

A Global Parametric Programming Optimisation Strategy for Multilevel Problems

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Abstract

In this paper, we outline the foundations of a general global optimisation strategy for the solution of multilevel hierarchical and general decentralised multilevel problems based on our recent developments in multiparametric programming theory. The core idea is to recast each optimisation subproblem in the multilevel hierarchy as a multiparametric programming problem and then transform the multilevel problem into a single-level optimisation problem. For decentralised systems, where more than one optimisation problem is present at each level of the hierarchy, Nash equilibrium is considered. A three person dynamic optimisation problem is presented to illustrate the mathematical developments.

1. Introduction

It is widely recognised that the successful design of large and complex systems involves some type of decomposition of the original problem into smaller and intercommunicating subsystems, typically arranged in a multilevel hierarchy. Such multilevel problems arise commonly in process systems engineering [9,12,13], with bilevel programming problems being the simplest and most studied [8,14]. Bilevel programming problems involve an optimisation hierarchy of two levels, of the following form:

$$\begin{aligned} & \min_{x,y} F(x,y) \\ \text{s.t. } & G(x,y) \leq 0 \\ & x \in X \\ & y \in \operatorname{argmin}\{f(x,y) : g(x,y) \leq 0, y \in Y\} \end{aligned} \tag{1}$$

where $X \subseteq \mathbb{R}^{nx}$, $Y \subseteq \mathbb{R}^{ny}$ and both are compact convex sets; F and f are real functions: $\mathbb{R}^{(nx+ny)} \rightarrow \mathbb{R}$; G and g are vectorial real functions, $G : \mathbb{R}^{(nx+ny)} \rightarrow \mathbb{R}^{nu}$ and $g : \mathbb{R}^{(nx+ny)} \rightarrow \mathbb{R}^{nl}$; $nx, ny \in \mathbb{N}$ and $nu, nl \in \mathbb{N} \cup \{0\}$. We also define the rational

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reaction set as follows:

$$M(x) = \{y \in Y : y \in \operatorname{argmin}\{f(x, y) : y \in C(x)\}\}; \quad (2)$$

where, $C(x) = \{y \in Y : g(x, y) \leq 0\}$;

Multilevel and decentralised optimisation problems, which typically arise in many engineering [4,9,12,13] and financial applications [2,10], involve a hierarchy of such optimisation levels, as in (1); where each optimisation level (or subproblem) controls a subset of the overall optimisation variables. When more than one subproblem is present at the same hierarchical optimisation level the problem is called a decentralised multilevel problem. Nash equilibrium is often a preferred strategy to coordinate such decentralised systems [13].

Despite their significance, general solution strategies for solving such complex problems are limited, especially due to the multi-layer nature, non-linearities and non-convexities occur [14]. In addition, the potential presence of logical decisions (which requires the inclusion of binary variables) increases further the complexity of the problem. Therefore, it is widely accepted that a global optimisation approach is needed for the solution of such multilevel optimisation problems [8].

Recently, Pistikopoulos and co-workers [1,5,6,11] have proposed novel solution algorithms, based on parametric programming theory [7], which open the possibility to address general classes of multilevel programming problems. The core idea of this approach is to recast each optimisation subproblem as a multiparametric programming problem, and hence obtain an analytical solution for the rational reaction set for each of the subproblems. These analytical expressions can then be used to compute, through direct comparison the Nash equilibrium between subproblems in the same optimisation level, for decentralised problems.

2. Methodology

The proposed approach is illustrated by considering a multiple person dynamic linear-quadratic optimisation problem [10], which involves the coordination of a number of controllers within a complex system.

Consider the dynamic system represented in Figure 1, where u, v^1 and v^2 are input variables, and x, y^1 and y^2 output variables:

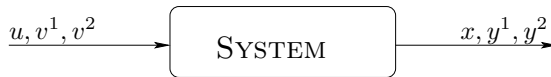


Figure 1. Schematic representation of the dynamic system

The discrete dynamic behaviour of this system is described by the following linear state transition model:

$$\begin{aligned} x_{t+1} &= x_t + u_t - 2v_t^1 + v_t^2, \\ y_{t+1}^1 &= y_t^1 + 2v_t^1, \\ y_{t+1}^2 &= y_t^2 + 2v_t^2, \end{aligned} \quad t = 0, 1, 2 \quad (3)$$

with constrains on the input and state variables as follows:

$$\begin{aligned} -30 &\leq v_0^1, v_1^1, v_2^1, v_0^2, v_1^2, v_2^2 \leq 30, \\ -20 &\leq u_0, u_1, u_2 \leq 20, \\ -10 &\leq x_0, y_0^1, y_0^2 \leq 10. \end{aligned} \quad (4)$$

