QUADRATIC APPROXIMATION IN PRICE-BASED COORDINATION OF CONSTRAINED SYSTEMS-OF-SYSTEMS

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Abstract

Price-based coordination can be used to coordinate (physically coupled) systems-of-systems (SoS) consisting of competing subsystems that are not willing to share every necessary detail that is required to compute the centralized solution. One example of such a SoS is a large integrated chemical production site that consists of a central site coordinator and individually optimized processing plants, operated by different business units and coupled by shared resource networks. One drawback of price-based coordination is its slow rate of convergence accompanied by many communication rounds between the coordinator and subsystems, which is the bottleneck in applying price-based strategies to practical applications. The rate of convergence can be improved with price updates computed from a quadratic approximation (QA) of the responses of the subsystems to price incentives announced by the coordinator. However, this update strategy relies on the assumption of unconstrained individual subsystems. In this contribution, we propose an algorithm for updating the price based on QA with a simple heuristic that uses subgradient updates, if one or more constraints of the subsystems become active. The advantages of the proposed algorithm are demonstrated via a numerical example.

Keywords

Price-based coordination; Systems-of-systems; Quadratic approximation; Shared resource allocation; Confidential communication

Introduction

Large integrated chemical production sites can be regarded as physically coupled SoS, since there are usually many processing plants on site that are managed and optimized by different competing business units or companies, who pursue their own economic interests. To operate the overall site resource and energy efficiently, a central coordination is required, because the optima of the individual subsystems are not necessarily coherent with the site-wide optimum. Employing a centralized optimization that coordinates the subsystems' decisions, however, is not always possible due to technical or managerial reasons. One of the reasons is the lack of information on the site level, e.g., because the subsystems are not willing to disclose their individual cost and constraint information for the sake of confidentiality.

For the distributed coordination of SoS there exists a variety of available techniques in order to recover the overall optimum (see e.g. Lasdon (2013)). The techniques mainly differ in the degree of autonomy they grant to the subsystems, the quality of shared information, and the direction of the communication. For instance, in proximal gradient methods additional reference signals are broadcasted (Parikh and Boyd, 2013) or in cutting plane methods the (aggregate) cost function values need to be known Gatsis and Giannakis (2013). Both might not be desirable in some applications. Distributed coordination strategies have attracted an increased interest in different fields, for instance in solving plant-wide MPC problems (Cheng et al., 2007), management of oxygen distribution grids (Martí et al., 2013), coordination of populations of PEV drivers (Grammatico et al., 2015) or the coordination of unmanned vehicles (Cao et al., 2013).

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Price-based coordination is a technique based on market theory that can be used to recover the centralized solution from decentralized ones under certain assumptions, while preserving a high level of confidentiality of to the subsystems. It is based on a Walrasian auction, in which there exists an equilibrium price such that the supply and the demand of goods are equal and no agent that takes part in the auction can further increase its profit (Walker, 1987). In the auction a central coordinator sends prices for the goods (shared resources) to the subsystems, which–after they have optimized based on the announced price—report their shared resource utilization to the coordinator (see Figure 1). The coordinator adjusts iteratively the prices in the so-called tatônnement process until the equilibrium price vector is found. The tatônnement process is known for a slow rate of convergence, which is a drawback for practical applications. In Wenzel et al. (2016) a novel price update strategy is proposed that uses QA of the network imbalance as a function of the announced prices. This approach however, relies on the assumption of inactive constraints of the individual subsystems, which is usually not the case in industrial environments, where plants are often operated at their bounds.

In this contribution we propose an extension of the strategy described in Wenzel et al. (2016) that uses a simple heuristic in which subgradient-based updates are used once the activeness of constraints is detected. The rest of the paper is structured as follows. First, the mathematical problem formulation is given, then price-based coordination is discussed in more detail. Afterwards, the proposed algorithm is described and simulation results for a numerical example are provided. Finally, conclusions are drawn and an outlook is given.

