Mixed Integer Bilevel Optimization through Multi-parametric Programming

S. Avraamidou[1,2], N. A. Diangelakis[1,2] and E. N. Pistikopoulos*[2,3]

[1]Centre for Process Systems Engineering, Department of Chemical Engineering, Imperial College London, London, UK SW7 2AZ

[2]Artie McFerrin Department of Chemical Engineering, Texas A&M University, College Station, TX 77843

[3]Texas A&M Energy Institute, Texas A&M University, College Station, TX 77843

Abstract

Optimization problems involving two decision makers at two different decision levels are referred to as bi-level programming problems. In this work, we present a novel algorithm for the exact and global solution of two classes of bi-level programming problems, namely (i) bi-level mixed-integer linear programming problems (B-MILP) and (ii) bi-level mixed-integer quadratic programming problems (B-MIQP) containing both integer and continuous variables at both optimization levels. Based on multi-parametric theory, the main idea is to recast the lower level problem as a multi-parametric programming problem, in which the optimization variables of the upper level problem are considered as parameters for the lower level. The resulting exact parametric solutions are then substituted into the upper level problem, which can be solved as a set of single-level deterministic mixed-integer programming problems. The algorithm will be further illustrated through two numerical examples.

Keywords

Bi-level programming, Multi-parametric programming, Mixed-Integer programming

Introduction

Optimization problems involving two decision makers at two different decision levels are referred to as bi-level programming problems: the first decision maker (upper level; leader) is solving an optimization problem which includes in its constraint set another optimization problem solved by the second decision maker (lower level; follower). This class of problems has attracted considerable attention across a broad range of research communities, including economics, sciences and engineering. It was applied to many and diverse problems that require hierarchical decision making such as transportation network planning (Migdalas (1995)), urban planning (Tam and Lam (2004)), economic planning (Gao et al. (2011)), design under uncertainty (Ierapetritou and Pistikopoulos (1996); Floudas et al. (2001)), and design and control integration (Tanartkit and Biegler (1996)).

*To whom all correspondence should be addressed *stratos@tamu.edu*

This work focuses on a novel approach for the solution of sub-classes of multi-parametric bi-level programming problems of the following general form:

$$
\begin{aligned}
\min_{x} \quad & F(x,y) \\
s.t. \quad & G(x,y) \leq 0 \\
& H(x,y) = 0 \\
& \min_{y} \quad f(x,y) \\
& s.t. \quad g(x,y) \leq 0 \\
& \qquad\quad h(x,y) = 0 \\
x_1, ...x_i \in \Re, & \quad y_1, ..., y_j \in \Re
\end{aligned} \tag{1}
$$

,where x is a vector of the upper level problem variables and y is a vector of the lower level problem variables.

Challenges and Previous work

Bi-level programming problems are very challenging to solve, even in the linear case (shown to be NP-hard by Hansen et al. (1992) and Deng (1998)). To strengthen this results Vicente et al. (1994) proved that even checking strict or local optimality is NP-hard. For classes of problems where the lower level problem also involves discrete variables, this complexity is further increased, typically requiring global optimization methods for its solution. Solution approaches for mixed integer bi-level problems with discrete variables in both levels mainly include reformulation approaches (Mitsos (2010); Saharidis and Ierapetritou (2009)), branch and bound techniques (Gumus and Floudas (2005)) or genetic algorithms (Nishizaki and Sakawa (2005)), all of which result in approximate solutions. It is worth mentioning that, to our knowledge, there do not exist any rigorous approaches for the exact solution of bi-level mixed-integer quadratic problems, with or without uncertainty, in the open literature.

