

HANDLING INTERACTIVE SYSTEMS IN PRIMAL-DUAL FEEDBACK-OPTIMIZING CONTROL

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Abstract

This paper aims to solve a steady-state optimization problem with simple feedback controllers using the Primal-Dual approach. Compared to solving the full original optimization (standard real-time optimization (RTO)), this approach controls the constraints without the need of an explicit model, and has less computation time. The main idea is to control the constraints in an upper slow layer by manipulating the Dual variables (Lagrange multipliers). Given values of dual variables, the problem in the lower layer is unconstrained with the Lagrange function as the cost function, and the Primal variables (inputs) as decision variables. The solution can be obtained by solving the equation set that results when the gradient of the cost is equal to zero. It is possible to solve this equation set using decentralized feedback control for a weakly interactive system where the pairing of Primal variables and the gradient is obvious. This results in a decomposition of the optimization problem which may have significant advantages for practical implementation. However, the main focus in this paper is to study interactive systems where the pairing is not obvious. In such cases, we may use the alternative approach where we solve the equation set analytically or numerically to find the Primal variables. Both strategies are applied to a continuously stirred tank reactor. The simulation results for this case study show that both approaches obtain similar performance.

Keywords

Feedback-optimizing control, Real-time optimization, Production optimization

Introduction

Real-time optimization (RTO) deals with the steady-state economic optimization of the entire plant based on a detailed process model. As input, it needs an estimate of the present state (including constraints) and as output (results from the steady state optimization), it gives setpoints to the control layer. The RTO-layer is operating at a slow time scale (often around an hour) and because disturbances may affect the operation on a faster time scale, it is desirable to put some of the optimization into the control layer, so that at least the control layer moves the inputs in the right economic direction when there are disturbances. This is the idea of feedback-optimizing control (Morari et al., 1980), which aims at translating optimization objectives into control objectives. A comprehensive review of RTO as a feedback control problem is given by Krishnamoorthy and Skogestad (2022).

In this paper, the starting point is that a constrained optimization problem can be translated into an unconstrained optimization problem using Lagrangian/Dual relaxation (see Uzawa (1960); Rantzer (2009); Krishnamoorthy (2021); Dirza et al. (2021); Krishnamoorthy and Skogestad (2022),

and Fig. 1). The formulation suggested by Dirza et al. (2021) enables automatic active constraint region switching.

The main idea is to control the constraints in the upper (slow timescale) layer by manipulating the Dual variables (Lagrange multipliers). This is called central constraint controller in this work. These may be controlled using simple single-loop I-controllers, one for each constraint. For inequality constraint, the I-controllers are complemented with selectors, one for each I-controller, in order to satisfy the complementary condition. Because of this complementary condition, there is only one single-loop pairing choice. With given values for dual variables, the problem in the lower layer is unconstrained with Lagrange function as the objective, and the Primal variables (Inputs) as the decision variables. For fixed values of dual variables, the solution can be obtained by solving the equation set where the gradient of Lagrange function is equal to zero, in order to satisfy the stationary condition. It is possible to solve this equation set using feedback control to translate the entire optimization problem into pure feedback control problem. The controller is also called gradient controller. For a weakly interactive system, the pair-

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require a dynamic model and in many cases a good pairing simply does not exist.

Alternative Strategy: Equation Solver

In this paper, we instead consider a general strategy using an equation solver to find \mathbf{u} such that $\nabla_{\mathbf{u}} \hat{\mathcal{L}}(\mathbf{u}, \hat{\mathbf{d}}, \boldsymbol{\lambda}) = \mathbf{0}$ as shown in Fig. 2.