In Process and Systems Engineering, the performance of the system is in most of the cases optimised regarding just one objective function (e.g. optimal control). But, it is also common to have conflicting goals during in the management of a dynamic process. Since the aim is to optimise the overall performance of the system, suitable cost-functions should be considered. For example, we consider a three-controller system [10]:

$$\mathbf{J}_1 = \min_{u_0, u_1, u_2} 4x_3 + 3y_3^1 + 2y_3^2 + \sum_{t=0}^2 \left\{ (u_t)^2 + (v_t^1)^2 - (v_t^2)^2 + 2u_t x_t + x_t^2 \right\}, \quad (5)$$

$$\mathbf{J}_2 = \min_{v_0^2, v_1^2, v_2^2} 2x_3 + 3y_3^2 + \sum_{t=0}^2 \left\{ 2 \cdot u_t v_t^2 + (v_t^1 + 1)^2 + (v_t^2 + 1)^2 \right\}, \quad (6)$$

$$\mathbf{J}_3 = \min_{v_0^1, v_1^1, v_2^1} x_3 + 2y_3^1 - 10y_3^2 + \sum_{t=0}^2 \left\{ -15u_t + (v_t^1 - 1)^2 - 2v_t^1 v_t^2 + (v_t^2)^2 \right\}. \quad (7)$$

Where J_1 , J_2 and J_3 correspond to Controllers 1,2 and 3, respectively.

Figure 2 further displays two possible configurations for the control structure of the considered system.P

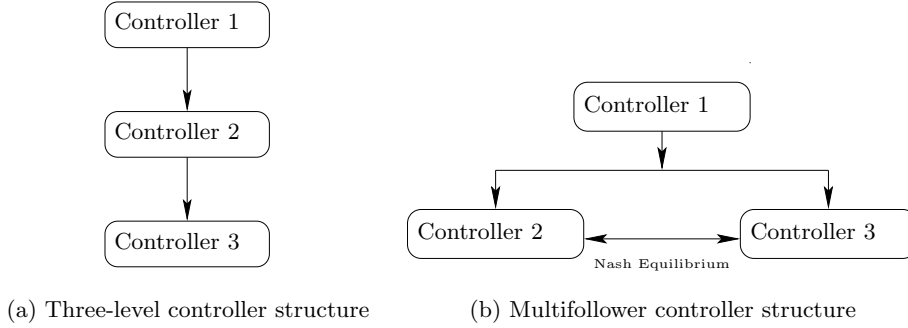


Figure 2. Three-controller multilevel problem

The objective then is to derive suitable optimal strategies for the two controller structures. Case (a) of Figure (2) corresponds to a three-level optimisation problem, whereas case (b) refers to a bilevel multifollower optimisation problem.

In the following subsections, we briefly describe the developed optimisation strategy for the global solution [6] of these two classes of optimisation problems.

2.1. Three-level programming problem

The steps of the proposed parametric global optimisation strategy for the three-level programming problem follow directly from the strategy adopted to the bilevel programming problem [6], and can be summarised as follows:

Step 1. Recast the lower optimisation problem, J_3 , as a multiparametric programming problem, with the control variables belonging to the other two levels being the parameters $(x_0, y_0^1, y_0^2, u_t, v_t^2)$. Solve the resulting problem using a multiparametric optimisation algorithm [5];

Step 2. Include the rational reaction set, $v_t^1 = f(x_0, y_0^1, y_0^2, u_t, v_t^2)$, into the optimisation problem corresponding to Controller 2, J_2 ;

Step 3. Recast the optimisation problem J_2 as a multiparametric programming problem, with the control variables belonging to the upper level being the parameters (x_0, y_0^1, y_0^2, u_t) , and solve it using a multiparametric optimisation algorithm;

Step 4. Include the rational reaction set from the two levels below,

$$v_t^1 = f(x_0, y_0^1, y_0^2, u_t, v_t^2(u_t)),$$

and

$$v_t^2 = f(x_0, y_0^1, y_0^2, u_t),$$

into the optimisation problem corresponding to the leader controller, J_1 ;

Step 5. Recast the multilevel optimisation problem in a single-level multiparametric programming problem, having as parameters the state-space (x_0, y_0^1, y_0^2) , and solve it using a multiparametric optimisation algorithm.

If overlapping regions were created at **Step 5**, the comparison method described in [1] is employed. The result for this problem is listed in Table 1.

