

Robust Controller Design for a Chemical Reactor

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Abstract

In this paper, the design of a robust static output feedback controller is presented. This problem is transformed to solving of linear matrix inequality (LMI) problems. A computationally simple LMI based non-iterative algorithm is used. The design procedure guarantees with sufficient conditions the robust quadratic stability and guaranteed cost. The presented approach is applied for robust controller design for an exothermic continuous-time stirred tank reactor (CSTR). The designed robust controller is able to stabilize the CSTR with four uncertain parameters in the entire operating area.

Keywords: robust control, static output feedback, LMI, chemical reactor

1. Introduction

Chemical reactors are the ones of most important plants in chemical industry. But their processing is connected with many different uncertainties. Some of them arise from varying or not exactly known parameters, as e.g. chemical kinetics or reaction activity. In other cases operating points change. Chemical reactors are also affecting by various types of perturbations. All these uncertainties can cause poor performance or even instability of closed-loop control system. Application of robust control approach can be one of the possibilities how to overcome all these problems (Alvarez-Ramirez and Fermat, 1999).

Robustness has been recognized as a key issue in the analysis and design of control systems for the last two decades. One of the up to now opened problems is also the problem of a robust static output feedback (Syrmos et al., 1997). Various approaches have been used to study the two aspects of the robust stabilization problem. The first aspect is related to conditions under which the linear system described in the state space can be stabilized via output feedback. The necessary and sufficient conditions can be found e.g. in Kučera and DeSouza (1995), Veselý (2004). The second aspect is related to finding a procedure for obtaining a stabilizing or robustly stabilizing control law. Recently, it has been shown that an extremely wide array of robust controller design problems can be reduced to the problem of finding a feasible solutions of LMIs, see e.g. Veselý (2002), Benton and Smith (1999), Yu and Chu (1999) and others.

In this paper, conditions for robust stabilization of linear continuous-time variant (LTV) systems via static output feedback are presented. The problem of robust controller design with the output feedback is reduced to LMI problems (Boyd, S. et al., 1994). A computationally simple LMI based non-iterative algorithm is used for the design of

robust static output feedback controller (Vesely, 2002). The designed robust controller is used to robust stabilization of an exothermic CSTR.

2. Problem formulation

Consider the LTV system in the form of a polytopic linear differential inclusion (PLDI) (Boyd et al., 1994)

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \\ \mathbf{y}(t) &= \mathbf{C}(t)\mathbf{x}(t)\end{aligned}\tag{1}$$

whose system matrix $\mathbf{S}(t) = \begin{bmatrix} \mathbf{A}(t) & \mathbf{B}(t) \\ \mathbf{C}(t) & 0 \end{bmatrix}$ varies within a fixed polytope of matrices:

$$\mathbf{S}(t) \in \text{Co}\{\mathbf{S}_1, \dots, \mathbf{S}_n\} := \left\{ \sum_{i=1}^n \alpha_i \mathbf{S}_i : \alpha_i \geq 0, \sum_{i=1}^n \alpha_i = 1 \right\}\tag{2}$$

where $\text{Co}\{\mathbf{S}_1, \dots, \mathbf{S}_n\}$ is the convex envelope of a set of linear time invariant (LTI)

models $\mathbf{S}_i = \begin{bmatrix} \mathbf{A}_i & \mathbf{B}_i \\ \mathbf{C}_i & 0 \end{bmatrix}$, $i = 1, \dots, n$, representing vertex systems.