Mathematical problem formulation

Consider N individually optimized subsystems with quadratic cost functions

$$\begin{aligned} \min_{x_i} \quad & (1/2)x_i^T P_i x_i + q_i^T x_i \\ \text{s.t.} \quad & G_i x_i \leq h_i, \end{aligned} \tag{1}$$

where $x_i \in \mathbb{R}^{n_{x_i}}$ is the state vector of the subsystem, the matrix $P_i \in \mathbb{R}^{n_{x_i} \times n_{x_i}}$ is assumed to be symmetric and positive definite ($P_i \succ 0$) and $q_i \in \mathbb{R}^{n_{x_i}}$ is a vector. The subsystem can exchange quantities with other subsystems via connection nodes. In physically coupled systems-of-systems these can be streams of material or energy. The vector of exchanged quantities with other subsystems $r_i \in \mathbb{R}^{n_{r_i}}$ is defined by a linear mapping

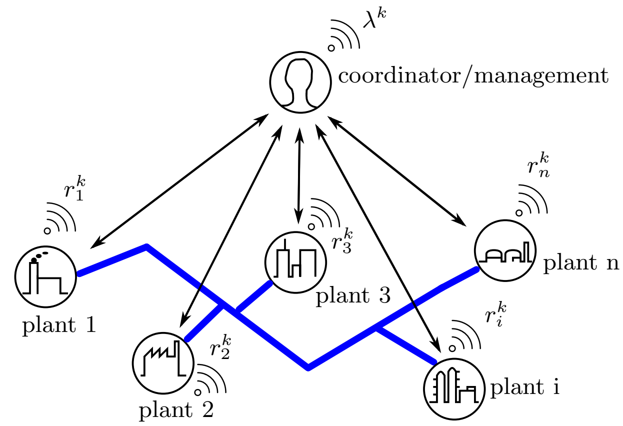


Figure 1. Schematic topology of the numerical example.

$$r_i = A_i x_i, \tag{2}$$

where $A_i \in \mathbb{R}^{n_{r_i} \times n_{x_i}}$ is a non-zero matrix. The overall optimization of the system-of-systems with N subsystems can be formulated as follows

$$\begin{aligned} \min_{x} \quad & (1/2)x^T P x + q^T x \\ \text{s.t.} \quad & Gx \leq h, \\ & \textstyle\sum_{i=1}^{N} r_i + \phi_r = 0, \end{aligned} \tag{3}$$

where $x \in \mathbb{R}^{n_x}$ is the vertical stack of the single state vectors, P and G are block-diagonal matrices constructed from P_i and G_i, and q and h are vertically stacked vectors of q_i and h_i, respectively. For the SoS it is crucial that the networks of the exchanged quantities are balanced. Note that the formulation of the network constraint requires that $n_r = n_{r_i}, \forall i$. If some of the subsystems are only connected to a subset of the available networks, the respective entries of r_i are set to zero. The offset ϕ_r represents a constant sink or source for the networks and can be interpreted, e.g., as the amount of electric power that is retrieved from the public grid.

Price-based coordination

We assume that each individual subsystem solves problem (1). We further assume that the subsystems do not share their individual cost functions, cost values, and constraints, but they share their resource utilization vector r_i. Communication is only allowed between the subsystems and the coordinator, but not between single subsystems as illustrated in Figure 1.

To apply coordination techniques to existing SoS that are individually optimized in a semi-automated fashion, possibly involving managers, engineers, or operating personnel, the change to the individual optimization problems should be minimal. This ensures that the individual costs functions maintain their structure and remain intuitive to understand, which increases the acceptance of the coordination strategy.

A distribution coordination strategy that reflects the above mentioned requirement is the use of Lagrangian relaxation. By relaxing the constraint in Eq. (3), the reformulated optimization can then be written as

$$\begin{aligned} \min_{x} \quad & \mathcal{L}(x,\lambda) \\ \text{s.t.} \quad & Gx \leq h, \end{aligned} \tag{4}$$

with the Lagrangian

$$\mathcal{L}(x,\lambda) := (1/2)x^T P x + q^T x + \lambda^T (\textstyle\sum_{i=1}^{N} r_i + \phi_r), \tag{5}$$

where $\lambda \in \mathbb{R}^{n_r}$ is the vector of Lagrange multipliers that can be interpreted as a price vector that is assigned to a particular shared resource vector r_i. Problem (4) can be distributed and solved in parallel for a given price vector λ^k, where k denotes the current iteration during the tatônnement process. The optimal decision of an individual subsystem x_i^* is defined as

$$\begin{aligned} x_i^*(\lambda^k) = \arg\min_{x_i} \quad & \mathcal{L}_i(x_i,\lambda^k) \\ \text{s.t.} \quad & G_i x_i \leq h_i. \end{aligned} \tag{6}$$