In this paper, we present a novel global optimization algorithm for the *exact* and *global* solution of different classes of bi-level programming problems, more specifically (i) having linear or convex quadratic optimization levels, (ii) containing continuous and/or integer variables in either or both optimization levels, and (iii) having right hand sitde uncertainty in one or both optimization levels. The algorithms are based on multi-parametric theory (Acevedo and Pistikopoulos (1997)) and our earlier results (Faisca et al. (2009, 2007); Oberdieck et al. (2016b)). The main idea is to recast the lower level problem as a multi-parametric programming problem, in which the optimization variables of the upper level problem are considered as parameters for the lower level. The resulting exact parametric solutions are then substituted into the upper level problem, which can be solved as a set of single-level deterministic mixed-integer programming problems.

Theory and Algorithm

A known property of the general bi-level programming problem is that the feasible set of the inner problem is parametric in terms of the decision variables of the outer problem. To effectively utilize this property, Pistikopoulos and co-workers have presented a series of algorithms based on multi-parametric programming theory, which can address different classes of continuous multilevel programming problems(Faisca et al. (2007)).

Expanding on the work of Pistikopoulos and co-workers, the approach presented here is based upon the Multi-parametric Mixed-integer Linear Programming (mp-MILP) and Multi-parametric Mixed-Integer Quadratic Programming (mp-MIQP) algorithm of Oberdieck and Pistikopoulos (2015). The proposed algorithm will be introduced through the general form of the B-MILP problem (2), but a similar approach can be used for the solution of B-MIQP problems. The main difference will be the substitution of the solvers from mp-MILP to mp-MIQP solvers. Finally, two numerical examples will be used to further illustrate the use of the algorithm.

$$
\begin{aligned}
\min_{x} \quad & F(x,y) = {c_1}^T x + {d_1}^T y \\
s.t. \quad & A_1 x + B_1 y \le b_1 \\
& A_{eq1} x + B_{eq1} y = b_{eq1} \\
& \min_{y} \quad f(x,y) = {c_2}^T x + {d_2}^T y \\
& s.t. \quad A_2 x + B_2 y \le b_2 \\
& \qquad\quad A_{eq2} x + B_{eq2} y = b_{eq2} \\
& x_1, ...x_i \in \Re, \quad y_1, ..., y_j \in \Re \\
& x_{i+1}, ..., x_{k_1} \in Z^+, \quad y_{j+1}, ..., y_{k_2} \in Z^+
\end{aligned}
\tag{2}
$$

As a first step, we establish bounds for all integer and continuous variables, by solving problems (2.1) for the upper level variables x, and similar problems for the lower level variables y, to obtain bounds on both x, $x^L \le x \le x^U$, and y, $y^L \le y \le y^U$.

$$
\begin{aligned}
x_l^L = \min \quad & x_l \\
s.t. \quad & A_1 x + B_1 y \le b_1 \\
& A_{eq1} x + B_{eq1} y = b_{eq1} \\
& A_2 x + B_2 y \le b_2 \\
& A_{eq2} x + B_{eq2} y = b_{eq2} \quad , \\
x_l^U = \min \quad & -x_l \\
s.t. \quad & A_1 x + B_1 y \le b_1 \\
& A_{eq1} x + B_{eq1} y = b_{eq1} \\
& A_2 x + B_2 y \le b_2 \\
& A_{eq2} x + B_{eq2} y = b_{eq2}
\end{aligned}
\tag{2.1}
$$

Then, the B-MILP is transformed into a binary B-MILP by expressing integer variables that are not binary, $x_{i+1}, ..., x_{n_1}$ and $y_{j+1}, ..., y_{n_2}$, in terms of binary 0-1 variables, $x'_{i+1}, ..., x'_{n_3} \in \{0,1\}$ and $y'_{j+1}, ..., y'_{n_4} \in \{0,1\}$, according to the formula in Floudas (1995)(Section 6.2.1, Remark 1). The acute accent will be omitted in the following steps for simplicity.

