

## A Problem Formulation for Optimal Mixed-Sized Box Packing

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### Abstract

In this paper, an approach for optimal mixed-size box packing is presented. The problem manages the packing of mixed-sized rectangular boxes to achieve optimal three-dimensional unit allocation in a limited space. Allocation of boxes of mixed size enables load plans for both multi-product and single-product problems. The problem formulation presented is a further extension of previous formulations applied on two-dimensional Facility Layout and two and three-dimensional Process Plant Layout problems.

**Keywords:** MILP, Optimisation, Packing problems

### 1. Introduction

Packing problems may be formulated as allocation problems in a three dimensional Euclidian space. The problem formulation presented in this work is concerned with the spatial allocation of mixed-sized boxes into fixed dimensional containers. The boxes may, however, be allocated into a container in such way that there is space between the boxes. Furthermore, the formulation allows all boxes to rotate in three dimensions and they may be allocated in multiple layers formed by different sized and shaped boxes in order to make the package as tight as possible. The formulation is not layer oriented and determines the allocation of each box simultaneously. The presented mathematical model is a further extension of previous work on process plant layout problems, Patsiatzis and Papageorgiou (2002), and on facility layout problems, Castillo and Westerlund (2005). Furthermore, the proposed mathematical model shares some similarities with the three dimensional palletisation method of Tsai, Malmstrom and Kuo (Tsai et al. 1993). The model proposed in this paper, however, offers some substantial practical differences in comparison with the model of Tsai et al.. The ability to use a set of different container types instead of one fixed pallet size in the packing pattern being the most important.

Minimising transportation costs, warehouse costs, container loading costs and other related layout problems is of great relevance for many industrial organisations. The considered packing problem formulation may easily be applied to other three dimensional allocation problems, thus, offering great possibilities for industrial applications. The applicability of the problem formulation presented in this paper is demonstrated through some illustrative examples.

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## 2. Problem Description

A typical packing problem consists of a number of different shaped boxes, which are to be optimally packed into containers. A number of different containers are available of which all or just a few may be used. Furthermore, each container has given dimensions and specific fixed cost, depending on the size and the shape of the container. In the considered approach, each container used will add the fixed container cost to the cost function regardless of how many boxes it contains or how great percentage of the total container volume is used. Other cost functions, differentiating for costs connected to bottom area, bottom width or bottom length may also be included.

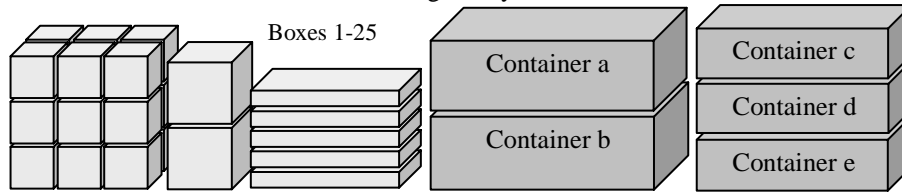


Figure 1. Boxes (light grey boxes) are to be packed into containers (dark grey boxes).

## 3. Problem Formulation

The problem formulation presented in this paper can be seen as an extension of previous research work on Process Plant Layout (PPL) and Facility Layout (FL) problems. The PPL formulations mentioned above are presented in Papageorgiou and Rotstein (1998) and Patsiatzis and Papageorgiou (2002). The FL formulations mentioned are presented in Castillo and Westerlund (2005).

The packing problem formulation can be stated as follows:

Given:

- A set of  $N$  boxes of different shape
- A set of  $K$  containers
- Cost data (fixed container costs)

Determine:

- Which containers to use
- Detailed layout (which boxes in what container, spatial allocation of boxes)

### 3.1 Objective function

The objective function may be formulated in different ways. If the costs related to each container is minimised, the objective function can be written as in equation (1).

$$\min \sum_{k=1}^K C_k \cdot D_k \quad (1)$$

$D_k$  is a binary variable equal to 1 if there are items allocated in container  $k$ ; 0 otherwise.  $C_k$  is a fixed container cost, specific for each container type. The objective function may easily be expanded with different logistic cost drivers.

### 3.2 Main constraints

In the proposed model, four major sets of constraints are used, volume, non-overlapping, box orientation and container constraints. The volume constraints are used to prevent the boxes to be allocated outside the containers and the non-overlapping constraints are used to prevent overlapping between boxes. The box orientation constraints are used in order to allow the boxes to rotate in three dimensions while the container constraints are used in order to define if a certain container is used or not.