The static output feedback problem can be formulated as follows. For the system (1) find a static output feedback $\mathbf{u}(t) = \mathbf{F}\mathbf{y}(t)$ such that the closed loop system (3) is stable.

$$\dot{\mathbf{x}}(t) = (\mathbf{A}(t) + \mathbf{B}(t)\mathbf{F}\mathbf{C}(t))\mathbf{x}(t) = \mathbf{A}_{\text{CL}}(t)\mathbf{x}(t)\tag{3}$$

3. Robust output feedback controller design

Consider the uncertain closed-loop system (3) with

$$\mathbf{A}_{\text{CL}}(t) \in \text{Co}\{\mathbf{A}_{\text{CL}1}, \dots, \mathbf{A}_{\text{CL}n}\} := \left\{ \sum_{i=1}^n \alpha_i \mathbf{A}_{\text{CL}i} : \alpha_i \geq 0, \sum_{i=1}^n \alpha_i = 1 \right\}, \quad \mathbf{A}_{\text{CL}i} = \mathbf{A}_i + \mathbf{B}_i \mathbf{F} \mathbf{C}_i\tag{4}$$

A sufficient condition for the asymptotic stability of the system (3) is feasibility, e. a. the existence of a quadratic Ljapunov function $V(\mathbf{x}) = \mathbf{x}(t)^T \mathbf{P} \mathbf{x}(t)$, $\mathbf{P} > 0$ such that

$\frac{dV(\mathbf{x}(t))}{dt} < 0$ along all state trajectories. If a $\mathbf{P} > 0$ exists, system (3) is quadratically

stable and following statement holds: system (3) is quadratically stable if and only if there exists a positive definite matrix $\mathbf{P} > 0$ such that following inequalities are satisfied

$$\mathbf{A}_{\text{CL}i}^T \mathbf{P} + \mathbf{P} \mathbf{A}_{\text{CL}i} < 0, \quad \mathbf{P} > 0, \quad i = 1, \dots, n\tag{5}$$

Consider the polytopic closed-loop system (3). Then the following two statements are equivalent (Vesely, 2002):

1. The system (3) is robust static output feedback quadratically stabilizable.
2. There exist a positive definite matrix $\mathbf{P} = \mathbf{P}^T > 0$ and a matrix $\mathbf{F}$ satisfying the following matrix inequality

$$(\mathbf{A}_i + \mathbf{B}_i \mathbf{F} \mathbf{C}_i)^T \mathbf{P} + \mathbf{P} (\mathbf{A}_i + \mathbf{B}_i \mathbf{F} \mathbf{C}_i) < 0, \quad \mathbf{P} > 0, \quad i = 1, \dots, n\tag{6}$$

Consider the polytopic closed-loop system (3). Then the following three statements are equivalent (Vesely, 2002):

1. The system (3) is simultaneously static output feedback stabilizable with guaranteed cost

$$\int_0^{\infty} (\mathbf{x}(t)^T \mathbf{Q} \mathbf{x}(t) + \mathbf{u}(t)^T \mathbf{R} \mathbf{u}(t)) dt \leq \mathbf{x}_0(t)^T \mathbf{P} \mathbf{x}_0(t) = J^*, \mathbf{P} > 0 \quad (7)$$

2. There exist matrices $\mathbf{P} > 0$, $\mathbf{Q} > 0$, $\mathbf{R} > 0$ and a matrix $\mathbf{F}$ such that the following inequalities hold

$$(\mathbf{A}_i + \mathbf{B}_i \mathbf{F} \mathbf{C}_i)^T \mathbf{P} + \mathbf{P}(\mathbf{A}_i + \mathbf{B}_i \mathbf{F} \mathbf{C}_i) + \mathbf{Q} + \mathbf{C}_i^T \mathbf{F}^T \mathbf{R} \mathbf{F} \mathbf{C}_i < 0, \quad i = 1, \dots, n \quad (8)$$

