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Spectral Galerkin Method in the Study of Mass Transfer in Laminar and Turbulent Flows

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Abstract

The aim of this article is to determine the effective diffusion coefficient in turbulent incompressible fluid flow with axial symmetry. To determine eigenfunctions and eigenvalues of a Sturm-Liouville problem, we use spectral Galerkin method and first kind Bessel function. The proposed calculation method may be also easily adapted to other movements with axial symmetry.

Keywords: Diffusion coefficient, Galerkin, eigenfunction, eigenvalue

1. Introduction

Many practical applications require the determination of the distribution of the concentration in the mixing area between two miscible moving fluids. Such an example is constituted by the succesive transport of the oil products. The reciprocal contamination of these two fluids is produced, in the case of laminar regime, both due to the molecular diffusion and to the convective diffusion. In turbulent regime, besides these two phenomena, we also observe the turbulent diffusion caused by fluctuations of the velocity. An approximate solution for the laminar regime can be found in [1,2].
Starting from the differential equation of the average concentration c for the miscible fluid flow through tubes with circular section in turbulent regime:

$$\frac{\partial c}{\partial t} + v_0 f(r)\frac{\partial c}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left(r \cdot D_r \cdot \frac{\partial c}{\partial r}\right), \tag{1}$$

in [3] the following dimensionless expression is established for the effective diffusion coefficient

$$\kappa = 2 \cdot P_0^2 \cdot \sum_{n=1}^{\infty} \frac{1}{\lambda_n^6 \left\|\Phi_n\right\|^2}\left[\frac{D}{D_0}\int_0^1 \rho \frac{\mathrm{d}\Phi_n}{\mathrm{d}\rho}\frac{\mathrm{d}f}{\mathrm{d}\rho}\mathrm{d}\rho - \frac{r_0 v_*^2}{D_0 v_0}\Phi_n(1)\right]^2. \tag{2}$$

In formula (1) t is the time, x the spatial coordinate along the tube axis, r the radial coordinate, v_0 the maximum velocity, $f(r)$ the velocity distribution function in tube cross-section and D_r the diffusion coefficient in radial direction. In formula (2) D is the molecular diffusion coefficient, D_0 a reference value of D_r, r_0 is the tube radius, P_0 is the Péclet number of molecular diffusion and

$$\rho = \frac{r}{r_0} \tag{3}$$

is the dimensionless radial coordinate.
The functions Φ_n and the numbers λ_n are the eigenfunctions and eigenvalues of singular Sturm-Liouville problem [4]:

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho \cdot D_\rho \cdot \frac{d\Phi}{d\rho}\right) + \lambda^2 \cdot \rho \cdot \Phi = 0, \tag{4}$$

$$\rho = 0, \frac{\mathrm{d}\Phi}{\mathrm{d}\rho} = 0 \quad ; \quad \rho = 1, \frac{\mathrm{d}\Phi}{\mathrm{d}\rho} = 0, \tag{5}$$

where

$$D_\rho = D_r / D_0. \tag{6}$$

It easy to see that the first eigenfunction is one and the first eigenvalue is zero. These values are not used in formula (2).

To use formula (2) the eigenfunctions and the eigenvalues of eq. (4) and (5) must be determined. Unfortunately, these values cannot be determined exactly, so there must be used approximate methods.

To approximately determine the eigenfunctions and the eigenvalues of this singular Sturm-Liouville problem we use the spectral Galerkin method. The key for efficient implementation of this method is to choose appropriate basis functions. In this article the basis functions (a complete functions system of $L_2[0,1]$) are made of Bessel functions of first kind and zero order.

The article is structured as follows: in section two we formulate the mathematical problem, section three presents the algorithm for determination of eigenvalues and eigenfunctions and the last section will contain some approximate solutions of the problem and some numerical results.

2. The Mathematical Problem

In the case of tubes, the turbulent diffusion in radial direction is different from the one in axial direction. That is because in axial direction the convective diffusion prevails, hence the turbulent diffusion in axial direction can be neglected. Under the given conditions, the diffusion coefficient in radial direction is:

$$D_r = D + D_{rt} \tag{7}$$

where D_{rt} is the turbulent diffusion coefficient in radial direction. For this coefficient, G. I. Taylor [5] proposed the expression:

$$D_{rt} = -\frac{r_0 v_*^2}{v_0} \cdot \frac{\rho}{f'(\rho)} \tag{8}$$

To calculate function $f(r)$ the logarithmic law is used [6]:

$$f(r) = 1 + 2.5 \cdot v_* / v_0 \cdot \ln(1 - r / r_0), \tag{9}$$

where v_* is the friction velocity [6].