As a next step, the lower level problem of the B-MILP, is transformed as a mp-MILP problem (2.2), in which the optimization variables of the upper level problem, x , are considered as parameters for the lower level.

$$\begin{aligned} \min_{y} \quad & {d_2}^T y + {c_2}^T x \\ s.t. \quad & B_2 y \leq b_2 - A_2 x \\ & B_{eq2} y = b_{eq2} - A_{eq2} x \\ & x^L \leq x \leq x^U \end{aligned} \tag{2.2}$$

The solution of (2.2) using POP®toolbox (Oberdieck et al. (2016a)), results to the complete profile of optimal solutions of the lower level problem as explicit functions of the variables of the higher level problem with corresponding boundary conditions (2.3).

$$y = \begin{cases} \xi_1 = m_1 + n_1 x & \text{if } H_1 x \leq h_1 \\ \xi_2 = m_2 + n_2 x & \text{if } H_2 x \leq h_2 \\ \vdots & \vdots \\ \xi_k = m_k + n_k x & \text{if } H_k x \leq h_k \end{cases} \tag{2.3}$$

,where $H_k x \leq h_k$ is referred to as critical region, CR^k, and k denotes the number of computed critical regions.

The computed solutions (2.3) are then substituted into the upper level problem, which can be solved as a set of single-level deterministic mixed-integer programming problems, (2.4).

$$\begin{aligned} z_1 = \min_{x} \quad & {c_1}^T x + {d_1}^T \xi_1(x) \\ s.t. \quad & A_1 x + B_1 \xi_1(x) \leq b_1 \\ & A_{eq1} x + B_{eq1} \xi_1(x) = b_{eq1} \\ & H_1 x \leq h_1 \\ z_2 = \min_{x} \quad & {c_1}^T x + {d_1}^T \xi_2(x) \\ s.t. \quad & A_1 x + B_1 \xi_2(x) \leq b_1 \\ & A_{eq1} x + B_{eq1} \xi_2(x) = b_{eq1} \\ & H_2 x \leq h_2 \\ & \vdots \\ z_k = \min_{x} \quad & {c_1}^T x + {d_1}^T \xi_k(x) \\ s.t. \quad & A_1 x + B_1 \xi_k(x) \leq b_1 \\ & A_{eq1} x + B_{eq1} \xi_k(x) = b_{eq1} \\ & H_k x \leq h_k \end{aligned} \tag{2.4}$$

The solutions of the above single level MILP problems correspond to all the local optimal solutions of the original B-MILP, as parametric programming has the ability to explore the whole parametric space and find all solutions. The final step of the algorithm is to compare all the local solutions to obtain the exact and global optimum.

The algorithm can be extended to problems including right-hand-side uncertainty on both lower and upper levels, with the solution of a single level mp-MILP instead of the MILP at the last step of the algorithm.

Table 1. Algorithm for the solution of B-MILP problems

Step 1: Establish integer/continuous variable bounds.
Step 2: Transform the B-MILP into a binary B-MILP.
Step 3: Recast the lower level as a mp-MILP, in which the optimization variables of the upper level problem are considered as parameters.
Step 4: Solve the resulting mp-MILP problems to obtain the optimal solution of the lower lever as explicit functions of the upper level variables.
Step 5: Substitute each multi-parametric solution into the upper level problem to formulate k single level MILP problems.
Step 6: Solve all k single level problems and compare their solutions to select the exact and global optimum.

The proposed algorithm, summarized also in Table 1, will be further illustrated through two numerical examples.

Numerical examples

Two numerical examples will be solved to illustrate the use of the proposed algorithm. The first one is a bi-level programming problem with a mixed-integer quadratic programming problem (MIQP) at the first level and a mixed-integer linear programming problem (MILP) at the second level. The second example illustrates the extension of the algorithm to tackle uncertainties and supply the parametric solution of the problem.