3. There exist matrices $\mathbf{P} > 0$, $\mathbf{Q} > 0$, $\mathbf{R} > 0$ and a matrix $\mathbf{F}$ such that the following inequalities hold

$$\mathbf{A}_i^T \mathbf{P} + \mathbf{P} \mathbf{A}_i - \mathbf{P} \mathbf{B}_i \mathbf{R}^{-1} \mathbf{B}_i^T \mathbf{P} + \mathbf{Q} \leq 0, \quad i = 1, \dots, n \quad (9)$$

$$(\mathbf{B}_i^T \mathbf{P} + \mathbf{R} \mathbf{F} \mathbf{C}_i) \Phi_i^{-1} (\mathbf{B}_i^T \mathbf{P} + \mathbf{R} \mathbf{F} \mathbf{C}_i)^T - \mathbf{R} \leq 0 \quad (10)$$

$$\text{where } \Phi_i = -(\mathbf{A}_i^T \mathbf{P} + \mathbf{P} \mathbf{A}_i - \mathbf{P} \mathbf{B}_i \mathbf{R}^{-1} \mathbf{B}_i^T \mathbf{P} + \mathbf{Q}), \quad i = 1, \dots, n \quad (11)$$

The design procedure for simultaneous static output feedback stabilization of the system (3) with guaranteed cost is based on statements formulated above.

1. Compute $\mathbf{S} = \mathbf{S}^T > 0$ from the following inequality

$$\begin{bmatrix} \mathbf{S} \mathbf{A}_i^T + \mathbf{A}_i \mathbf{S} - \mathbf{B}_i \mathbf{R}^{-1} \mathbf{B}_i^T & \mathbf{S} \sqrt{\mathbf{Q}} \\ \mathbf{S} \sqrt{\mathbf{Q}} & \mathbf{I} \end{bmatrix} < 0, \quad \gamma \mathbf{I} < \mathbf{S}, \quad i = 1, \dots, n \quad (12)$$

when $\gamma > 0$ is any non-negative constant and $\mathbf{S} = \mathbf{P}^{-1}$.

2. Compute $\mathbf{F}$ from the following inequality

$$\begin{bmatrix} -\mathbf{R} & \mathbf{B}_i^T \mathbf{P} + \mathbf{R} \mathbf{F} \mathbf{C}_i \\ (\mathbf{B}_i^T \mathbf{P} + \mathbf{R} \mathbf{F} \mathbf{C}_i)^T & -\Phi_i \end{bmatrix} < 0, \quad i = 1, \dots, n \quad (13)$$

If the solutions of (12), (13) are not feasible, either the system (3) is not stabilizable with a prescribed guaranteed cost, or it is necessary to change $\mathbf{Q}$, $\mathbf{R}$ and γ in order to find a feasible solution.

4. Simulation results

Consider a continuous-time stirred tank reactor (CSTR) with the first order irreversible parallel exothermic reactions according to the scheme $A \xrightarrow{k_1} B$, $A \xrightarrow{k_2} C$, where B is the main product and C is the side product. Under the condition of perfect mixing, the dynamic mathematical model of the controlled system has been obtained by mass balances of reactants, energy balance of the reactant mixture and energy balance of the coolant. Using usual simplifications, the model of the CSTR can be described by four nonlinear differential equations

$$\frac{dc_A}{dt} = -\left(\frac{q_r}{V_r} + k_1 + k_2\right)c_A + \frac{q_r}{V_r}c_{Af} \quad (14)$$

$$\frac{dc_B}{dt} = -\frac{q_r}{V_r}c_B + k_1c_A + \frac{q_r}{V_r}c_{Bf} \quad (15)$$

$$\frac{dT_r}{dt} = -\frac{h_1k_1 + h_2k_2}{\rho_r c_{pr}}c_A + \frac{q_r}{V_r}(T_{rf} - T_r) + \frac{A_h U}{V_r \rho_r c_{pr}}(T_c - T_r) \quad (16)$$

$$\frac{dT_c}{dt} = \frac{q_c}{V_c}(T_{cf} - T_c) + \frac{A_h U}{V_c \rho_c c_{pc}}(T_r - T_c) \quad (17)$$

with initial conditions $c_A(0)$, $c_B(0)$, $T_r(0)$ and $T_c(0)$. Here, t is time, c are concentrations, T are temperatures, V are volumes, ρ are densities, c_p are specific heat capacities, q are volumetric flow rates, h are reaction enthalpies, A_h is the heat transfer area and U is the heat transfer coefficient. The subscripts denote $_r$ the reactant mixture, $_c$ the coolant, $_f$ feed values and the superscript s the steady-state values. The reaction rates k_1 , k_2 are expressed as

$$k_j = k_{0j} \exp\left(\frac{-E_j}{RT_r}\right), j = 1, 2 \quad (18)$$

where k_0 are pre-exponential factors, E are activation energies, R is the gas constant. The values of all parameters and feed values are in Table 1.