Every subsystem reports its optimal resource utilization $r_i^*(\lambda^*)$ (Eq. (2)) to the coordinator, which announces a new price vector λ^{k+1}. The simplest price update is the subgradient price update

$$\lambda^{k+1} = \lambda^k + \alpha^k \left(\textstyle\sum_{i=1}^{N} r_i^* + \phi_r \right), \tag{7}$$

with the step size parameter $\alpha^k \in \mathbb{R}$. The choice of the step size parameter α^k is not trivial, if no information about the subsystems is present. If the prices have different orders of magnitude the update step can be normalized as in Lau et al. (2007). The step size parameter needs to be small enough to ensure convergence, but from a practical point of view it should be large enough to reduce the number of communication rounds (Bertsekas, 2009). The coordinator iteratively updates the prices until the network balance is achieved and the equilibrium price vector λ^* is found.

Link between prices and network imbalance

One of the drawbacks of the tatônnement process is its slow rate of convergence. In Wenzel et al. (2016) a novel price update strategy based on QA is proposed. It relies on the assumption that around the site-wide (overall) optimum the individual subsystems are unconstrained. In this case, the residual of the network constraint for the optimization problem in Eq. (4) is a quadratic function of the price vector:

$$f_r(\lambda) = \left\| \textstyle\sum_{i=1}^{N} r_i^* + \phi_r \right\|_2^2 = \lambda^T H \lambda + s^T \lambda + t, \tag{8}$$

with the matrix $H \in \mathbb{R}^{n_\lambda \times n_\lambda}$, the vector $s \in \mathbb{R}^{n_\lambda}$ and the scalar $t \in \mathbb{R}$. Based on a minimum number of $n_{q,min} = (n_r+1)(n_r+2)/2$ points the function $f_r(\lambda)$ can be approximated. The next price vector λ^{k+1} is found be minimizing the QA of f_r (QA price update)

$$\lambda^{k+1} = \arg\min_{\lambda \in \Lambda^k} f_r(\lambda), \tag{9}$$

where Λ^k is a search space based on the distribution of points (Gao et al., 2016). The QA is updated with new information, i.e., responses of the subsystems to announced prices, until convergence is achieved. This is referred to as recursive quadratic approximation (RQA).

However, the assumption of inactive constraints around the overall optimum is not always fulfilled, especially in real world industrial environments, where it is common to operate at least some of the systems within a system-of-systems at their maximum capacity and Eq. (8) becomes piece-wise quadratic. Therefore, in the following section an extension to the approach in Wenzel et al. (2016) is proposed.

Proposed price update algorithm

A QA of Eq. (8) fails, if the subsystems are insensitive to changing price vectors λ^k, because the optimal decision of at least one subsystem is governed by active constraints in Eq. (6). Thus, we propose a simple heuristic that checks whether the responses of the subsystems change with the announced prices. If this is not the case, the algorithm switches to subgradient-based updates. Hence, Algorithm 1 is a combination of two different price update steps. At first, the initial price vector is sent to the subsystems. These perform their individual optimization based on the announced price vector, do a private update of their resource utilization r_i^*, and communicate the result to the coordinator. The coor-

Algorithm 1 Proposed price update algorithm.