Example 1: MIQP-MILP

Consider the following example consisting of a mixed-integer linear programming problem at the lower level, and a mixed-integer quadratic programming problem at the higher level.

$$\begin{aligned} \min_{x_{1,2},y_3} \quad & 4{x_1}^2 - {x_2}^2 + 2x_2 + x_3 y_3 + 5y_1 - 6y_3 \\ s.t. \quad & y_1 + y_2 + y_3 \leq 1 \\ & \min_{x_3,y_{1,2}} \quad -x_1 + x_2 - 2x_3 - y_1 + 5y_2 \\ & s.t. \quad 6.4x_1 + 7.2x_2 + 2.5x_3 \leq 11.5 \\ & \qquad -8x_1 - 4.9x_2 - 3.2x_3 \leq 5 \\ & \qquad 3.3x_1 + 4.1x_2 + 0.02x_3 ... \\ & \qquad ... + 4y_1 + 4.5y_2 \leq 1 \\ & \qquad -10 \leq x_{1,2} \leq 10 \\ & x_1, x_2, x_3 \in \Re, \quad y_1, y_2, y_3 \in \{0, 1\} \end{aligned}$$

As the problem is already in the form of a binary mixed-integer bi-level programming problem, Steps 1 and 2 are not needed, therefore we proceed to Step 3.

Following Step 3, the lower level problem is reformulated as a mp-MILP problem (3.1), in which the optimization variables of the upper level problem that appear in the lower level, i.e. x_1, x_2, are considered as parameters.

$$\begin{aligned} \min_{x_3, y_{1,2}} \quad & -2x_3 - y_1 + 5y_2 - x_1 + x_2 \\ s.t. \quad & 2.5x_3 \leq 11.5 - 6.4x_1 - 7.2x_2 \\ & -3.2x_3 \leq 5 + 8x_1 + 4.9x_2 \\ & 0.02x_3 + 4y_1 + 4.5y_2 \leq 1 - 3.3x_1 - 4.1x_2 \\ & -10 \leq x_{1,2} \leq 10 \end{aligned}$$

The above problem is then solved using a mp-MILP algorithm, and yields the optimal parametric solution shown in Table 2.

The solutions obtained for every critical region are then substituted into the upper level problem to formulate five new single level MIQP problems. More specifically, the value of the optimization variable of the lower level, x_3, is substituted in the upper level in terms of the upper level optimization variables, x_1 and x_2; and the definition of the critical region is substituted in the upper level as a new set of constraints.

$$\begin{aligned} z_1 = & \min_{x_{1,2}, y_3} 4{x_1}^2 - {x_2}^2 + 2x_2 \\ & \quad - (-165x_1 - 205x_2 + 50)\, y_3 - 6y_3 \\ s.t. \quad & -y_3 \leq 1 \\ & -0.624x_1 - 0.780x_2 \leq -0.175 \\ & 0.624x_1 + 0.781x_2 \leq 0.198 \\ & x_1 \leq 10 \\ & \vdots \\ z_5 = & \min_{x_{1,2}, y_3} 4{x_1}^2 - {x_2}^2 + 2x_2 \\ & \quad - (-2.56x_1 - 2.88x_2 + 4.6)\, y_3 - 6y_3 \\ s.t. \quad & -y_3 \leq 1 \\ & 0.044x_1 + 0.999x_2 \leq 4.565 \\ & 0.626x_1 + 0.780x_2 \leq 0.175 \\ & -0.624x_1 - 0.781x_2 \leq 0.570 \\ & -10 \leq x_1 \leq 10 \\ & -x_2 \leq 10 \end{aligned} \tag{3}$$

Solving all the single level MIQP problems in (3) results to the solution in Table 3.

As a final step the solutions of all the single level problems are compared. The solution with the minimum objective value corresponds to the global minimum of the initial bi-level programming problem.