### 3.2.1 Volume constraints

Each box is to be allocated inside a container volume. No part of any box is to be allocated outside the container. A container  $k$  containing one box,  $i$ , is shown in figure 2.  $CCX_i$ ,  $CCY_i$  and  $CCZ_i$  represent variable dimensions of a container in which box  $i$  is allocated which can be defined as functions of binary variables  $\beta_{k,i}$ , defining in which specific container  $k$ , with the given dimensions:  $CX_k$ ,  $CY_k$  and  $CZ_k$ , the box,  $i$  is placed, according to eq. (4) and (4b).  $X_i$ ,  $Y_i$  and  $Z_i$  are variables defining the dimensions of box  $i$  and the centroid of box  $i$  is allocated at coordinates  $y_i$ ,  $z_i$ ,  $x_i$ .

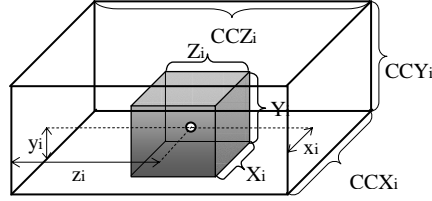


Figure 2. Box  $i$  allocated inside container  $k$ .

The volume constraints are given in equations (2) – (3).

$$x_i \geq \frac{1}{2} X_i, \quad y_i \geq \frac{1}{2} Y_i, \quad z_i \geq \frac{1}{2} Z_i \quad (2)$$

$$x_i \leq CCX_i - \frac{1}{2} X_i, \quad y_i \leq CCY_i - \frac{1}{2} Y_i, \quad z_i \leq CCZ_i - \frac{1}{2} Z_i \quad (3)$$

$$CCX_i = \sum_{k=1}^K CX_k \cdot \beta_{k,i}, \quad CCY_i = \sum_{k=1}^K CY_k \cdot \beta_{k,i}, \quad CCZ_i = \sum_{k=1}^K CZ_k \cdot \beta_{k,i} \quad (4)$$

$$\sum_{k=1}^K \beta_{k,i} = 1 \quad (4b)$$

### 3.2.2 Non-overlapping constraints

The non-overlapping constraints are used to avoid overlap of boxes allocated in the same container. The non-overlapping constraints used in the packing model are further extensions of the non-overlapping considerations presented in Patsiatzis and Papageorgiou (2002) and in Castillo and Westerlund (2005). The model includes constraints designed to prevent boxes from occupying the same physical location. The boxes may overlap each other in one or two dimensions, one dimension should however always not be violated. Non-overlapping is guaranteed if at least one of the inequalities in equations (5a-5c) is active. To get the disjunctive non-overlapping constraints written in conjunctive form, big-M constraints are used as shown in equations 5a-5e.  $P_{ij}$  and  $Q_{ij}$  are binary variables, used in the big-M non-overlapping constraints in x-dimension.  $R_{ij}$  and  $S_{ij}$  are corresponding variables used in y-dimension and  $T_{ij}$  and  $U_{ij}$  in the z-dimension.  $M_x$ ,  $M_y$  and  $M_z$  are appropriate upper bounds.  $G_{ij}$  is a binary variable, equal to 1 if  $i$  and  $j$  are allocated in the same container; 0 otherwise. The constraints (5a) – (5e) are valid for all values on  $i, j$  defined by:  $1 \leq i < j \leq N$ .

$$\frac{x_i + x_j}{2} \leq (x_i - x_j) + M_x (P_{ij} + Q_{ij}), \quad \frac{x_i + x_j}{2} \leq (x_j - x_i) + M_x (P_{ij} - Q_{ij} + 1) \quad (5a)$$

$$\frac{y_i + y_j}{2} \leq (y_i - y_j) + M_y (R_{ij} + S_{ij}), \quad \frac{y_i + y_j}{2} \leq (y_j - y_i) + M_y (R_{ij} - S_{ij} + 1) \quad (5b)$$

$$\frac{z_i + z_j}{2} \leq (z_i - z_j) + M_z (T_{ij} + U_{ij}), \quad \frac{z_i + z_j}{2} \leq (z_j - z_i) + M_z (T_{ij} - U_{ij} + 1) \quad (5c)$$

$$P_{i,j} + R_{i,j} + T_{i,j} + G_{i,j} \leq 3 \quad (5d)$$

$$\beta_{k,i} + \beta_{k,j} \leq G_{i,j} + 1, \quad k = 1, \dots, K \quad K \neq 1 \quad (5e)$$