We formulate the equation solver as a steady-state optimization problem that drives each element of vector $\nabla_{\mathbf{u}} \hat{\mathcal{L}}(\mathbf{u}, \hat{\mathbf{d}}, \boldsymbol{\lambda})$ to zero. This strategy assumes that the decision variable, \mathbf{u} , still explicitly appears in this function. This assumption is satisfied for our case study. Therefore, we define the objective function as a norm of the vector $\nabla_{\mathbf{u}} \hat{\mathcal{L}}(\mathbf{u}, \hat{\mathbf{d}}, \boldsymbol{\lambda})$. This results in the following (unconstrained-) optimization problem,

$$\min_{\mathbf{u}} \quad \|\nabla_{\mathbf{u}} \hat{\mathcal{L}}(\mathbf{u}, \hat{\mathbf{d}}, \boldsymbol{\lambda})\| \quad (5a)$$

where $\nabla_{\mathbf{u}} \hat{\mathcal{L}}(\mathbf{u}, \hat{\mathbf{d}}, \boldsymbol{\lambda})$ is the estimated gradient of Lagrange function that can be obtained from eq. (3).

Given $\boldsymbol{\lambda}$, estimated differential state, $\hat{\mathbf{x}}$, and estimated disturbance, $\hat{\mathbf{d}}$, we solve unconstrained problem (5) in order to obtain the calculated optimal input \mathbf{u}^* .

Remark 1: To estimate the steady-state gradients in (3), which is now inside the equation solver, we use the model-based gradient estimation framework (see Srinivasan et al. (2011)).

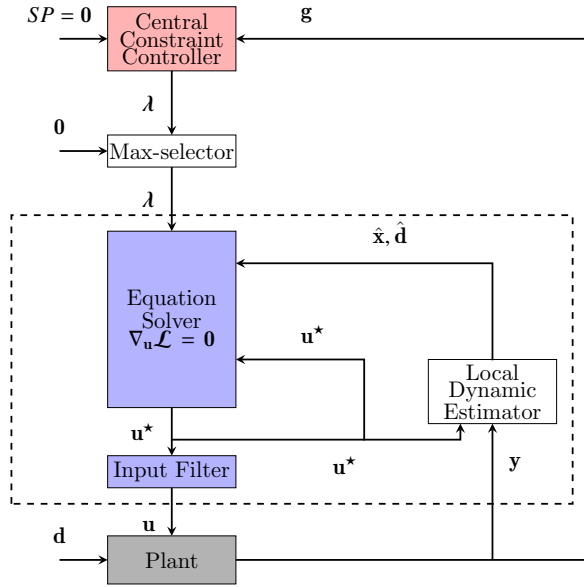


Figure 2: Block diagram of the primal-dual scheme with an equation solver and input filter in the lower layer.

Input filter: In order to ensure a smooth (not too aggressive) implemented input, \mathbf{u} , we suggest to use a first-order input filter combined with input rate limiter as follows,

$$\mathbf{u}(k) = \mathbf{u}(k-1) + \mathbf{K}_{\mathbf{u}} \Delta \mathbf{u}(k) \quad (6)$$

where $\Delta \mathbf{u}^{\min} \leq \mathbf{K}_{\mathbf{u}} \Delta \mathbf{u}(k) \leq \Delta \mathbf{u}^{\max}$, $\Delta \mathbf{u}(k) = \mathbf{u}^*(k) - \mathbf{u}(k-1)$, the diagonal matrix $\mathbf{K}_{\mathbf{u}} < \mathbf{I}$ is the filter gain. Note that $\mathbf{K}_{u_i} = 1/(1 + \tau_{f_i}/\Delta t)$, where τ_{f_i} is the filter time constant, and Δt is the sampling time. For the case with relatively mild

disturbance, the input rate limiters may not be necessary. The input filter may be needed to achieve smooth changes of the inputs to the plant. It may also be needed to achieve closed-loop stability of the system because of the feedback from the dynamic plant to the equation solver through the estimator block.

Case Study

In this section, we apply both approaches to a continuously stirred tank reactor (CSTR) used in Economou et al. (1986), and Jäschke and Skogestad (2011). This CSTR has a reversible exothermic reaction $A \rightleftharpoons B$. A system of coupled ordinary differential equations model the process (i.e., Reactant mass balance, Product mass balance, and Energy balance, respectively):

$$\frac{dC_A}{dt} = f_1(\mathbf{x}, \mathbf{u}, \mathbf{d}) = \frac{F}{V} (C_{A,in} - C_A) - r \quad (7)$$

$$\frac{dC_B}{dt} = f_2(\mathbf{x}, \mathbf{u}, \mathbf{d}) = \frac{F}{V} (C_{B,in} - C_B) + r \quad (8)$$

$$\frac{dT}{dt} = f_3(\mathbf{x}, \mathbf{u}, \mathbf{d}) = \frac{F}{V} (T_{in} - T) + \frac{-\Delta H_{rx}}{\rho c_p} r \quad (9)$$