Table 2. Example 1: Parametric solution of the lower level problem

CR	Definition	Variables
1	$-0.624x_1 - 0.780x_2 \leq -0.175$ $0.624x_1 + 0.781x_2 \leq 0.198$ $x_1 \leq 10$	$x_3 = -165x_1 - 205x_2 + 50$ $y_1 = 0$ $y_2 = 0$
2	$0.624x_1 + 0.781x_2 \leq -0.570$ $-0.624x_1 - 0.780x_2 \leq 0.594$ $x_1 \leq 10$	$x_3 = -2.56x_1 - 2.88x_2 + 4.6$ $y_1 = 0$ $y_2 = 0$
3	$-0.626x_1 - 0.780x_2 \leq 0.596$ $0.624x_1 + 0.781x_2 \leq -0.570$ $-0.626x_1 - 0.780x_2 \leq 0.594$ $x_1 \leq 10$	$x_3 = -165x_1 - 205x_2 + 50$ $y_1 = 1$ $y_2 = 0$
4	$0.626x_1 + 0.780x_2 \leq -0.596$ $0.044x_1 + 0.999x_2 \leq 4.565$ $-10 \leq x_1 \leq 10$ $-x_2 \leq 10$	$x_3 = -2.56x_1 - 2.88x_2 + 4.6$ $y_1 = 1$ $y_2 = 0$
5	$0.044x_1 + 0.999x_2 \leq 4.565$ $0.626x_1 + 0.780x_2 \leq 0.175$ $-0.624x_1 - 0.781x_2 \leq 0.570$ $-10 \leq x_1 \leq 10$ $-x_2 \leq 10$	$x_3 = -2.56x_1 - 2.88x_2 + 4.6$ $y_1 = 0$ $y_2 = 0$

Table 3. Example 1: Single level solutions

CR	Variables	Objective
1	$x_1 = 0.370$ $x_2 = -0.042$ $y_{1,2} = 0, y_3 = 1$	-7.962
2	$x_1 = 0.458$ $x_2 = -1.129$ $y_{1,2} = 0, y_3 = 1$	-2.015
3	$x_1 = 0.423$ $x_2 = -1.133$ $y_{2,3} = 0, y_1 = 1$	2.286
4	$x_1 = 0$ $x_2 = -10$ $y_{2,3} = 0, y_1 = 1$	-115
5	$x_1 = 0.449$ $x_2 = -1.088$ $y_{1,2} = 0, y_3 = 1$	-1.969

Example 2: mpMIQP-MILP

The algorithm introduced in this paper can be also extended to bi-level problems with uncertainty in one or both levels.

For simplicity, the same problem solved in Example 1 was also used for this example, but x_2 will be now considered as an uncertainty instead of an optimization variable. Therefore, the problem considered in this example is the following:

$$\begin{aligned}
\min_{x_1,y_3} \quad & 4{x_1}^2 + x_3y_3 + 5y_1 - 6y_3 - {x_2}^2 + 2x_2 \\
s.t. \quad & y_1 + y_2 + y_3 \leq 1 \\
& \min_{x_3,y_{1,2}} \quad -x_1 - 2x_3 - y_1 + 5y_2 + x_2 \\
& s.t. \quad 6.4x_1 + 7.2x_2 + 2.5x_3 \leq 11.5 \\
& \qquad -8x_1 - 4.9x_2 - 3.2x_3 \leq 5 \\
& \qquad 3.3x_1 + 4.1x_2 + 0.02x_3 ... \\
& \qquad\qquad ... + 4y_1 + 4.5y_2 \leq 1 \\
& \qquad -10 \leq x_{1,2} \leq 10 \\
& x_1, x_2, x_3 \in \Re, \quad y_1, y_2, y_3 \in \{0,1\}
\end{aligned}$$

Again for this case the problem is already bounded and in a binary form. Starting from Step 3, the lower level of the programming problem is transformed into a mp-MILP problem. Both the higher level variables and parameters are being treated as parameters for the lower level.

