 = I_{n_x}$ and $\check{x}_0 = [10, 5, 8, 8]^T$. The measurement noise covariant matrix was selected as $R = I_{n_y}$, and the process noise covariance matrix was selected as $Q = \text{diag}[0.015, 0.015, 1, 1, 1, 1]$. The nonlinear programs were solved using Ipopt (Wächter and Biegler (2006)) and embedded using JuMP (Dunning et al. (2017)). For the Extended Kalman Filter update of the arrival cost a forward Euler discretization was used to find the model Jacobians. As it is

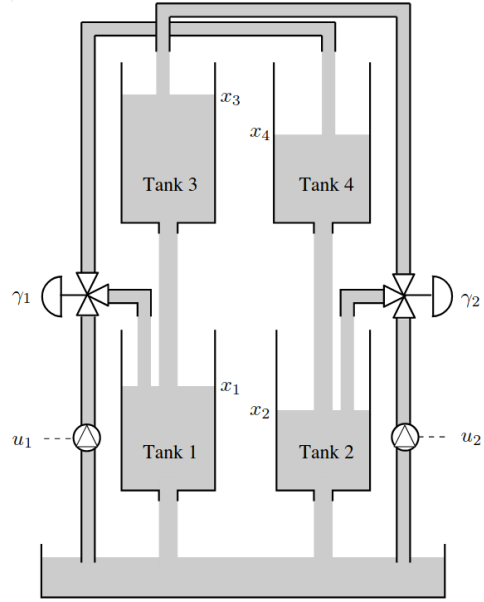


Figure 1: A schematic figure of the quad-tank process (Raff et al. (2006)).

assumed that the uncertain variables w_1 and w_2 can only take on values between -0.15 and 0.15 these bounds are implemented as inequality constraints in the MHE.

Simulation setup

For simulating the system 10 sets of random noise are generated for the process and measurement noise. The system was simulated for 1500 seconds with a sampling time of 10 seconds, leading to a total of 150 iterations for every simulation run. A tracking multi-stage scenario model predictive controller (Lucia et al. (2013)) (msMPC), tracking the levels in tank 1 and 2 to be 10 cm, was used to compute the control actions u for the simulation. True state feedback was used for the msMPC so that the state trajectories in the simulation would not be dependent on the estimates of the MHE estimators. This was done so that the two different arrival cost methods could be more accurately compared with each other and the full information estimation problem.

Simulation Results

The result section is divided into two parts. In the first part the estimation results of the moving horizon estimator with the parametric NLP sensitivity are presented. In the second part the parametric NLP sensitivity method is compared with the EKF approach to see which is better at approximating the full information problem.

Sensitivity based arrival cost MHE

In Figure 2 the simulation results for the Moving Horizon Estimator with the proposed Arrival Cost update method is shown for one set of uncertainty. The estimator is capable of recreating the true state trajectories of x_1 and x_2 . This can be seen as the estimates are closer to the true states than both

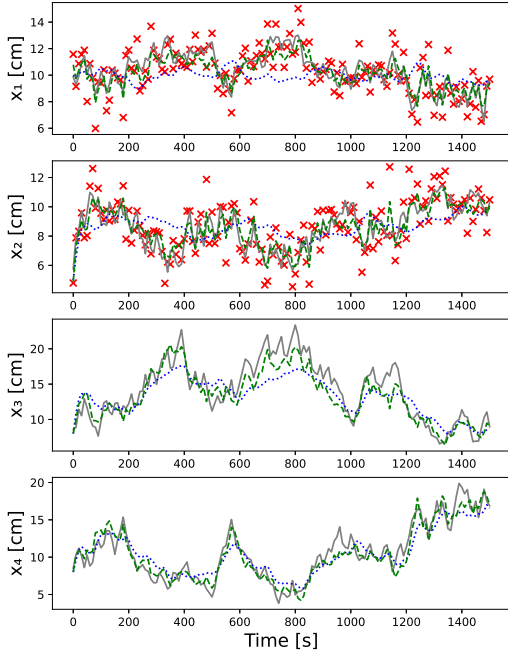


Figure 2: Estimation results using the parametric NLP sensitivity method for the arrival cost update. The true state trajectories are given as the solid grey line. The estimated state trajectories are given as the dashed green line. The blue dotted line is the deterministic state trajectory. The red crosses are the measured states.

the measured and the open loop state trajectories. The MHE also estimates the unmeasured states x_3 and x_4 to a certain degree, however at some points there is a small off-set.