Using (6), (7), (8) and (9) we obtain finally:

$$D_\rho = D / D_0 \cdot [1 + h \cdot \rho \cdot (1 - \rho)] \tag{10}$$

where

$$h = r_0 \cdot v_* / (2{,}5 \cdot D). \tag{11}$$

With these formulae eq. (4) becomes:

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\left\{\rho \cdot [h \cdot \rho \cdot (1-\rho)+1]\frac{\mathrm{d}\Phi}{\mathrm{d}\rho}\right\} + \rho \cdot \frac{D_0}{D} \cdot \lambda^2 \cdot \Phi = 0\,. \tag{12}$$

3. The Galerkin Method

For the determination of eigenfunctions and eigenvalues of Sturm-Liouville problem (12), (5) we will apply the Galerkin method. For this we consider the bilinear forms a and b defined on $H^1(0,1)\times H^1(0,1)$

$$a(u,v) = \int_0^1 \{\rho[h \cdot \rho \cdot (1-\rho)+1]\} \cdot \frac{\mathrm{d}u}{\mathrm{d}\rho} \cdot \frac{\mathrm{d}v}{\mathrm{d}\rho} \cdot \mathrm{d}\rho\,, \tag{13}$$

$$b(u,v) = \frac{D_0}{D}\int_0^1 \rho \cdot u \cdot v \cdot \mathrm{d}\rho\,. \tag{14}$$

We look for the eigenpair (λ, Φ) which satisfies

$$\Phi \in H^1(0,1),\ \Phi \neq 0 \quad ; \quad a(\Phi, v) = \lambda^2 \cdot b(\Phi, v),\ (\forall)\, v \in H^1(0,1) \tag{15}$$

Equation (15) is called a variational formulation of Sturm-Liouville problem (12), (5) [7].
We look for the solution of (15) under approximate form

$$\Phi_n(\rho) = \sum_{k=0}^{m} a_{k,n} J_0(\alpha_k \cdot \rho),\ n = \overline{0,m}\,, \tag{16}$$

where the numbers α_n are the roots of equation

$$J_1(\alpha) = 0\,, \tag{17}$$

m is the approximation level of functions Φ_n; J_0 and J_1 are the Bessel functions of first kind, zero order and first order, respectively.
The unknown coefficients a_k are calculated by solving the homogeneous linear system:

$$\sum_{k=0}^{m}\left(-a\left(J_0(\alpha_k\rho),J_0(\alpha_j\rho)\right)+\lambda^2 b\left(J_0(\alpha_k\rho),J_0(\alpha_j\rho)\right)\right)\cdot a_k=0\ ,\ j=\overline{0,m} \tag{18}$$

The solutions of the equation:

$$\left|-a\left(J_0(\alpha_k\rho),J_0(\alpha_j\rho)\right)+\lambda^2 b\left(J_0(\alpha_k\rho),J_0(\alpha_j\rho)\right)\right|=0 \tag{19}$$

are the approximate values, for the m approximation level, for the first m eigenvalues of Sturm-Liouville problem (12), (5).

4. Approximate Solutions and Numerical Results

By using (16) we obtain for the effective diffusion coefficient the approximate formula:

$$\kappa=2P_0^2\sum_{n=1}^{m}\frac{v_*}{\lambda_n^6\left\|\Phi_n\right\|^2 v_0 D_0}\left[2.5D\sum_{k=1}^{m}\alpha_k a_{k,n}\int_0^1\frac{\rho J_1(\alpha_k\rho)}{1-\rho}\mathrm{d}\rho-r_0 v_*\Phi_n(1)\right]^2 \tag{20}$$

For m=1 this formula becomes

$$\kappa=\frac{2\cdot P_0^2\cdot v_*}{\lambda_1^6\cdot\left(a_{0,1}^2+0.081\cdot a_{1,1}^2\right)\cdot v_0\cdot D_0}\cdot\left[6.302\cdot a_{1,1}-r_0\cdot v_*\cdot\left(a_{0,1}-0.403\cdot a_{1,1}\right)\right]^2 \tag{21}$$

The eigenvalues

$$\mu_n^2=\lambda_n^2\cdot D_0/D \tag{22}$$

of Sturm-Liouville problem (12), (5) are presented in table 1. These eigenvalues are obtained for level nine of approximation.

Table 1 The eigenvalue for the Sturm-Liouville problem

h	μ_1^2	μ_2^2	μ_3^2	μ_4^2	μ_5^2	μ_6^2	μ_7^2	μ_8^2
0.001	14.68	49.22	103.51	177.55	271.33	384.85	518.12	671.13
0.01	14.71	49.31	103.68	177.83	271.75	385.44	518.91	672.14
0.1	15.00	50.13	105.33	180.59	275.92	391.32	526.78	682.32
1	17.87	58.29	121.57	207.77	316.90	448.97	603.99	781.96
5	30.40	93.28	190.29	321.86	488.13	689.20	925.09	1195.9

Formulae (23) and (24) contain some eigenfunctions calculated for the same level of approximation and for $h = 0.2$.

$$\Phi_2(\rho) = -0.003J_0(\alpha_1\rho) + 0.001J_0(\alpha_2\rho) + 1.022J_0(\alpha_3\rho) - 0.002J_0(\alpha_4\rho) + 0.008J_0(\alpha_5\rho) + 0.001J_0(\alpha_7\rho), \tag{23}$$

$$\Phi_3(\rho) = -0.019J_0(\alpha_1\rho) - 0.002J_0(\alpha_2\rho) + 0.002J_0(\alpha_3\rho) + 1.008J_0(\alpha_4\rho) - 0.002J_0(\alpha_5\rho) + 0.011J_0(\alpha_7\rho) + 0.001J_0(\alpha_8\rho). \tag{24}$$

The integrals (13) and (14) are calculated by using the Gauss-type quadrature rules. The main advantage of the method presented here is that the coefficients of the system (18) can be easily handled without extra effort, while its main disadvantage is that the corresponding linear system has full matrix. We used numerical algorithms and programmes in Pascal to determine the eigenvalues and eigenvectors of Sturm-Liouville problem [8]. The algorithms proposed here are stable and offer a good precision even for a low level of approximation.

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