### 3.2.3 Orientation constraints

The dimensions of each box are determined using special orientation constraints in order to allow the boxes to rotate. The box orientation is determined as shown in equation (6) where the  $b$  variables are binary variables. The original box dimensions,  $L_i$ ,  $W_i$  and  $H_i$  are used to determine the orientation variables for each box,  $X_i$ ,  $Y_i$  and  $Z_i$ .

$$\begin{cases} X_i = b_{xiL} \cdot L_i + b_{xiW} \cdot W_i + b_{xiH} \cdot H_i, & b_{xiL} + b_{xiW} + b_{xiH} = 1 \\ Y_i = b_{yiL} \cdot L_i + b_{yiW} \cdot W_i + b_{yiH} \cdot H_i, & b_{yiL} + b_{yiW} + b_{yiH} = 1 \\ Z_i = b_{ziL} \cdot L_i + b_{ziW} \cdot W_i + b_{ziH} \cdot H_i, & b_{ziL} + b_{ziW} + b_{ziH} = 1 \\ b_{xiL} + b_{yiL} + b_{ziL} = 1, & b_{xiW} + b_{yiW} + b_{ziW} = 1, & b_{xiH} + b_{yiH} + b_{ziH} = 1 \end{cases} \quad (6)$$

### 3.2.4 Container constraints

In order to define if a certain container is used or not, equation (7) is defined.

$$\sum_{i=1}^N \beta_{k,i} \leq N \cdot D_k \quad k = 1, \dots, K \quad (7)$$

## 3.3 Additional constraints

Packing problems are large scale combinatorial optimisation problems, requiring significant computational effort for their solution. In order to cut down the CPU time for these problems, a couple of formulation enhancement methods are considered. Observe, however, that although a large number of binary variables are included in the formulation above, many of the binaries are involved in different special order sets, minimising the combinatorial complexity.

### 3.3.1 Non-over-packing constraints

These constraints are used to prevent *over-packing* attempts. Each container may only contain a selection of boxes with an aggregated box volume equal to or less than the total volume of the container. Pre-determining that the total volume of the boxes inside each container may not be greater than the container volume cuts down the CPU time by preventing any attempts to allocate more boxes inside a container than it in practice can embody. The constraints used are shown in equation (8).

$$\sum_{i=1}^N (L_i \cdot W_i \cdot H_i) \cdot \beta_{k,i} \leq (CX_k \cdot CY_k \cdot CZ_k) \cdot D_k \quad k = 1, 2, \dots, K \quad (8)$$

### 3.3.2 Symmetry-breaking constraints

Due to the geometry of the packing problems considered in this paper, equivalent symmetrical solution will be received for each problem, implying the existence of multiple optima's. These symmetries consume additional CPU time in already tedious problems. By applying further developed versions of the symmetry-breaking constraints, applied to PPL problems, presented in Westerlund and Papageorgiou (2004), the problem symmetries are avoided. An example of equivalent symmetrical solutions for the same packing problem is demonstrated in figure 3.

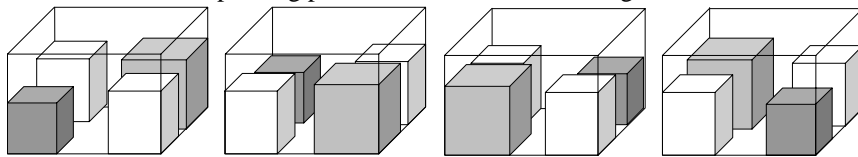


Figure 3. Symmetrical solutions to the same packing problem.

The symmetry-breaking constraints used in this paper are forcing one box in each container to be placed in a specified corner. Other boxes are relaxed by using the big-M formulation in equation (9) where the big-M is an appropriate upper bound.

$$\left. \begin{aligned} x_j &\leq \frac{1}{2}CCX_j + M'_x \cdot \sum_{i=1}^{j-1} G_{i,j} \\ y_j &\leq \frac{1}{2}CCY_j + M'_y \cdot \sum_{i=1}^{j-1} G_{i,j} \\ z_j &\leq \frac{1}{2}CCZ_j + M'_z \cdot \sum_{i=1}^{j-1} G_{i,j} \end{aligned} \right\} \quad j = 1, \dots, N \quad (9)$$

Observe that the sum on the RHS is excluded when  $j=1$ .