The states \mathbf{x} , which are C_A , C_B , and T denote the concentrations of the two components in the reactor and the reactor temperature, whereas the independent variables $C_{A,in}$, $C_{B,in}$, T_{in} and F are the inlet concentrations, the feed inlet reactor temperature, and the feed rate. Further, V is the reactor volume, $-\Delta H_{rx}$ is the reaction enthalpy, ρ is the density, and c_p is the heat capacity. The reaction rate r is defined by $r = k_1 C_A - k_2 C_B$, where $k_1 = C_1 e^{-\frac{E_1}{RT}}$ and $k_2 = C_2 e^{-\frac{E_2}{RT}}$. Moreover, C_1 and C_2 are the Arrhenius factors for the reaction constants k_1 and k_2 . Further, E_1 and E_2 are the activation energy, and R is the ideal gas constant.

This process has two MVs (primal variables), the inlet temperature and the feed rate $\mathbf{u} = [u_1 \ u_2]^T = [T_{in} \ F]^T$. The expected disturbances are the inlet concentration of the two components $\mathbf{d} = [d_1 \ d_2]^T = [C_{A,in} \ C_{B,in}]^T$. The objective of this CSTR operation is to maximize the throughput rate F and the product concentration C_B while minimizing the heating cost associated with the inlet temperature T_{in} . For this process, we have two constraints ($n_{\mathbf{g}} = 2$), that are associated with maximum reactor temperature T^{\max} , and minimum product concentration C_B^{\min} . The steady-state optimization problem is formulated as,

$$\min_{T_{in}, F} \quad J = -F - p_{c_B} C_B + (p_{T_{in}} T_{in})^2 \quad (10a)$$

$$\text{s.t.} \quad g_1 : \frac{T}{T^{\max}} - 1 \leq 0, \quad (10b)$$

$$g_2 : 1 - \frac{C_B}{C_B^{\min}} \leq 0 \quad (10c)$$

where p_{c_B} is 2.009, and $p_{T_{in}}$ is 1.657×10^{-3} .

The gradient of the Lagrangian function of problem (10) is as follows.

$$\nabla_{\mathbf{u}} \mathcal{L} = \begin{bmatrix} -p_{c_B} \nabla_{T_{in}} C_B + 2p_{T_{in}} T_{in} \\ -1 - p_{c_B} \nabla_F C_B \end{bmatrix} + \begin{bmatrix} T^{\max-1} \nabla_{T_{in}} T & -C_B^{\min-1} \nabla_{T_{in}} C_B \\ T^{\max-1} \nabla_F T & -C_B^{\min-1} \nabla_F C_B \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \quad (11)$$

where λ_1 and λ_2 are Lagrange multipliers associated with constraint g_1 and g_2 , respectively. We see that the inlet temperature ($u_1 = T_{in}$) appears explicitly in (11) but it may seem that the expression is independent of the throughput rate ($u_2 = F$). However, it actually appears in expressions for the gradients, for example, $\nabla_{T_{in}} T = \frac{F}{V} \left(\frac{\partial f_3}{\partial T} \right)^{-1}$. Thus, it's possible to solve $\nabla_{\mathbf{u}} L = 0$ with respect to \mathbf{u} using an equation solver.

As disturbance, the inlet concentration of component A $C_{A,in}$ varies in the range 0.28 to 0.42 mol/L, and the inlet concentration of component B $C_{B,in}$ varies in the range 0.0253 to 0.0493 mol/L. For this system, we have at most 4 (2^{n_g}) constraint regions (but only two appear for the disturbances considered): Fully unconstrained (never), Only \mathbf{g}_1 active (never), Only \mathbf{g}_2 active (R-I), and both \mathbf{g}_1 and \mathbf{g}_2 are active (R-II).