Required: N, ϕ_r, k_{max}, ε_r, ε_ξ, λ^0, α^0
Initialize: $k = 0$, $n_{\mathcal{S}_\lambda} = 0$, $\mathcal{S}_\lambda$, $\Delta_\xi^0 > \varepsilon_\xi$
while $\neg$ *Convergence* **do**
  for all subsystems $i = 1 : N$ **do**
    Find $x_i^*(\lambda^k)$ by solving problem (6),
    $r_i^* \leftarrow A_i x_i^*(\lambda^k)$.
  end for
  $\xi^k \leftarrow \sum_{i=1}^N r_i^* + \phi_r$ ▷ Evaluate responses.
  if $(k == k_{max}) \vee (\|\xi^k\|_2^2 < \varepsilon_r)$ **then**
    $Convergence = True$
  end if
  if $(\Delta_\xi^k < \varepsilon_\xi) \wedge (k > 0)$ **then** ▷ see Eq. (10).
    $n_{\mathcal{S}_\lambda} \leftarrow 0$
    $\mathcal{S}_\lambda \leftarrow \varnothing$
  else
    $\mathcal{S}_\lambda \leftarrow \mathcal{S}_\lambda \cup (\lambda^k, \xi^k)$
    $n_{\mathcal{S}_\lambda} \leftarrow n_{\mathcal{S}_\lambda} + 1$
  end if
  if $(k < n_q - 1) \vee (n_{\mathcal{S}_\lambda} < n_q - 1)$ **then**
    $\lambda^{k+1} \leftarrow \lambda^k + \alpha^k \xi^k$ ▷ See Eq. (7), $\alpha^k = \alpha^0$.
  else
    Find λ^{k+1} by solving Eq. (9).
  end if
  $k \leftarrow k + 1$
end while

dinator evaluates the responses of the subsystems and checks for convergence. If convergence is not achieved, a simple heuristic is used to decide whether the next price can be found by solving Eq. (9). The heuristic evaluates for $k > 0$, whether there is a change from the last response to the current response according to

$$\Delta_\xi^k = \left\|(\xi^k - \xi^{k-1})/\xi^{k-1}\right\|_2, \ \xi^k = \textstyle\sum_{i=1}^N r_i^*(\lambda^k) + \phi_r. \tag{10}$$

If the change is smaller than the user defined value ε_ξ, the number of already found points for the QA $n_{\mathcal{S}_\lambda}$ is set to zero and the set of considered points $\mathcal{S}_\lambda$ is emptied. This is done, because the recent collected data might not be valid for the next piece-wise quadratic part of the response surface with a different set of active constraints.. If the change is large enough, the current tuple (λ^k, ξ^k) is taken into the set $\mathcal{S}_\lambda$. In the next step the algorithm evaluates the possibility to perform a QA. Two conditions have to be fulfilled. On the one hand, the minimum number of required points has to be available and on the other hand, the set $\mathcal{S}_\lambda$ must contain enough points. If one of the conditions is not met, a subgradient update is performed according to Eq. (7). If both conditions are met, then the new price vector λ^{k+1} is found by solving Eq. (9). The algorithm iterates until either the maximum number of iterations k_{max} is reached or the convergence criterion $\|\xi^k\|_2^2 < \varepsilon_r$ is fulfilled.

Table 1. Model coefficients for the numerical example.

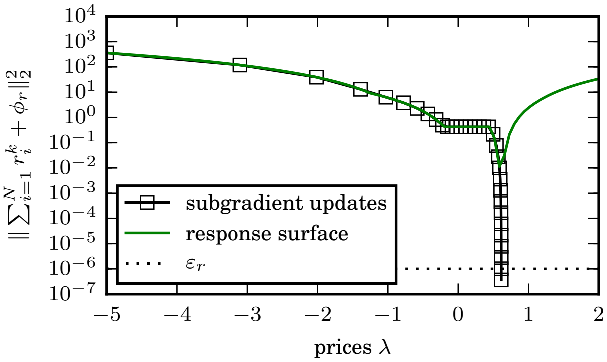
Plant	P_i	q_i	G_i	h_i	A_i
Plant 1	2.0	1.5	−1.0	−0.5	2.0
Plant 2	3.0	1.2	1.0	−0.8	1.0
Plant 3	1.0	−0.5	1.0	−0.4	2.0
Plant 4	4.0	−1.8	−1.0	−0.6	3.0
Plant 5	0.5	0.8	1.0	−1.1	0.5