Table 1. Parameters and inputs of the chemical reactor

$V_r = 0.23 \text{ m}^3$	$\rho_r = 1020 \text{ kg m}^{-3}$	$A_h = 1.51 \text{ m}^2$	$c_{Af} = 4.22 \text{ kmol m}^{-3}$
$V_c = 0.21 \text{ m}^3$	$\rho_c = 998 \text{ kg m}^{-3}$	$U = 42.8 \text{ kJ m}^{-2} \text{ min}^{-1} \text{ K}^{-1}$	$c_{Bf} = 0 \text{ kmol m}^{-3}$
$q_r^s = 0.015 \text{ m}^3 \text{ min}^{-1}$	$c_{pr} = 4.02 \text{ kJ kg}^{-1} \text{ K}^{-1}$	$g_1 = E_1 / R = 9850 \text{ K}$	$T_{rf} = 310 \text{ K}$
$q_c^s = 0.004 \text{ m}^3 \text{ min}^{-1}$	$c_{pc} = 4.182 \text{ kJ kg}^{-1} \text{ K}^{-1}$	$g_2 = E_2 / R = 22019 \text{ K}$	$T_{cf} = 288 \text{ K}$

Model uncertainties of the over described reactor follows from the fact that there are four only approximately known physical parameters in this reactor: $h_1 \in [-8.8 \times 10^4; -8.4 \times 10^4]$, $h_2 \in [-5.7 \times 10^4; -5.3 \times 10^4]$, $k_{10} \in [1.5 \times 10^{11}; 1.6 \times 10^{11}]$, $k_{20} \in [4.95 \times 10^{26}; 12.15 \times 10^{26}]$. The nominal values of these parameters are mean values of intervals.

The steady state behavior of the chemical reactor with nominal values and also with all 16 combinations of minimal and maximal values of 4 uncertain parameters was studied at first. It can be stated the reactor has always three steady states, two of them are stable and one is unstable. The maximal concentration of the product B is obtained in the unstable steady state. So, the main operating point is described by unstable steady-state values of state variables. The situation for the nominal model is shown in Figure 1, where Q_{GEN} is the heat generated by chemical reactions and Q_{OUT} is the heat removed by the jacket and the product stream. The main operating point for the nominal model is $[c_A^s, c_B^s, T_r^s, T_c^s] = [1.8614 \text{ kmol.m}^{-3}, 1.0113 \text{ kmol.m}^{-3}, 338.41 \text{ K}, 328.06 \text{ K}]$.

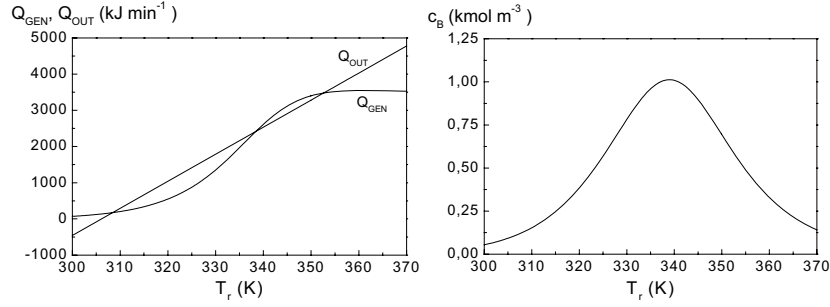


Figure 1. Steady state behavior of the chemical reactor for nominal values of uncertain parameters