For this example, the reformulated lower level will be exactly the same as in Example 1, and can be solve at Step 4 with the same methodology. Continuing to Step 5, the lower level solutions, corresponding to each critical region, are substituted into the higher level problem.

$$\begin{aligned}
z_1 = & \min_{x_1,y_3} 4{x_1}^2 + 165x_1y_3 + 44y_3 \\
& \quad -205x_2y_3 - {x_2}^2 + 2x_2 \\
s.t. \quad & -y_3 \leq 1 \\
& -0.624x_1 \leq -0.175 + 0.780x_2 \\
& 0.624x_1 \leq 0.198 - 0.781x_2 \\
& x_1 \leq 10 \\
& \vdots \\
z_5 = & \min_{x_1,y_3} 4{x_1}^2 - 2.56x_1y_3 - 1.4y_3 \\
& \quad -{x_2}^2 - 2.88x_2y_3 + 2x_2 \\
s.t. \quad & -y_3 \leq 1 \\
& 0.044x_1 \leq 4.565 - 0.999x_2 \\
& 0.626x_1 \leq 0.175 - 0.780x_2 \\
& -0.624x_1 \leq 0.570 + 0.781x_2 \\
& -10 \leq x_1 \leq 10 \\
& -x_2 \leq 10
\end{aligned} \tag{4}$$

The five resulting single level problems in this case, are in the form of mp-MIQP problems, with the uncertainty x_2 being a parameter for the problem. Therefore, the mpMIQP algorithm was used for their solution. Each critical region formed in Step 4 is now divided into smaller critical regions as another parametric programming problem is solved within them. A summary of the resulting parametric solutions is presented in Table 4.

As a last step, the solutions generated from each critical region have to be compared and the minima through the parametric space are chosen as the final solution of the mixed integer bi-level programming problem with uncertainty. Figure 1 illustrates all the solutions through the parametric space, and Table 5 summarizes the final solution.

Table 4. Example 2: Single level mp-MIQP solution

CR	**Definition**	**Objective**
1.1	$-4.824 \leq x_2 \leq 7.733$	$2.136{x_2}^2 - 408.010 - 8.154$
1.2	$7.733 \leq x_2 \leq 7.812$	$-{x_2}^2 - 203x_2 - 1406$
...	...	...
5.1	$0.290 \leq x_2 \leq 1.241$	$-{x_2}^2 - 0.88x_2 - 2.219$
5.2	$-4.824 \leq x_2 \leq 0.290$	$2.096{x_2}^2 - 2.674x_2 - 1.959$
5.3	$-4.882 \leq x_2 \leq -4.824$	$0.001{x_2}^2 + 0.0092x_2 + 0.0002$
5.4	$1.2407 \leq x_2 \leq 8.7160$	$2.136{x_2}^2 - 8.661x_2 - 2.607$

Table 5. Example 2: Final solution

CR	**Definition**	**Objective**
4.1	$-5.014 \leq x_2 \leq -4.882$	$0.011{x_2}^2 + 0.009x_2 + 0.0002$
3.1	$-4.882 \leq x_2 \leq -4.840$	$2.136{x_2}^2 - 2.575x_2 + 6.669$
5.3	$-4.840 \leq x_2 \leq -4.824$	$0.011{x_2}^2 + 0.009x_2 + 0.0002$
5.2	$-4.824 \leq x_2 \leq -0.015$	$2.096{x_2}^2 - 2.674x_2 - 1.959$
1.1	$-0.015 \leq x_2 \leq 7.733$	$2.136{x_2}^2 - 408.010x_2 - 8.154$
1.2	$7.733 \leq x_2 \leq 7.812$	$-{x_2}^2 - 203x_2 - 1406$
3.4	$7.812 \leq x_2 \leq 8.799$	$2.096{x_2}^2 - 310.374x_2 + 4.065$
3.3	$8.799 \leq x_2 \leq 8.802$	$-{x_2}^2 - 203x_2 - 701$
4.2	$8.802 \leq x_2 \leq 10$	$-{x_2}^2 - 0.880x_2 - 1.095$

Conclusion and Future Work

This paper introduces a novel algorithm for the solution of a range of classes of mixed integer bi-level programming problems with or without integer variables and uncertainty on both optimization levels. The ability of the algorithm to give the parametric solution of such problems can be utilized for the implementation of a hierarchical model predictive controllers, scheduling and control integration or planning and scheduling integration. We are currently working on developments in all three above mentioned areas.

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