Numerical example

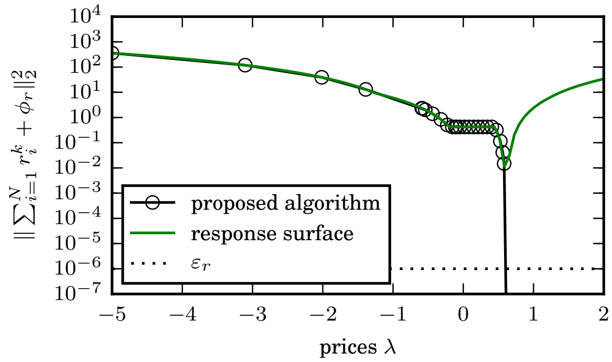
In this section, we present a numerical example of five processing plants that are connected by one shared resource as depicted in Figure 1. The shared resource can either be sent to the network ($r_i < 0$) or taken from the network ($r_i > 0$). First, the models are described in detail and the numeric settings of the simulation study are given, then a comparison of price updates based on subgradients and the proposed algorithm is done.

The coefficients of the $N = 5$ models of the form given in Eq. (1) are listed in Table 1. The maximum number of iterations is set to $k_{max} = 100$, the price update parameter $\alpha^k = 0.1$, $\forall k$, has a fixed value. Note that this value has not been optimized, but has been chosen in the range where convergence is ensured (for $\alpha^k > 0.1$) the simulation results showed diverging behavior). It is assumed that the coordinator has no knowledge about the subsystem models and constraints. The tolerance for convergence is set to $\varepsilon_r = 1 \times 10^{-6}$. For the decision of switching between QA and subgradient price updates the tolerance is adjusted to $\varepsilon_\xi = 0.1$. The external sink and source term is set to zero ($\phi_r = 0$).

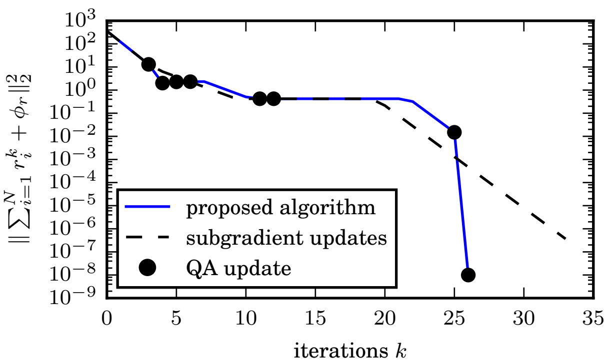
The simulation results for the coordination of the five subsystems are shown in Figure 2. The initial price is set to $\lambda^0 = -5$, in order to investigate the behavior of the algorithm for different subsets of constraints being active. This is visible in Figures 2(a) and 2(b), where the residuals of the network balance (response surface) are plotted on a log-scale against the prices λ in the interval $[-5, 2]$. It is clearly visible that, e.g., around $\lambda = 0.3$, the response surface is flat, i.e., the response of the subsystems does not change with changing prices and the right hand side of Eq. (8) reduces to a constant. The results for the classical subgradient updates are shown in Figure 2(a). The algorithm is able to find the equilibrium price, but requires numerous update steps when being close to the optimal value. In contrast, the proposed algorithm converges with few iteration when having collected enough information close to the optimal value (see Figure 2(b)). The proposed algorithm has advantages in



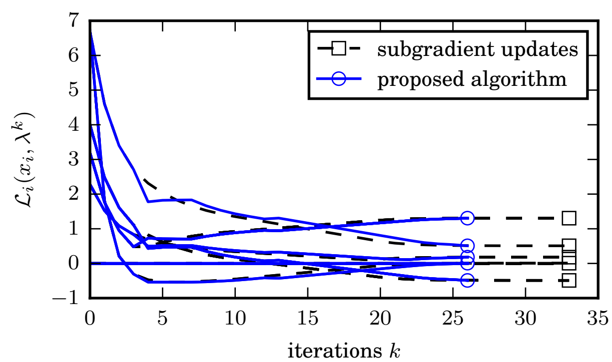
(a) Iterations of the subgradient updates on the response surface of the residuals against the prices.