#### 4. Illustrative examples

To illustrate the benefit of the formulation presented, two illustrative examples are considered. In both cases, a number,  $N$ , of boxes are to be optimally allocated into a number,  $K$ , of containers. The objective function in equation (1) is used, where  $C_k$  for each container is listed in tables 2 and 4. The dimensions of the boxes and containers are displayed in tables 1 – 4 and the solution data for the examples is shown in table 5. The examples were solved using Ilog Cplex 8.0 (2002) CPU times are given in seconds.

Table 1. Dimensions of boxes, example 1.

Box	Wi	Hi	Li
1	1	2	1
2	1	2	1
3	1	2	1
4	2	2	2
5	2	2	2
6	2	2	2
7	2	3	2
8	2	3	2
9	2	3	2
10	2	4	2
11	2	4	2
12	2	4	2

Table 2. Dimensions and cost of containers, example 1.

Container	Cxk	Cyk	Czk	CCk
1	4	5	4	8
2	4	5	4	8
3	4	5	4	8
4	4	6	4	10
5	6	6	6	25

Table 3. Dimensions of boxes, example 2.

Box	Wi	Hi	Li
1	2	2	2
2	2	2	2
3	2	2	2
4	2	2	2
5	2	2	3
6	2	2	3
7	2	2	3
8	2	2	3
9	2	2	3
10	2	2	3
11	3	3	1
12	3	3	1
13	1	2	5

Table 4. Dimensions and cost of containers, example 2.

Container	Cxk	Cyk	Czk	CCk
1	3	3	7	80
2	3	3	7	80
3	4	4	7	110
4	4	4	7	110

Table 5. Solution data for example 1 and 2.

Example	Objective function	CPU time	Boxes in container 1	Boxes in container 2	Boxes in container 3	Boxes in container 4	Boxes in container 5
1	16	2.7	10,11,12	1,2,3,4,5,6,7,8,9	-	-	-
2	190	67.5	5,8,12	-	1,2,3,4,6,7,9,10,11,13	-	-

The optimal container packing is shown in figures 4-11. Figures 4 and 5 show the optimal box positions for example 1 and figures 8 and 9 for example 2 when minimising the total cost, as in eq. (1). Figures 6 and 7 show the final box positions for example 1 after a second optimisation step and figures 10 and 11 showing ditto for example 2. The second optimisation step, illustrated in figures 6-7 and 10-11, is an optional step where the boxes are further packed inside the containers to achieve the tightest possible packing pattern.

When the solution from the first step is obtained, the orientation of each box as well as the container content of each container is fixed. Subsequently, all boxes in each container are, separately, packed tighter by maximising the free space in all dimensions.

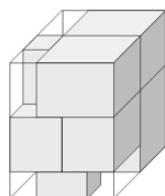


Fig.4. Cont. 1, step 1.

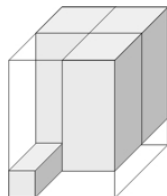


Fig.5. Cont. 2, step 1.

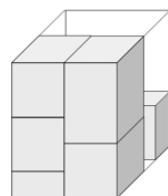


Fig.6. Cont. 1, step 2.

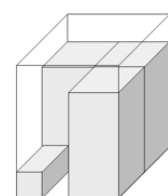


Fig.7. Cont. 2, step 2.

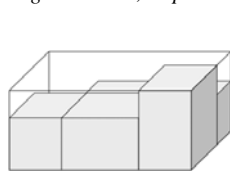


Fig.8. Cont. 1, step 1.

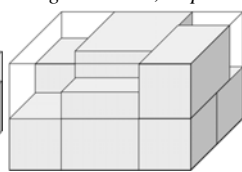


Fig.9. Cont. 3, step 1.

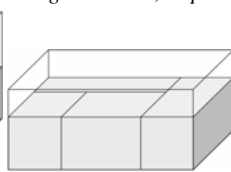


Fig.10. Cont. 1, step 2.

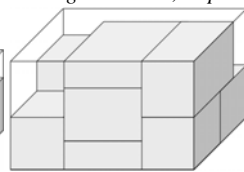


Fig.11. Cont. 3, step 2.

## 5. Conclusions

In this paper, a problem formulation for mixed-size box packing is presented. The applicability of the formulation is demonstrated through two illustrative examples. In addition to the actual problem formulation, some formulation enhancement strategies are also considered in order to cut down the computational effort needed for solution. By virtue of the universal nature of the formulation, the considered method may be applied on a wide variety of problems, *inter alia*, container loading or palletisation problems. Moreover, the formulation may easily be extended to cover problems in four or more dimensions by expanding the formulation, specifically the constraints in equation (5a) – (5e) to cover as many dimensions as required, thereby enabling taking into account a time dimension or other logistical aspects.

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