For the purpose of this work, we compare the performance of the following approaches:

- C1: Steady state optimization solver assuming known disturbances and constraint values (baseline).
- C2: Primal-Dual with PI Feedback control in the lower/gradient layer.
- C3: Primal-Dual with Equation solver in the lower/gradient layer.

To obtain the ideal steady-state optimal solutions as the baseline (C1), we solve the 'whole' steady state optimization problem (10) at every seconds. We assume that the disturbances are perfectly known in this approach, making it unrealistic in practice.

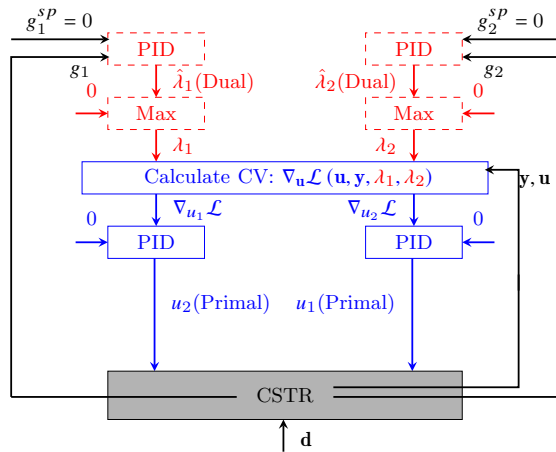


Figure 3: Control structure based on RGA-based pairing, where the gradient controllers has off-diagonal pairing. Red lines and blocks represent controllers, logic and information that belong to Central Constraint Controller(s). Blue lines and blocks belong to Gradient Controller(s).

To solve problem (10) using Primal-Dual approach, the upper (slow) feedback control layer computes the dual variables (Lagrange multipliers) whereas lower (fast) layer should find the primal variable \mathbf{u} which make $\nabla_{\mathbf{u}} L = \mathbf{0}$ in (11). As discussed, this may be done using feedback control

or equation solver. We first consider the use of two single-loop feedback controllers. The system is interactive, and from RGA analysis (Dirza and Skogestad, 2022b), it turns out that we need to use the *off-diagonal pairing*. Fig. 3 shows the resulting control structure, where $\nabla_{u_1} L \longleftrightarrow u_2$ and $\nabla_{u_2} L \longleftrightarrow u_1$, is used to drive the system to the steady-state optimal solution. Appendix A provides the detail of both central constraint controllers and gradient controllers used in this approach. To summarize, we find that the pairing in the gradient controllers layer is not obvious, and it may be time consuming if we have large numbers of possible pairing.

To obtain $\nabla_{\mathbf{u}} L = \mathbf{0}$ in (11) with the Equation Solver (C3), we use the norm of the gradient of Lagrange function (11) as the objective function of the unconstrained steady-state optimization problem (5). To obtain implemented input, we use an input filter with $\Delta t = 1$ seconds and $\mathbf{K}_{u_i} = 0.01$ corresponding to $\tau_{f_i} = 99$ seconds. In this case study, we do not use any input rate limiter as we assume a relatively mild disturbance.

We use an Extended Kalman Filter (EKF) with augmented differential states and parameters/disturbances in the (local-) dynamic estimator (Simon, 2006). To estimate the steady-state gradients in (3), we use the model-based gradient estimation framework proposed in Krishnamoorthy et al. (2019). PI controllers are tuned using the SIMC (Simple Internal Model Control) tuning method introduced by Skogestad (2003). The local gradient controllers of C2, and the central constraint controller of both C2 and C3 are designed with a sampling time of 1 second.

The plant simulator is developed using the CasADi ver. 3.5.1 toolbox (Andersson et al. (2019)) in MATLAB R2019b, and is simulated using the IDAS integrator. The resulting NLP problems, i.e., problem (10) and (5), are solved using IPOPT v3.12.2. The simulations are performed on a 2.11 GHz processor with 16 GB memory for 80 minutes (4800 seconds) simulation time.