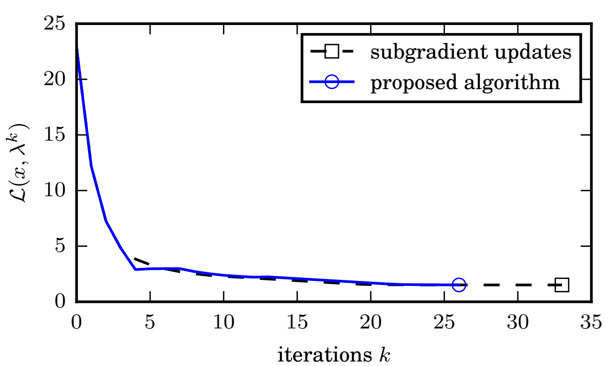
(b) Iterations of the proposed algorithm on the response surface of the residuals against the prices.



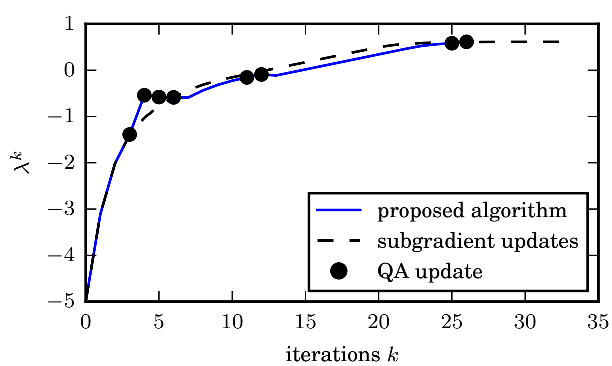
(c) Residuals of the network balance against the number of iterations. Every filled circle represents a price update based on QA.



(d) Evolution of the Lagrangian values of the single subsystems for both price updates. The squares and circles denote the termination of the coordination.



(e) Comparison of the Lagrangian values of both price updates for the system-of-systems. The squares and circles denote the termination of the coordination.



(f) Evolution of the price for the shared resource for both price updates. Filled circles represent a price update based on QA.

Figure 2. Simulation results for the numerical example.

the regions where the assumption of unconstrained subsystems holds. Hence, the proposed algorithm requires less iterations. Figures 2(d) and 2(e) show the evolution of the Lagrangian values for the single subsystems and for the system-of-systems. The results show that some subsystems increase the Lagrangian value and some decrease it. The overall Lagrangian value of the system-of-systems (Figure 2(e)) is gradually decreased. Note, that the feasibility is only given upon convergence, i.e., the lower cost (profit) of some of the subsystems at the beginning of the auction is only of hypothetical nature. Figure 2(f) shows the evolution of the price of the shared resource against the iterations. Both algorithms have similar trajectories within the first 25 iterations. This is because the subsystems are at their bounds most of the time and only a few update steps are performed based on QA. In Figures 2(c) and 2(f) the update steps based on QA are shown as filled circles. After iteration 21, the proposed algorithm restarts to collect information and performs a quadratic update step as soon as $n_q = 3$ points are found. The optimal price is found in the subsequent iteration.

Conclusions and outlook

In this contribution, a price update algorithm for price-based coordination of systems-of-systems based on QA has been proposed with an extension to individually constrained subsystems. A simple heuristic is suggested that switches between QA and subgradient updates in the case of active individual constraints. The algorithm shows a good performance in the simulation study and outperforms simple subgradient updates especially in regions of inactive individual constraints. Thus the proposed strategy combines the use of an accelerated update step where possible and the robustness of subgradient updates in the case of price-insensitive responses of the subsystems. Future research will incorporate the application of the algorithm to a higher number of competing subsystems and more shared resources. Further, an analysis needs to be performed to characterize the properties of the proposed algorithm. Additionally, the approach can be extended to consider subsystems with general strictly convex cost functions, such that the network residual cannot be assumed to be a quadratic function even in the unconstrained case (see Eq. (8)). This limiting assumption could be overcome by a suitable choice of regression points (cf. Gao et al., 2016).

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