Design of a robust stabilizing controller is based on having a linear state space model (1) of the controlled system. Linearized mathematical model has been derived under the assumption that the control inputs are the reactant flow rate q_r and the coolant flow rate q_c and the controlled output is the temperature of reaction mixture T_r . The other input variables are considered to be constant. The matrices of the nominal linearized model in the main operating are

$$A_0 = \begin{pmatrix} -0.1479 & 0 & -0.0226 & 0 \\ 0.0354 & -0.0652 & 0.0057 & 0 \\ 1.3763 & 0 & 0.2118 & 0.0685 \\ 0 & 0 & 0.0737 & -0.0928 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 10.2546 & 0 \\ -4.3968 & 0 \\ -123.5131 & 0 \\ 0 & -190.7612 \end{pmatrix},$$

$C_0 = (0 \ 0 \ 1 \ 0)$. This model is unstable, the eigenvalues of A_0 are $-0.0652, 0.1195, -0.0741+0.0310i, -0.0741-0.0310i$. For 4 uncertain parameters, we have obtained $2^4=16$ linearized mathematical models, which differ in coefficients of A_i, B_i . These systems represent vertices of the uncertain polytopic system and they all are unstable. It was further important to find a robust static output feedback, which would be able to stabilize the whole uncertain system with the guaranteed cost expressed by (7), where $Q = q_{const} \cdot \text{diag}(1, 1, 1 \times 10^{-5}, 1 \times 10^{-5})$, $R = r_{const} \cdot \text{diag}(1 \times 10^3, 1 \times 10^3)$. The parameters of matrices Q, R have been chosen according to the values of state variables and control inputs.

For finding a stabilizing output feedback controller it is necessary to solve two sets of LMIs (12), (13), each set consisting of 16 LMIs. The feasibility of the solution of (12) assures that the reactor is robust static output feedback quadratically stabilizable and the feasibility of the solution of (13) gives robust static output stabilizing controller with guaranteed cost for the whole uncertain system.

For solving the LMIs, the LMI MATLAB toolbox was used. There are three parameters, which influence solution and can be changed: $q_{const}, r_{const}, \gamma$. In dependence on the choice of these parameters, it was possible to find several stabilizing controllers, which stabilize the polytopic system with 16 vertices and also stabilize the reactor. For all stabilizing controllers all closed loop systems obtained for the nominal system and also for all vertices of the polytopic system are stable, e. a. all eigenvalues of all state matrices (4) of all 17 closed loop systems have negative real parts.

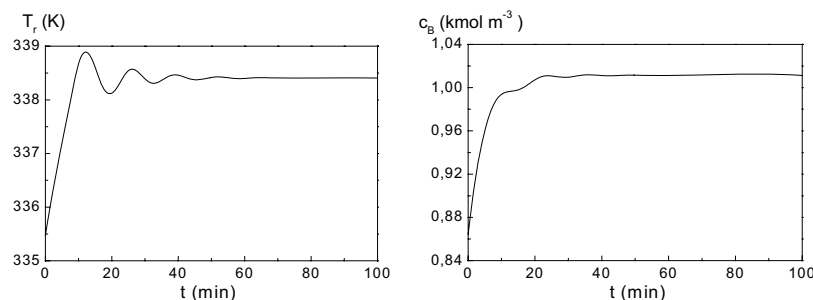


Figure 2. Closed-loop response of the CSTR with robust output feedback controller

Some of the simulation results obtained with the robust static feedback controller $F = [0.0023 \ 0.0186]^T$ are shown in Figure 2 for the nominal values of uncertain parameters.

Conclusions

In this paper, the possibility to stabilize an exothermic chemical reactor with uncertainties working in the unstable operating point via static output feedback controller is studied. The robust controller design is converted to solving of LMI problems. A computationally simple LMI based non-iterative algorithm is used for the design of robust static output feedback controller. This algorithm is based on linear state-space representation of a controlled system. The design procedure guarantees with sufficient conditions the robust quadratic stability and guaranteed cost. The designed robust controller is able to stabilize the exothermic CSTR for the entire operating area not only for a single operating point.

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