Fig. 4 compares the simulation results of the three approaches. This result indicates that both Primal-Dual with Feedback control (C2) and Primal-Dual with Equation solver (C3) can reach the optimal steady-state solution, and their trajectories are very similar. This similarity also leads to similar objective trajectories shown in Fig. 5. Therefore, the accumulated cost is also very similar, i.e., 31034.30 [price unit] for C2 and 31034.36 [price unit] for C3. These results confirm that both Primal-Dual with Equation solver (C3) and Primal-Dual with Feedback control (C2) obtain similar performance in handling a (highly-) interactive system such as CSTR in this work.

Table 1: Average Computation Time

Approaches	Comp. Time [Sec]
Full optimization solver (C1)	4.4807×10^{-2}
Primal-Dual-Feedback control (C2)	0.2406×10^{-2}
Primal-Dual-Equation solver (C3)	1.4354×10^{-2}

Table 1 depicts the average computation of each approach we consider. This result shows that the use of equation solver (C3) may need more computation time to provide the optimal

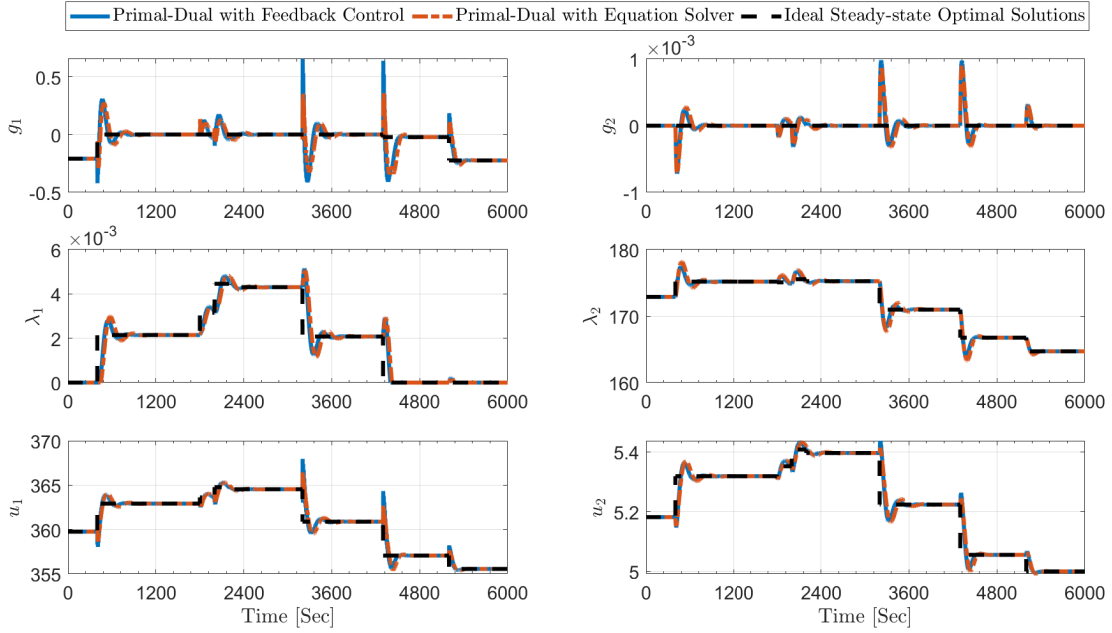


Figure 4: Comparison of Primal-Dual with Feedback control (C2), Primal-Dual with Equation Solver (C3), and Ideal Steady state Optimal Solutions (C1). The constraints are compared on the top plots. The associated dual variables are compared on the middle plots. The primal variables (implemented inputs) are compared on the bottom plots.

input, \mathbf{u}^* , than the use of feedback control (C2). However, it is still much faster than solving the full optimization problem (C1). This happens because solving an (unconstrained-) optimization problem is usually easier than solving constrained optimization problem.

Discussion

The main advantages of solving the equations using feedback controllers (C2) are simpler implementation and shorter computational times. The I-controllers also do the filtering of the input which needs to be added when the equations are solved numerically (C3). However, the use of feedback controllers (C2) is less general as it does not work for highly interactive systems because we may get instability because of changes in the sign of the gain (as may be identified by negative steady state RGA elements). For weakly interactive systems, where the expressions for $\nabla_{\mathbf{u}}\mathcal{L}$ are decoupled or weakly coupled, we may also use local solvers for the equations $\nabla_{\mathbf{u}}\mathcal{L} = \mathbf{0}$. This may allow for the subsystems being optimized independently and with different rates. This may have significant advantages in practical implementations.