Table 1
Solution to the three-level optimisation problem

Critical Region 1	Critical Region 2
$u_0 = 6.84615 - 0.76928x_0$	$u_0 = -0.333333 - 1.8519x_0$
$u_1 = -20$	$u_1 = -1.33333 + 2.8148x_0$
$u_2 = 15.2308 + 0.15388x_0$	$u_2 = -2 - 2.4444x_0$
$-10 \leq x_0 \leq -6.63161$	$-6.63161 \leq x_0 \leq 7.36377$
Critical Region 3	Critical Region 4
$u_0 = -1.53333 - 1.6889x_0$	$u_0 = -9 - 0.72732x_0$
$u_1 = 8.26667 + 1.5111x_0$	$u_1 = 20$
$u_2 = -20$	$u_2 = -20$
$7.36377 \leq x_0 \leq 7.76466$	$7.76466 \leq x_0 \leq 10$
$v_0^1 = v_0^2 = -2 - 0.5u_0; v_1^1 = v_1^2 = -2 - 0.5u_1; v_2^1 = v_2^2 = -2 - 0.5u_2$	

2.2. Bilevel multifollower programming problem

The solution steps for the bilevel multifollower optimisation problem are as follows:

Step 1. Recast optimisation subproblems corresponding to Controller 2 and Controller 3 as multiparametric programming problems, with parameters being the set of variables out of their control, $(x_0, y_0^1, y_0^2, u_t, v_t^2)$ and $(x_0, y_0^1, y_0^2, u_t, v_t^1)$, respectively. Then solve each one using a multiparametric optimisation algorithm;

Step 2. Compute the Nash equilibrium point (see Appendix I), through direct comparison of the two explicit analytical rational reaction sets,

$$\begin{cases} v_t^1 = f_1(x_0, y_0^1, y_0^2, u_t, v_t^2) \\ v_t^2 = f_2(x_0, y_0^1, y_0^2, u_t, v_t^1) \end{cases} ; \quad (8)$$

Step 3. Incorporate both expressions into Controller 1, J_1 , and formulate a multiparametric optimisation with the state-space (x_0, y_0^1, y_0^2) being the parameter.

The unique solution for this problem, in the analysed state space $(-10 \leq x_0, y_0^1, y_0^2 \leq 10)$, is shown in Table 2.

Table 2
Solution to multifollower problem

Critical Region 1
$u_0 = 1 - x_0$
$u_1 = -8 + x_0$
$u_2 = 5 - x_0$
$v_0^1 = v_0^2 = -6 + x_0$
$v_1^1 = v_1^2 = 3 - x_0$
$v_2^1 = v_2^2 = -10 + x_0$
$-10 \leq x_0 \leq 10$

The complexity of this solution procedure clearly depends on the complexity of the underlying parametric programming algorithms, as studied in our previous work [5].

3. Concluding Remarks

A novel global optimisation based strategy has been described for the solution of the hierarchical multilevel and decentralised multilevel systems problems to global optimality. Based on recent developments in parametric programming theory and algorithms [5,11], each subproblem of the optimisation hierarchy is interpreted as a multiparametric programming problem with the variables from the other subproblems being the parameters. The approach has been successfully tested using a three person dynamic problem illustrative example, for the two different optimisation strategies.

While the illustrative example involves the same model for the three objective functions (controllers), the proposed optimisation strategy is equally applicable for the case when different models are involved (i.e. all control subproblems are treated in a decentralised fashion).

The developed algorithm can address linear and quadratic (controller) objective functions, and a linear model for the dynamic system. Extensions towards general nonlinear models are currently under development.

4. Acknowledgments

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5. Appendix I

In this section, the computation of the Nash equilibrium point, using the analytical expressions for the rational reaction sets, is briefly described. Being u_t (Controller 1), v_t^2 (Controller 2) and v_t^1 (Controller 3) the optimisation variables, the Nash equilibrium for the lower level (u, v_t^{2*}, v_t^{1*}), Figure 2(b), is reached when [3]:

$$\begin{aligned} J_2(u, v_t^{2*}, v_t^{1*}) &\leq J_2(u, v_t^2, v_t^{1*}) & \forall v_t^2 \in V_t^2 \\ J_3(u, v_t^{2*}, v_t^{1*}) &\leq J_3(u, v_t^{2*}, v_t^1) & \forall v_t^1 \in V_t^1 \end{aligned} \quad (9)$$

As mentioned before, this equilibrium is easily computed since the expressions for the rational reaction sets are explicitly obtained. Thus, this equilibrium point is equivalent to the solution of the following system:

$$\begin{cases} v_t^1 = f_1(u_t, v_t^2) \\ v_t^2 = f_2(u_t, v_t^1) \end{cases} \quad (10)$$

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