This is an expected results as perfect information of the noise characteristics was assumed in tuning the MHE. Especially considering the noise for v_1, v_2, w_3, w_4, w_5 and w_6 were assumed to be normal distributed. Since it is assumed the uncertain parameters w_1 and w_2 can only take on values within a certain bound it is reasonable to expect the constraints on them improved the estimation as well. It should be noted that because the process noise is not additive, selecting its covariance matrix as the tuning matrix Q is not the best strategy to achieve the MAP estimate due to the nonlinear propagation of the uncertainty through the process model. Tuning strategies similar to those found in (Valappil and Georgakis (2000)) (Elsheikh et al. (2021)) (Tuveri et al. (2021)) could perhaps be used to improve the performance.

Arrival Cost Comparison

We compare our approach to updating the arrival cost with the standard approach of using an EKF. 10 simulations were conducted with different noise realization for both arrival cost update schemes as well as the full information problem. To evaluate which scheme is the best at approximating the

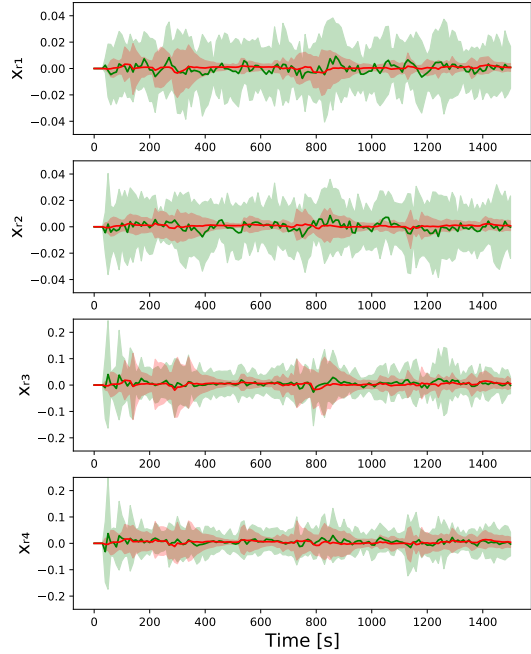


Figure 3: The mean approximation residuals for the estimated states. The EKF approach is shown in green, and the parametric sensitivity scheme in red. The shaded area represents the mean ± 2 times the standard deviation.

full information problem, residuals between the full information estimates and the MHE estimations are calculated as

$$x_r = \hat{x}_{Full} - \hat{x}_{MHE} \quad (18)$$

x_r denotes the approximation residuals, \hat{x}_{Full} is the vector of filtered full information estimates $[\hat{x}_{0|0,Full}, \dots, \hat{x}_{N|N,Full}]$ and \hat{x}_{MHE} is the vector of filtered MHE estimates $[\hat{x}_{0|0,MHE}, \dots, \hat{x}_{N|N,MHE}]$. The mean and variance of the approximation residuals were calculated for the 10 simulations and plotted in figure 3.

The desired result is that the mean approximation residual should be zero, as no estimation bias would be introduced by the sliding window approximation. However, due to the small amount of simulations being conducted (10), some deviation from the zero mean is to be expected. As seen in figure 3 both arrival cost schemes have a mean close to zero, however the parametric sensitivity scheme sticks closer to zero for all 4 states, indicating that the sliding window approximation does not introduce a bias.

In figure 3 we see that the variance in the approximation residuals are larger for the EKF method compared to the proposed parametric sensitivity method. This is especially true for the estimates of the measured states x_1 and x_2 . It should be noted that the approximation for the measured states are better for both methods compared to the unmeasured states x_3 and x_4 . This means that in this case the proposed

parametric sensitivity method seems to be a better approach to approximate the full information problem, and therefore also a better estimator. One of the reasons for the improved performance is likely that the linearization which takes place in the parametric sensitivity scheme is around a smoothed estimate and not a filtered estimate. One can assume that the smoothed MHE estimate is closer to the optimal smoothed full information estimate than the filtered estimate, meaning the linearization should be better. This philosophy is what is the basis for the smoothed EKF scheme (Robertson et al. (1996)) and the QR-based method (Kühl et al. (2011)), meaning the proposed parametric sensitivity method should have similar performance to these methods.

Discussion and Conclusion

One way in which our proposed method differs from previous methods in literature (see (Elsheikh et al. (2021)) for review) is the treatment of the inequality constraints in the ideal arrival cost formulation (4). In the other methods described in literature inequality constraints of the MHE are not included in the update of the Arrival Cost. Our proposed method includes the strongly active inequalities in the update rule. If this improves the estimate has not been investigated and is to be studied in future work.

In this paper a new method for calculating the arrival cost for a MHE estimator was presented and applied to a quad-tank case study. The new method is based on parametric nonlinear programming concepts and was shown in our case study to outperform the extended Kalman filter approach at approximating the full information problem. As future work a more in depth analysis of how inequality constraints affect the update should be looked into, as well comparing with other arrival cost update schemes.

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