In terms of convergence, a nice theorem from Arrow et al. (1958) proves the convergence of the proposed approach for the equality constraint case when the Lagrangian function \mathcal{L} is strictly convex in \mathbf{u} . If \mathcal{L} is not strictly convex, for example J depends linearly on the input \mathbf{u} , the approach with an equation solver will not work if the resulting gradient equations (3) do not depend explicitly on \mathbf{u} . In such cases we may sometimes use a trick, for example, adding the square of the constraint to the Lagrange function Krishnamoorthy (2021). If we instead use a controller in the lower layer, such tricks may not be necessary as we have found numerically in other case studies Dirza et al. (2021).

Conclusion

In this paper, we compares the primal-dual approach with the use of equation solver for the unconstrained part (problem (5)) with the primal-dual with simple feedback controllers for the unconstrained part (gradient controllers). The problem with using simple feedback controllers is that the expected diagonal pairing is not necessarily the best, In fact, for the CSTR case study we found that we have to use the off-diagonal pairing in order to avoid pairing on negative steady-state RGA-values. The general strategy with the use of equation solver for the unconstrained part plus a simple dynamic filter is therefore recommended for interactive processes. The advantage of the general strategy compared to solving the full optimization problem is first that the constraints maybe controlled in the upper layer, without needing to use and update the model. Second, the computation time can be significantly shorter.

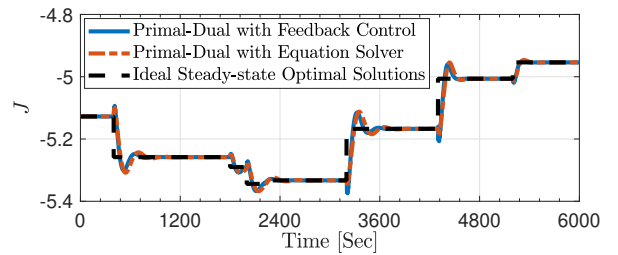


Figure 5: The cost comparison of Primal-Dual with Feedback control (C2), Primal-Dual with Equation solver (C3), and Ideal Steady state Optimal Solutions (C1).

Acknowledgments

The authors gratefully acknowledge the financial support

from SUBPRO, which is financed by the Research Council of Norway, major industry partners and NTNU.

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Appendix A. Controllers

For central constraint controllers (CCC), we consider the following I-controllers for $i = 1, 2$,

$$\lambda_i^{k+1} = \max \left[0, \lambda_i^k - \frac{1}{K_{\lambda_i} \tau_{\lambda_i}} g_i \right] \quad (12)$$

For gradient controllers (GC), we consider the following I-controllers.

$$u_1^{k+1} = u_1^k - \frac{1}{K_{u_1} \tau_{u_1}} \nabla_{u_2} \mathcal{L}(\mathbf{u}, \hat{\mathbf{d}}, \boldsymbol{\lambda}) \quad (13)$$

$$u_2^{k+1} = u_2^k - \frac{1}{K_{u_2} \tau_{u_2}} \nabla_{u_1} \mathcal{L}(\mathbf{u}, \hat{\mathbf{d}}, \boldsymbol{\lambda}) \quad (14)$$

where $\boldsymbol{\lambda} = [\lambda_1 \quad \lambda_2]^\top$. Further, $K_{u_1}, K_{u_2}, K_{\lambda_1}$ and K_{λ_2} are the step response gains, and $\tau_{u_1}, \tau_{u_2}, \tau_{\lambda_1}$ and τ_{λ_2} are the desired time constants (tuning parameters). Table 2 provides the values we use in this case study.

Table 2: *Controllers Parameters*

Parameters	CCC 1	CCC 2	GC 1	GC 2
	K_{λ_1}	K_{λ_2}	K_{u_1}	K_{u_2}
Gains	−189.9965	0.0002	−0.1411	−0.1437
	τ_{λ_1}	τ_{λ_2}	τ_{u_1}	τ_{u_2}
Time constant	40	40	1	1